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A Nonembeddable Noetherian Ring

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We provide an example of a (left and right) Noetherian C-algebra which cannot be embedded into any Artinian ring.

INTRODUCTION

An old problem in the theory of Noetherian rings, and a natural extension of Goldie's Theorems, is that of determining whether all Noetherian rings can be embedded into Artinian ones. For the case of right Noetherian rings, a nonembeddable example was found by Small [6]. However, the proof that his ring cannot be embedded depends on the fact that it does not satisfy the ascending chain condition on left annihilators; obviously, such a criterion is not available for the case of (left and right) Noetherian rings.

Among Noetherian rings, on the other hand, many of the standard classes are known to admit embeddings, since the rings in question satisfy primary decomposition and the primary factors have Artinian quotient rings. (Here, a ring R is said to be *primary* if it has a unique associated prime when considered as a right R-module.) Examples for which this approach works include fully bounded rings and factor rings of both enveloping algebras of solvable Lie algebras and of group rings of polycyclic-by-finite groups. This can be seen by combining [4, Section 4] with [4, Theorem 1.8, parts (1) and (6)].

Guided by these positive results, an obvious Noetherian ring to test for

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nonembeddability is a particular factor ring of the universal enveloping algebra $U(\mathrm{sl}(2, \mathbf{C}))$ first considered by Brown [2]. This ring does not satisfy primary decomposition and admits no obvious embedding; indeed, it is this ring which we prove cannot be embedded.

The idea of the proof is the following. Let R be any ring and suppose that R can be embedded into a right Artinian ring, S. Then one may consider the function d which is defined on finitely presented right R-modules by

$$d(M) = \frac{\mathrm{length}(M \otimes_R S)_S}{\mathrm{length}\ S_S} \tag{0.1}$$

for each such module M. We assume that our particular ring can be embedded and then obtain bounds on $d(M)$ for enough modules M to get the desired contradiction.

We might remark that a function which satisfies the elementary properties of d is called a faithful Sylvester rank function; see Remark 3.2 for the formal definition. One of the main results of [5] is that a necessary and sufficient condition for an algebra R over a field k to admit an embedding into a right Artinian ring—indeed, into a simple Artinian ring—is that R admit a faithful Sylvester rank function. (Necessity is immediate!) Recently, Blair and Small [1] have used this result to show that Krull homogeneous Noetherian k-algebras, among others, can be embedded.

Throughout this paper, all rings contain an identity element which is inherited by any subring, and all modules are unitary. The symbol $\subset$ stands for strict inclusion, while $M^{(n)}$ denotes the direct sum of n copies of the module M.

1. The Ring

In this section, we collect various facts from the literature about the ring we want to consider.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of the Lie algebra $\mathfrak{g} = \mathrm{sl}(2, \mathbf{C})$. We denote by e, f, and h the standard basis elements of $\mathfrak{g}$; thus, $[h, e] = 2e$, $[h, f] = -2f$, and $[e, f] = h$. Write $\Omega = 4fe + h^2 + 2h$ for the Casimir element of $U(\mathfrak{g})$ and $K = \mathfrak{g}U(\mathfrak{g})$ for the augmentation ideal.

The ring which interests us is $U = U(\mathfrak{g})/\Omega K$. The facts we need are the following.

U is a Noetherian domain of Rentschler–Gabriel Krull dimension 1, written K-dim $U = 1$. (1.1a)

U has exactly two proper, nonzero ideals, both completely prime. They are

$$P = \Omega U/\Omega K \subset Q = K/\Omega K. \tag{1.1b}$$

Note that $U/Q \cong \mathbf{C}$, while K-dim $U/P = 1$.

We have the identities $Q^2 = Q$ and $QP = PQ = 0$. In particular, P is the nilradical (and the Artin radical) of U, and U is subdirectly irreducible. (1.1c)

Of the above results, K-dim $U/P = 1$ can be found in [7], while the others are listed in [2, Example 6.4] and can be proved directly using [3].

While U is the ring that interests us, its factor ring $V = U/P$ is a rather more tractable object and many of our computations will take place in this ring. Write $\bar{e}$, $\bar{f}$, and $\bar{h}$ for the images of e, f, and h in V. The facts we will use about V are the following.

