Applied Mathematics Letters 22 (2009) 871–875

Contents lists available at ScienceDirect

Applied Mathematics Letters

journal homepage: www.elsevier.com/locate/aml

Oscillation of a family of q-difference equations

Jia Baoguo [a,b], Lynn Erbe [a], Allan Peterson [a,*]

[a] Department of Mathematics, University of Nebraska-Lincoln, Lincoln, NE 68588-0130, USA

[b] School of Mathematics and Computer Science, Zhongshan University, 510275, Guangzhou, China

ARTICLE INFO

Article history:
Received 3 July 2008
Accepted 11 July 2008

Keywords:
Classification
Oscillation
Nonoscillation
q-difference equation

ABSTRACT

We obtain the complete classification of oscillation and nonoscillation for the q-difference equation

$$x^{\Delta\Delta}(t)+\frac{b(-1)^n}{t^c}x(qt)=0,\quad b\neq 0,$$

where $t=q^n\in\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, $c,b\in\mathbb{R}$. In particular we prove that this q-difference equation is nonoscillatory, if $c>2$ and is oscillatory, if $c<2$. In the critical case $c=2$ we show that it is oscillatory, if $|b|>\frac{1}{q(q-1)}$, and is nonoscillatory, if $|b|\leq\frac{1}{q(q-1)}$.

1. Introduction

Let $\mathbb{T}$ be a time scale (i.e., a closed nonempty subset of $\mathbb{R}$) with $\sup\mathbb{T}=\infty$. Consider the second order dynamic equation on time scale

$$x^{\Delta\Delta}(t)+p(t)x^{\sigma}(t)=0, \tag{1.1}$$

where σ is the jump operator and $f^{\sigma}=f\circ\sigma$ (composition of f with σ), p is right-dense continuous functions on $\mathbb{T}$ and

$$\int_{t_0}^{\infty}p(t)\Delta t:=\lim_{t\to\infty}\int_{t_0}^{t}p(s)\Delta s\quad\text{exists (finite).}$$

When $\mathbb{T}=\mathbb{R}$ the dynamic equation (1.1) is the differential equation

$$x''+p(t)x=0, \tag{1.2}$$

and when $\mathbb{T}=\mathbb{Z}$ the dynamic equation (1.1) is the difference equation

$$\Delta^2x(t)+p(t)x^{\sigma}(t)=0. \tag{1.3}$$

When $\mathbb{T}=q^{\mathbb{N}_0}$, $q>1$, the dynamic equations (1.1) are called q-difference equations, which have important applications in quantum theory [8,6]. Our main results are for a family of q-difference equations. For $\mathbb{T}=\mathbb{R}$, in [10,4], Willett and Wong proved, respectively, the following theorems.

* Corresponding author.
E-mail addresses: mcsjbg@mail.sysu.edu.cn (J. Baoguo), lerbe2@math.unl.edu (L. Erbe), apeterso1@math.unl.edu, apeterso@math.unl.edu (A. Peterson).

0893-9659/$ – see front matter
doi:10.1016/j.aml.2008.07.014

Theorem A (*Willett–Wong, [10,4]*). *Suppose that*

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)\mathrm{d}s \leq \frac{1}{4}\bar{P}(t),$$

for large t, where $\bar{P}(t) = \int_t^\infty P^2(s)Q_P(s,t)\mathrm{d}s$, $Q_P(s,t) = \exp\left(2\int_t^s P(\tau)\mathrm{d}\tau\right)$. Then the differential equation (1.2) is nonoscillatory.

Theorem B (*Willett–Wong, [10,4]*). *If $\bar{P}(t) \not\equiv 0$ satisfies*

$$\int_t^\infty \bar{P}^2(s)Q_P(s,t)\mathrm{d}s \geq \frac{1+\epsilon}{4}\bar{P}(t),$$

for some $\epsilon > 0$ and large t. Then the differential equation (1.2) is oscillatory.

