JOURNAL OF COMBINATORIAL THEORY, Series A **65**, 164–171 (1994)

Note

A Geometry of Rank 5 Associated with $PGO_5(3)$

MICHEL DEHON*

Département de Mathématique, C.P. 216, Université Libre de Bruxelles, Boulevard du Triomphe, B-1050 Brussels, Belgium

Communicated by Francis Buekenhout

Received May 21, 1992

We construct a thin, residually connected, primitive, and flag-transitive geometry of rank 5. Its residues of type $\{i, i+1 (\text{mod } 5)\}$ $(i = 0, ..., 4)$ are hexagons; the other rank 2 residues are triangles.

1. INTRODUCTION

Let I be a set of n elements (called *types*) and let $\Gamma = (X, *, t)$ be a triple such that X is a set, $*$ is a reflexive and symmetric relation on X (called the *incidence relation*) and t is a surjective function from X to I. The type of a subset $Y \subset X$ is the set $t(Y)$ and for every $i \in I$, the elements of $t^{-1}(i)$ are called the *i-varieties*. A *flag* of Γ is a set of pairwise incident elements of X; a flag of type I is called a *chamber*. Γ is a *geometry* if

1. $x * y$ and $t(x) = t(y)$ implies $x = y$,
2. every flag of Γ is contained into a chamber.

The rank of a geometry Γ is $n = |I|$. If F is a flag of Γ, the triple $\Gamma_F = (X_F, *_F, t_F)$ where X_F is the set of varieties of type belonging to $I - t(F)$ which are incident to all the elements of F, $*_F$, and t_F are the restrictions of $*$ and t to X_F is a geometry of rank $n - |t(F)|$ which is called the *residue* of F in Γ. The *type* of Γ_F is the set $I - t(F)$.

A permutation α of X is an *automorphism* of Γ if it preserves the incidence and the types; i.e., if

(a) $x * y \Leftrightarrow \alpha(x) * \alpha(y)$, for every $x, y \in X$

(b) $t(\alpha(x)) = t(x)$, for every $x \in X$.

0097-3165/94 $6.00

A permutation verifying (a) but not necessarily (b) is called an *extended automorphism* of Γ. Clearly, the set $\mathrm{Aut}(\Gamma)$ of all the automorphisms of Γ is a subgroup of the group of all the extended automorphisms.

Let $G \subset \mathrm{Aut}(\Gamma)$, we say that G is *flag-transitive* if, for every $J \subset I$, G acts transitively on the flags of type J. Γ is *flag-transitive* if $\mathrm{Aut}(\Gamma)$ is flag-transitive. In a flag-transitive geometry, all the residues of a given type are isomorphic.

The *diagram* of a flag-transitive geometry Γ of rank n is the complete graph on the set of types of Γ, provided with the following information:

— on each vertex i, we indicate the cardinality of the residue of a flag of type $I - \{i\}$, diminished by 1

— on each edge $\{i, j\}$, we indicate parameters concerning the incidence graph of the residue of a flag of type $I - \{i, j\}$; namely, from i to j, the diameter of this incidence graph starting from an i-variety, the gonality of the graph (i.e., the half of the length of a smallest circuit) and the diameter starting from a j-variety.

Usually, the edges labelled 2, 2, 2 are omitted, the labels 3, 3, 3 are omitted, and the labels n, n, n $(n \geqslant 4)$ are replaced by a simple n.

Now, let Γ be a rank n flag-transitive geometry and $G = \mathrm{Aut}(\Gamma)$. Γ is *firm* if every non-maximal flag is contained in at least two chambers, it is *thin* if every flag of rank $n-1$ is contained in exactly two chambers. Γ is *residually connected* if the incidence graphs of Γ and of all its residues of rank $\geqslant 2$ are connected graphs. As it can be rather long to check if a geometry is residually connected, the following result that we can apply using a CAYLEY algorithm is useful:

THEOREM. [3] *Let $F = \{x_1, ..., x_n\}$ be a chamber of Γ and, for every $i = 1, ..., n$, let G_i be the stabilizer of x_i in G. For every $J \subset I$, we define $G_J = \bigcap_{j \in J} G_j$. Γ is residually connected if and only if for every $J \subset I$ such that $|J| \leqslant n-2$, G_J is generated by its subgroups $G_{J \cup \{k\}}$ $(k \in I - J)$.*

Finally, it is well known that G acts primitively on the set of all i-varieties of Γ if and only if the stabilizer of a fixed i-variety is a maximal subgroup of G. Γ is called *primitive* if G acts primitively on the i-varieties for every $i \in I$.