V is a primitive domain with exactly one proper nonzero ideal, $M = Q/P = \bar{e}V + \bar{f}V + \bar{h}V$. (1.2a)

Set $A = \bar{e}V + (\bar{h} - 2)V$ and $B = \bar{e}V + \bar{h}V$. Then A and B are projective right ideals of V satisfying $A \oplus B \cong V \oplus V$. (1.2b)

Furthermore, $A/AM \cong V/M \oplus V/M$, while $B/BM = 0$. (1.2c)

Of the above results, (1.2a) follows from (1.1), while (1.2b) and (1.2c) are established within the proofs of [9, Lemma 3.1 and Corollary 2.8].

2. Epimorphisms between Various V-Modules

Suppose that U embeds in an Artinian ring, and consider the function d on U-modules defined by (0.1). One immediately sees that $d(X) \leqslant d(Y)$ whenever X is a factor of the right U-module Y. In this section, we therefore obtain various epimorphisms between U-modules (in fact, between V-modules). These will be used in the next section to bound the function d and obtain the desired contradiction. The key to this approach is to use results from [8] that give conditions under which a projective V-module will map onto a second V-module.

We begin with the notation and terminology needed to state these results. Fix a Noetherian ring R and finitely generated right R-modules F and G, with F projective. F is said to *cover* G if every simple factor module of G is also a simple factor of F. Equivalently, $F^{(n)}$ maps onto G for some positive integer n. If J is a prime ideal of R, let $\rho(G, J)$ denote the reduced rank of the R/J-module G/GJ. Then we define

$$\begin{aligned} \hat{g}^F(G, J) &= 0, && \text{if } \rho(G, J) = 0 \\ &= \frac{\rho(G, J)}{\rho(F, J)}, && \text{otherwise.} \end{aligned}$$

Finally, write $g^F(G) = n$ if n is the smallest integer such that $F^{(n)}$ maps onto G, with $g^F(G) = \infty$ if no such integer exists. It is easy to prove, for any prime ideal J, that $g^F(G) \geqslant \hat{g}^F(G, J)$. However, we need to bound the $g^F(G)$ from above by the $\hat{g}^F(G, J)$. For this we use the following result, which is a slightly weakened version of [8, Corollary 4.6].

THEOREM 2.1. *Let F and G be finitely generated right modules over a Noetherian ring R, with F projective. Suppose that F covers G. Then*

$$g^F(G) \leqslant K\text{-dim } R + \sup\{\hat{g}^F(G, J) \mid J \text{ is a prime ideal}\}.$$

Now we apply this theorem to show that appropriate direct sums of the projective V-modules A and B from Section 1 map onto various modules.

PROPOSITION 2.2. *Keep the notation of Section 1. Let $n \geqslant 2$ be an integer. Then $A^{(n)}$ maps onto the right V-module $V^{(n-1)} \oplus (V/M)^{(n-1)}$.*

Proof. Set $G_n = V^{(n-1)} \oplus (V/M)^{(n-1)}$. Note that, as $A \nsubseteq M$ and M is the unique proper ideal of V, A is a progenerator and so covers any finitely generated V-module. Since V and V/M are domains, it is immediate that

$$\rho(G_n, 0) = \rho(V^{(n-1)}, 0) = n - 1$$

while

$$\rho(G_n, M) = 2(n-1).$$

Also, $\rho(A, 0) = 1$ while, by (1.2c), $\rho(A, M) = 2$. Thus, for both of the prime ideals $J = 0$ and $J = M$ of V, we have $\hat{g}^A(G_n, J) \leqslant n - 1$. Finally, K-dim $V = 1$, by (1.1b). Thus, Theorem 2.1 may be applied to conclude that $g^A(G_n) \leqslant n$, as required.

PROPOSITION 2.3. *Keep the notation of Section 1. Then $B^{(n)}$ maps onto $Q^{(n-1)}$, for each integer $n \geqslant 2$.*

Proof. Here Q is the unique maximal ideal of U, but since $QP = 0$ by (1.1c), Q is a right module over $V = U/P$.

We first need to check that B covers Q, a fact which is not completely trivial. First note that, as M is the only proper ideal of V, the trace ideal of B is either M or V (in fact, it is M). Certainly, then, B covers M. Thus $M = \bar{\phi}(B^{(r)})$ for some homomorphism $\bar{\phi}$ and integer r. Since B is projective as a V-module and $M = Q/P$, pull back $\bar{\phi}$ to a map ϕ from $B^{(r)}$ to Q. Thus, $Q = \phi(B^{(r)}) + P$. By (1.1c), $Q = Q^2 = QM$ and $PQ = 0$. Therefore,

$$Q = QM = (\phi(B^{(r)} + P)\, M = \phi(B^{(r)})\, M = \phi(B^{(r)}),$$

where the final equality follows from (1.2c). Thus B does indeed cover Q.