As applications of Theorems A and B, Willett [10] considered the very sensitive differential equation

$$x'' + \frac{\mu \sin \nu t}{t^\eta}x = 0 \tag{1.4}$$

for $|\frac{\mu}{\nu}| \neq \frac{1}{\sqrt{2}}$, $\mu \neq 0$, $\nu \neq 0$, η constants and proved that (1.4) is nonoscillatory, if $\eta > 1$ and is oscillatory, if $\eta < 1$. When $\eta = 1$, (1.4) is oscillatory, if $|\frac{\mu}{\nu}| > \frac{1}{\sqrt{2}}$, and is nonoscillatory, if $|\frac{\mu}{\nu}| < \frac{1}{\sqrt{2}}$.

Wong proved the following very nice result.

Theorem C (*Wong, [4]*). *If there exists a function $\bar{B}(t)$ such that*

$$\int_t^\infty [\bar{P}(s)+\bar{B}(s)]^2 Q_P(s,t)\mathrm{d}s \leq \bar{B}(t),$$

for large t, then the differential equation (1.2) is nonoscillatory.

As applications of Theorem C, Wong proved that Eq. (1.4) is nonoscillatory, for $|\frac{\mu}{\nu}| = \frac{1}{\sqrt{2}}$.

In [1,2], we extended Theorems A–C to the time scale case using the so-called 'second-level Riccati equation' (see [3] for the discrete case) or what Wong refers to as a new Riccati integral equation in the continuous case. Using this approach, one is able to handle various critical cases. These ideas are novel in treating the case when $P(t) := \int_t^\infty p(s)\mathrm{d}s$ is not of one sign for large t.

A special case of results in [1,2], is that the difference equation

$$\Delta^2 x(n) + \frac{b(-1)^n}{n^c}x(n+1) = 0, \quad b \neq 0, \tag{1.5}$$

where $b, c \in \mathbb{R}$ is nonoscillatory, if $c > 1$ and is oscillatory, if $c < 1$. Also if $c = 1$, then Eq. (1.5) is oscillatory, if $|b| > 1$ and is nonoscillatory, if $|b| \leq 1$.

Lemma 1.1 (*[2, Theorem 3.2]*). *Assume that $\int_{t_0}^\infty p(t)\Delta t$ is convergent, $P(t) = \int_t^\infty p(s)\Delta s$, $1 \pm \mu(t)P(t) > 0$, for large t. If $\int_T^\infty P^2(t) \times \frac{e_P(t,T)}{e_{-P}(t,T)}\Delta t$ is convergent and*

$$\bar{P}(t) := \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \tag{1.6}$$

satisfies

$$\frac{1}{4}\bar{P}(t) \geq \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s. \tag{1.7}$$

for large t, then (1.1) is nonoscillatory.

2. Main theorem

Our main concern in this paper is the q-difference equation

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c}x(qt) = 0, \quad b \neq 0, \tag{2.1}$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $b, c \in \mathbb{R}$ and our main result is the following complete classification of (2.1). Since the graininess function for $\mathbb{T} = q^{\mathbb{N}_0}$ is unbounded, we cannot use Theorem 4.1 in [2], when we consider the oscillation of the q-difference equation (2.1).

Theorem 2.1. *The q-difference equation (2.1) is nonoscillatory, if $c > 2$, and is oscillatory, if $c < 2$. If $c = 2$, then Eq. (2.1) is oscillatory, if $|b| > \frac{1}{q(q-1)}$, and is nonoscillatory, if $|b| \le \frac{1}{q(q-1)}$.*

Proof. First consider the case $c > 2$. Note that for $t = q^{2k}$

$$\begin{aligned} P(t) &= \int_t^\infty p(\tau)\Delta\tau = \sum_{j=2k}^{\infty} p(q^j)\mu(q^j) \\ &= \frac{b(q-1)q^{2k}}{q^{2kc}}\left[1 - \frac{q}{q^c} + \frac{q^2}{q^{2c}} - \cdots\right] \\ &= b\,\frac{q^{c-1}(q-1)}{q^{2k(c-1)}(q^{c-1}+1)}. \end{aligned}$$