In this paper we construct a thin, residually connected, primitive, and flag-transitive geometry Γ having the diagram shown in Fig. 1.

The group $G = \mathrm{Aut}(\Gamma)$ is the group $PGO_5(3)$ of the projectivities stabilizing a non-degenerate quadric in $PG(4, 3)$. This group of order 51840 is, among others, isomorphic to the automorphism group of the Schäfli graph constructed on the 27 lines of a general cubic surface and to the group generated by the 36 reflections in the set of minimal vectors of the

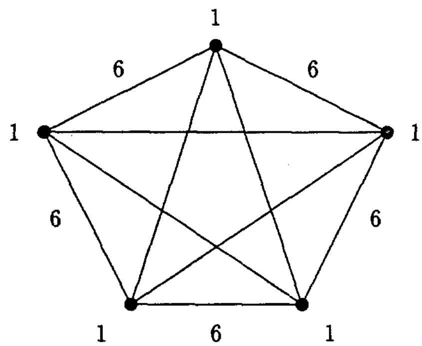

FIGURE 1.

E_6 lattice. The stabilizer of a chamber of Γ is the identity and the extended automorphism group of Γ induces the dihedral group D_5 on the set of types.

Since we have to work intensively in a generalized quadrangle to construct Γ, let me also recall the following definition:

A *generalized quadrangle of order* (s, t) is an incidence structure $S=(P, D)$ consisting of a set P of *points* and a set D of *lines* such that

1. every pair of points is contained in at most one line,

2. every line contains exactly $s+1$ points and every point is contained in exactly $t+1$ lines,

3. for every point p and every line d not containing p, there exists exactly one point p' and one line d' such that p, $p' \in d'$, and $p' \in d$.

2. THE GEOMETRY

Let Q be a non-degenerate quadric in the four-dimensional projective space over $GF(3)$. It is well known that the 40 points and the 40 lines of $PG(4, 3)$ which are contained in Q have a structure of generalized quadrangle $S = (P, D)$ of order $(3, 3)$. This generalized quadrangle is not self-dual. The 81 points of $PG(4, 3) - Q$ are divided into two orbits for $PGO_5(3)$: there are 36 points each of which is contained in 10 tangent hyperplanes of Q, and each of the 45 remaining points lies in 16 tangent hyperplanes.

The group $G = PGO_5(3)$ of all the projectivities stabilizing Q has order 51840, it is the full automorphism group of S.

Let D_i $(i=0, ..., 4)$ be five disjoint copies of the set D of lines of S, let $X = \bigcup_{i=0}^{4} D_i$ and let us denote by $d_{i,j}$ $(i=0, ..., 4; j=1, ..., 40)$ the elements of D_i in such a way that, for every j, the elements $d_{0,j}, \ldots, d_{4,j}$ correspond to the same line of D which is denoted by d_j. For every j, $j' = 1, \ldots, 40$ we define $x_{j,j'} = |d_j \cap d_{j'}|$ in S. For convenience, let us take the convention that all the additions and subtractions appearing in this paper have to be computed in $\mathbb{Z}_5$ (except when they correspond obviously to operations in $GF(3)$).

We define an incidence relation $*$ on D as

$$d_{i,j} * d_{k,l} \Leftrightarrow (x_{j,l} \text{ and } i-k = \pm 1) \quad \text{or} \quad (x_{j,l} = 0 \text{ and } i-k = \pm 2)$$

and an application $t \colon D \to \{0, 1, 2, 3, 4\}$ by $t(d_{i,j}) = i$.