Now we want to apply Theorem 2.1. Since $QM = Q$, it follows that $g^B(Q, M) = 0$. Further, $g^B(Q, 0) = 1$, since both $Q/P = M$ and B are nonzero right ideals of the domain V. It now follows from Theorem 2.1 that $g^B(Q^{(n-1)}) \leqslant n$, and the conclusion of the lemma is immediate.

Remark 2.4. It is really only in this proposition that we use full strength of the ideal structure of U. The fact that Q is a V-module implies that (a) $QP = 0$. On the other hand, since $BM = B$, the fact that B covers Q forces $QM = Q$ or, equivalently, (b) $Q^2 = Q$. It is routine to verify that a ring with just two nonzero prime ideals $P \subset Q$ must be nonprimary and have a nontrivial subdirectly irreducible factor ring if it satisfies properties (a) and (b).

3. The Ring U Cannot Be Embedded

We combine the earlier results to show that U cannot be embedded into any Artinian ring. We begin by elaborating on the concept of Sylvester rank functions, mentioned in the introduction.

LEMMA 3.1. *Let R be a ring and suppose that R can be embedded into a right Artinian ring S. For a finitely presented right R-module M, define*

$$d(M) = \frac{\operatorname{length}(M \otimes_R S)_S}{\operatorname{length} S_S}$$

to be the normalized length of the right S-module induced by M. Then d satisfies the following properties:

$$d(R) = 1. \tag{3.1a}$$

There exists an integer n such that $d(X) \in \frac{1}{n}\mathbf{Z}$ for every finitely presented right R-module X. (3.1b)

If $0 \to X \to Y \to Z \to 0$ *is a short exact sequence of finitely presented right R-modules, then* $d(Z) \leqslant d(Y) \leqslant d(X) + d(Z)$. *If the sequence splits then* $d(Y) = d(X) + d(Z)$. (3.1c)

$d(R/J) < 1$ *for any nonzero right ideal J of R.* (3.1d)

Proof. This is immediate.

Remark 3.2. For any ring R, a function on finitely presented right R-modules satisfying properties (3.1a)–(3.1c) is called a *Sylvester* (*module*) *rank function.* The rank function is *faithful* if it also satisfies (3.1d). Again, we mention a remarkable result of Schofield [5]; namely, a k-algebra R can be embedded into a right Artinian ring if and only if R admits a faithful Sylvester rank function. Indeed, [5, Theorems 7.10 and 7.11] prove that in this case R actually embeds into a simple Artinian ring.

We are now ready to prove our main result.

THEOREM 3.3. *Let* U *be the factor ring of* $U(\mathrm{sl}(2, \mathbf{C}))$ *described in Section* 1. *Then* U *cannot be embedded into any right Artinian ring.*

Proof. Suppose that U can be embedded into a right Artinian ring S, and let d be the faithful Sylvester rank function defined in Lemma 3.1. We aim to use the results of the last section to obtain a contradiction. Thus, set

$$d(U/P) = a \qquad \text{and} \qquad d(U/Q) = b.$$

By (3.1d), $a < 1$ and $b < 1$. By (3.1c), $d(P) \geqslant 1 - a$ and $d(Q) \geqslant 1 - b$. Now consider the modules A and B of Section 1, but as U-modules. From (1.2b) we have

$$d(A) + d(B) = d((U/P) \oplus (U/P)) = 2a.$$

By Proposition 2.2 combined with (3.1c), we obtain

$$nd(A) \geqslant (n-1)(a+b)$$

for each integer $n \geqslant 2$ and, therefore,

$$d(A) \geqslant a + b.$$

Similarly, Proposition 2.3 implies that

$$d(B) \geqslant d(Q) \geqslant 1 - b.$$

Combining the displayed inequalities gives

$$2a = d(A) + d(B) \geqslant (a+b) + (1-b) = a+1.$$

Equivalently, $a \geqslant 1$, contradicting (3.1d).

We end by remarking that the standard filtration on $U(\mathfrak{g})$ induces a filtration on U and the associated graded ring, $\mathrm{gr}(U)$ is then commutative Noetherian. As such, $\mathrm{gr}(U)$ automatically embeds into an Artinian ring. Thus, Theorem 3.3 shows that for embedding problems one cannot draw any conclusions from properties of an associated graded ring.

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Note added in proof. The first author has recently extended the results of this paper to show that the enveloping algebra of any semisimple Lie algebra has a nonembeddable factor ring.

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