Similarly, we have

$$P(q^{2k+1}) = -b\,\frac{q^{c-1}(q-1)}{q^{(2k+1)(c-1)}(q^{c-1}+1)}$$

and hence in general

$$P(t) = P(t^n) = b\,\frac{(-1)^n q^{c-1}(q-1)}{q^{n(c-1)}(q^{c-1}+1)} = b\,\frac{(-1)^n q^{c-1}(q-1)}{t^{c-1}(q^{c-1}+1)}. \tag{2.2}$$

Since $c > 2$, we get that

$$\lim_{t\to\infty} \mu(t)P(t) = \lim_{n\to\infty} b\frac{(-1)^n q^{c-1}(q-1)^2}{t^{c-2}(q^{c-1}+1)} = 0,$$

which implies that for large t, $\pm P$ are positively regressive.

By the definition of the exponential [5, Definition 2.30] we have for $s \ge t$

$$\begin{aligned} e_{\pm P}(s,t) &= \exp \int_t^s \frac{1}{\tau(q-1)} \ln\left(1 \pm \frac{b(q-1)^2(-1)^{\frac{\ln\tau}{\ln q}}}{\tau^{c-2}(1+q^{(1-c)})}\right)\Delta\tau \\ &= \exp\left[\sum_{i=n}^{m-1} \ln\left(1 \pm \frac{b(q-1)^2(-1)^i}{q^{i(c-2)}(1+q^{1-c})}\right)\right]. \end{aligned} \tag{2.3}$$

Note that $\ln(1 \pm x) \sim \pm x$, so when $c > 2$, the two series

$$\sum_{i=n}^{\infty} \ln\left(1 \pm \frac{b(q-1)^2(-1)^i}{q^{i(c-2)}(1+q^{(1-c)})}\right) \tag{2.4}$$

are absolutely convergent.

Using properties of the exponential [5, Theorem 2.36], we have

$$e_{\frac{2P}{1-\mu P}}(s,t) = \frac{e_P(s,t)}{e_{-P}(s,t)}.$$

By (2.3), (2.4) and $\lim_{t\to\infty} \mu(t)P(t) = 0$, given $0 < \epsilon < 1$, there exists a large N, so that when $s = q^m \ge t = q^n \ge q^N$,

$$1 - \epsilon \le e_{\frac{2P}{1-\mu P}}(s,t)\frac{1}{1-\mu(s)P(s)} \le 1 + \epsilon. \tag{2.5}$$

So from (2.2), we get that

$$\begin{aligned} \bar{P}(t) &= \int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{P^2(s)}{1-\mu(s)P(s)}\Delta s \le (1+\epsilon)\int_t^\infty P^2(s)\Delta s \\ &\le (1+\epsilon)b^2\frac{[q^{c-1}(q-1)]^2}{(q^{c-1}+1)^2}\sum_{i=n}^{\infty} q^i(q-1)\frac{1}{q^{2i(c-1)}} \\ &= (1+\epsilon)b^2\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^n, \end{aligned} \tag{2.6}$$

for large t. It follows that

$$\bar{P}(\sigma(t)) \le (1+\epsilon)b^2 \frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2} \cdot \frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^{n+1}.$$

So

$$\int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s \le (1+\epsilon)^3 b^4\left[\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\right]^2 \times \sum_{i=n}^{\infty}\left[\frac{q^{i+1}}{q^{2(i+1)(c-1)}}\cdot\frac{q^i}{q^{2i(c-1)}}q^i(q-1)\right]$$

$$= (1+\epsilon)^3 b^4\left[\frac{q^{4(c-1)}(q-1)^7}{(q^{c-1}+1)^4}\right]\cdot\left[\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\right]^2 \frac{\frac{q^{3n+1}}{q^{(4n+2)(c-1)}}}{1-\frac{q^3}{q^{4(c-1)}}}. \tag{2.7}$$