Let us prove the following results:

LEMMA 1. *G acts transitively on all the 5-tuples $(d_1, d_2, d_3, d_4, d_5)$ of lines of S such that $x_{i,j} = 1$ if $i-j = \pm 1$ and $x_{i,j} = 0$ if $i-j = \pm 2$.*

Proof. Let $(d_1, d_2, d_3, d_4, d_5)$ be such a 5-tuple. Every intersection of Q with a three-dimensional subspace of $PG(4, 3)$ consists of four lines meeting in a common point (tangent hyperplane) or of a non-degenerate three-dimensional quadric. None of these three-dimensional quadrics can contain a configuration of five lines intersecting as $d_1, \ldots, d_5$. Thus $\{d_1, \ldots, d_5\}$ is not contained in a proper subspace of $PG(4, 3)$. The subspace of $PG(4, 3)$ generated by $\{d_i \cap d_{i+1} \mid i = 1, \ldots, 5\}$ contains the lines d_i because the points $d_i \cap d_{i+1}$ are different; so this set of five points generates $PG(4, 3)$. Now, let α be a projectivity stabilizing Q and each line d_i; α fixes the points $d_i \cap d_{i+1}$ $(i = 1, \ldots, 5)$ and the pole of the hyperplane H generated by d_1, d_2, d_3. So α fixes a basis of $PG(4, 3)$ and is the identity.

On the other hand, let us compute the total number of configurations $(d_1, d_2, d_3, d_4, d_5)$ verifying the hypothesis. We have 40 choices for d_1. There are 12 lines intersecting d_1; let us choose one of them as d_2. There are nine lines intersecting d_2 in a point different from $d_1 \cap d_2$ and all these lines are disjoint from d_1, so we choose one of them as d_3. Among the nine lines intersecting d_3 at a point distinct from $d_2 \cap d_3$ there are three lines which also intersect d_1, so we have six possibilities for d_4. Finally, each point of d_4 is contained in one line intersecting d_1; for d_5, we cannot choose the line joining $d_1 \cap d_2$ to a point of d_4 or the line joining $d_3 \cap d_4$ to a point of d_1, so two possibilities remain. The total number of configurations is thus $40.12.9.6.2 = 51840$ and the lemma is proved because this is equal to $|G|$ and the only element of G fixing a possible configuration of five lines in the identity.

It follows from this lemma that G acts transitively on the chambers of Γ.

LEMMA 2. *Each flag of Γ which is not a chamber is contained in at least two chambers.*

Proof. It is easy to compute the number of chambers containing a given flag of Γ, using the axioms defining a generalized quadrangle as in the proof of Lemma 1. We obtain the results

Rank	Description of the flag	Number of chambers containing such a flag
4	$\{d_{i_1,j_1}, d_{i_2,j_2}, d_{i_3,j_3}, d_{i_4,j_4}\}$ $i_4 = i_3 + 1 = i_2 + 2 = i_1 + 3$	2
3	$\{d_{i_1,j_1}, d_{i_2,j_2}, d_{i_3,j_3}\}$ $i_3 = i_2 + 1 = i_1 + 2$	12
	$\{d_{i_1,j_1}, d_{i_2,j_2}, d_{i_3,j_3}\}$ $i_3 = i_2 + 2 = i_1 + 3$	6
2	$\{d_{i_1,j_1}, d_{i_2,j_2}\}$ $i_2 = i_1 + 1$	108
	$\{d_{i_1,j_1}, d_{i_2,j_2}\}$ $i_2 = i_1 + 2$	48
1	$\{d_{i_1,j_1}\}$	1296

LEMMA 3. *Γ is flag-transitive.*

Proof. We have to prove that if two flags F and F' have the same type, then there exists an element $g \in G$ such that $g(F) = F'$. This follows from Lemma 1 if F and F' are chambers. If not, let F_1 (resp., F_1') be a chamber containing F (resp., F'); there exists $g \in G$ such that $g(F_1) = F_1'$ and we have $g(F) = F'$.

LEMMA 4. *Γ is residually connected.*

Proof. This can be verified by running the following CAYLEY program:

G: permutation group (40);
$G \cdot$ generators:

$a = (1, 2, 5, 13)\ (3, 8, 10, 22)\ (4, 11, 15, 6)\ (7, 18, 25, 34)\ (9, 20, 23, 12)$ $(14, 19, 21, 26)\ (16, 27, 31, 35)\ (17, 30, 24, 33)\ (28, 36, 40, 37)$ $(29, 32, 38, 39)$,

$b = (1, 3, 9, 21, 10, 23)\ (2, 6, 16, 28, 4, 12)\ (5, 14)\ (7, 13, 22, 34, 40, 39)$ $(8, 19, 18, 32, 24, 25)\ (11, 17, 31, 15, 26, 20)\ (27, 29, 37, 35, 38, 33)$ $(30, 36)$,

```
  c = (1, 4) (2, 7) (3, 10) (5, 15) (6, 17) (8, 9) (11, 24) (12, 20) (13, 25)
      (14, 19) (16, 29) (18, 33) (21, 26) (22, 23) (27, 35) (28, 36) (30, 34)
      (31, 38) (32, 39) (37, 40);

X = empty;
X = append (X, stabilizer (G, [1, 4, 30, 34]));
X = append (X, stabilizer (G, [15, 19, 28, 34]));
X = append (X, stabilizer (G, [10, 20, 28, 37]));
X = append (X, stabilizer (G, [11, 14, 25, 37]));
X = append (X, stabilizer (G, [4, 11, 23, 35]));
```

"These statements define the group G acting on the 40 *points of S and a sequence X containing the stabilizers of* 5 *lines of S corresponding to the varieties of a chamber of Γ"*

```
print G eq⟨X[1], X[2], X[3], X[4], X[5]⟩;
T = [1, 2, 3, 4, 5];
for i = 1 to 5 do
  R = empty;
  for each j in T − [i] do R = append (R, X[i] meet X[j]); end;
  print X[i] eq⟨R[1], R[2], R[3], R[4]⟩;
end;
for i = 1 to 5 do
  for j = i + 1 to 5 do
          Z = X[i] meet X[j];
          R = empty;
          for each k in T − [i, j] do R = append (R, Z meet X[k]); end;
          print Z eq⟨R[1], R[2], R[3]⟩;
  end;
end;
for i = 1 to 5 do
  for j = i + 1 to 5 do
          for k = j + 1 to 5 do
            Z = X[i] meet X[j] meet X[k];
            R = empty;
            for each l in T − [i, j, k] do R = append (R, Z meet X[l]);
              end;
            print Z eq⟨R[1], R[2]⟩;
          end;
  end;
end;
```

The geometry Γ is thin since each flag of rank 4 is contained in exactly two chambers. It is primitive because the stabilizer of a variety in G is the stabilizer of a line of S; this is a group of order 1296 which is maximal in G. Let us now construct the diagram of Γ.

LEMMA 5. *Every residue of type* $\{i, i+1\}$ $(i=0, ..., 4)$ *of* Γ *is an hexagon.*

Proof. Since all the non-degenerate quadrics of $PG(4, 3)$ are equivalent under the action of G we can suppose that Q is the quadric

$$X_1X_3+X_1X_4+X_2X_4+X_2X_5+X_3X_5=0.$$

All the flags of type $\{j, j+1, j+2\}$ of Γ are equivalent under the extended automorphisms group of the geometry; it is thus sufficient to prove the result for the residue of the flag $F=\{d_{0,0}, d_{1,1}, d_{2,2}\}$ so that d_0, d_1, and d_2 are the following lines of Q:

$$d_0 : X_1=X_2=X_3=0$$
$$d_1 : X_1=X_2=X_5=0$$
$$d_2 : X_1=X_4=X_5=0.$$

We have $x_{0,1}=x_{1,2}=1$ and $x_{0,2}=0$ so F is effectively a flag. The 3-varieties incident to d_0, d_1, and d_2 correspond to the six lines of Q which have one point in common with d_2 and no point in common with d_0 and d_1. The equations of these lines are

$$d_a\begin{cases}X_1+2X_5=0\\X_2+2X_3=0\\X_4=0\end{cases}\qquad d_c\begin{cases}X_1+2X_4=0\\X_2+X_3+X_4=0\\X_5=0\end{cases}\qquad d_e\begin{cases}X_1+X_3=0\\X_3+2X_4=0\\X_4+X_5=0\end{cases}$$

$$d_g\begin{cases}X_1+X_4=0\\X_1+X_2+2X_3=0\\X_5=0\end{cases}\qquad d_i\begin{cases}X_1+2X_4=0\\X_2+X_3+X_4+X_5=0\\X_4+2X_5=0\end{cases}\qquad d_k\begin{cases}X_3=0\\X_4=0\\X_5=0\end{cases}$$

The 4-varieties incident to d_0, d_1, and d_2 correspond to the six lines of Q which have one point in common with d_0 and no point in common with d_1 and d_2. The equations of these lines are

$$d_b\begin{cases}X_1+2X_2=0\\X_1+2X_3=0\\X_2+X_3+X_4=0\end{cases}\qquad d_d\begin{cases}X_1+X_3=0\\X_2=0\\X_3+X_4+2X_5=0\end{cases}\qquad d_f\begin{cases}X_1+X_3=0\\X_1+X_4=0\\X_2+X_3=0\end{cases}$$

$$d_h\begin{cases}X_1+2X_3=0\\X_2=0\\X_3+X_4+X_5=0\end{cases}\qquad d_j\begin{cases}X_1+2X_2=0\\X_3=0\\X_4+2X_5=0\end{cases}\qquad d_l\begin{cases}X_2=0\\X_3=0\\X_4=0.\end{cases}$$

It is easy to verify that $x_{a,b}=x_{b,c}=\cdots=x_{k,l}=x_{l,a}=1$ and that all the other intersections of a line of $\{d_a, d_c, d_e, d_g, d_i, d_k\}$ with a line of $\{d_b, d_d, f_f, d_h, d_j, d_l\}$ are empty; this proves the lemma.

LEMMA 6. *Every residue of type* $\{i, i+2\}$ $(i=0, ..., 4)$ *of* Γ *is a triangle.*

Proof. We can suppose that $i=0$. Let $F=\{d_{1,j_1}, d_{3,j_3}, d_{4,j_4}\}$ be a flag; we have $x_{j_3,j_4}=1$ and $x_{j_1,j_3}=x_{j_1,j_4}=0$. In S, let p be the unique point of d_{j_1} which is collinear with $d_{j_3}\cap d_{j_4}$ and let $\{a, b, c\}=d_{j_1}-\{p\}$. We denote by d_{a_0} (resp., d_{b_0}, d_{c_0}) the unique line containing a (resp., b, c) and one point of $d_{j_4}-d_{j_3}$. Similarly, let d_{a_2} (resp., d_{b_2}, d_{c_2}) be the unique line containing a (resp., b, c) and one point of $d_{j_3}-d_{j_4}$. d_{0,a_0}, d_{0,b_0}, and d_{0,c_0} are the 0-varieties of Γ_F and d_{2,a_2}, d_{2,b_2}, d_{2,c_2} are its 2-varieties. We have $x_{a_0,a_2}=x_{b_0,b_2}=x_{c_0,c_2}=1$ and $x_{a_0,b_2}=x_{a_0,c_2}=x_{a_1,c_0}=x_{a_1,c_2}=x_{a_2,c_0}=x_{a_2,c_1}=0$ since the six lines intersect d_{j_1} and S contains no triangle. Γ_F is a triangle because a variety of type 0 and a variety of type 2 are incident in Γ_F if and only if the corresponding lines of S have an empty intersection.

We have thus proved the theorem:

THEOREM. Γ *is a thin, residually connected, primitive, and flag-transitive geometry of rank 5. Its residues of type* $\{i, i+1\}$ $(i=0, ..., 4)$ *are hexagons; its residue of type* $\{i, i+2\}$ *are triangles.*

ACKNOWLEDGMENT

The author thanks F. Buekenhout for stimulating discussions.

REFERENCES

1. F. BUEKENHOUT, The geometry of the finite simple groups, *in* "Buildings and the Geometry of Diagrams," Lecture Notes in Math., Vol. 1181, pp. 1–78, Springer, New York, 1986.
2. J. CANNON, "A Language for Group Theory," University Sydney, 1982.
3. M. HERMAND, "Géométries Langage CAYLEY et Groupe de Hall-Janko," Thèse de doctorat, Université Libre de Bruxelles, 1990.
4. J. W. P. HIRSCHFELD, "Projective Geometries over Finite Fields," Oxford Mathematical Monographs, 1979.

Printed in Belgium
Uitgever: Academic Press, Inc.
Verantwoordelijke uitgever voor België:
Hubert Van Maele
Altenastraat 20, B-8310 Sint-Kruis