Similarly to the proof of (2.6), we also have

$$\frac{1}{4}\bar{P}(t) > \frac{(1-\epsilon)b^2}{4}\cdot\frac{q^{2(c-1)}(q-1)^3}{(q^{c-1}+1)^2}\cdot\frac{q^{2(c-1)}}{q^{2(c-1)}-q}\left[\frac{q}{q^{2(c-1)}}\right]^n, \tag{2.8}$$

for large t. Note that when $c > 2$,

$$\lim_{n\to\infty}\frac{\frac{q^{3n+1}}{q^{(4n+2)(c-1)}}}{\frac{q^n}{q^{2n(c-1)}}} = 0.$$

From (2.7) and (2.8), we have that, for sufficiently large t,

$$\int_t^\infty e_{\frac{2P}{1-\mu P}}(s,t)\frac{\bar{P}(s)\bar{P}(\sigma(s))}{1-\mu(s)P(s)}\Delta s < \frac{1}{4}\bar{P}(t).$$

By Lemma 1.1, Eq. (2.1) is nonoscillatory.

Next we consider the case $c = 2$, that is we consider

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^2}x(qt) = 0 \tag{2.9}$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$. Expanding out Eq. (2.9) we obtain

$$x(q^{n+2}) - [q+1-bq(q-1)^2(-1)^n]x(q^{n+1}) + qx(q^n) = 0. \tag{2.10}$$

When $b = \frac{q+1}{q(q-1)^2}$, we get from (2.10) when $n = 2k$ is even $x(q^{2k+2}) = -qx(q^{2k})$, which implies that (2.10) is oscillatory. Similarly, when $b = -\frac{q+1}{q(q-1)^2}$, (2.10) is also oscillatory.

Let $d_n = q+1-bq(q-1)^2(-1)^n$ in Eq. (2.10). If we suppose that $b > \frac{q+1}{q(q-1)^2}$, we have $d_{2k} < 0$. From (2.10), we get for $n = 2k$

$$x(q^{2k+2}) + qx(q^{2k}) = d_{2k}x(q^{2k+1}) \tag{2.11}$$

which implies that (2.9) is oscillatory. Similarly, when $b < -\frac{q+1}{q(q-1)^2}$, (2.10) is also oscillatory.

Therefore in the following, we can assume that $|b| < \frac{q+1}{q(q-1)^2}$, so we have $d_n > 0$. Assume $x(t) = x(q^n)$ is a solution of (2.10) satisfying $x(t) = x(q^n) \neq 0$ for all large n. Then from (2.10), we get that

$$\frac{q}{d_{n+1}d_n}\cdot\frac{d_{n+1}x(q^{n+2})}{qx(q^{n+1})} + \frac{qx(q^n)}{d_n x(q^{n+1})} = 1.$$

Let $y(n) := \frac{d_n x(q^{n+1})}{qx(q^n)}$ and $A := \frac{q}{d_{n+1}d_n} = \frac{q}{(q+1)^2-b^2q^2(q-1)^4} > 0$ is a positive constant. We get

$$Ay(n+1) + \frac{1}{y(n)} = 1. \tag{2.12}$$

Letting $y(n) = \frac{z(n+1)}{z(n)}$, we get the second order difference equation (see [9, p. 82])

$$Az(n+2) - z(n+1) + z(n) = 0. \tag{2.13}$$

The characteristic equation of (2.13) is $\lambda^2 - \frac{1}{A}\lambda + \frac{1}{A} = 0$.

When $\frac{1-4A}{A^2} < 0$, that is $|b| > \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has complex roots $\lambda = re^{i\theta}$, $\theta \neq k\pi$, k an integer. So (2.13) has an oscillatory solution $z(n) = r^n \sin n\theta$. This means $y(n) = \frac{z(n+1)}{z(n)} = \frac{r\sin(n+1)\theta}{\sin n\theta}$ is an oscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^{n+1})}{qx(q^n)}$, we get that (2.10) has an oscillatory solution. Hence, we get that (2.10) is oscillatory.

When $\frac{1-4A}{A^2} \geq 0$, that is $|b| \leq \frac{1}{q(q-1)}$, the characteristic equation of (2.13) has a real root $\lambda = \frac{1+\sqrt{1-4A}}{2A} > 0$. So (2.13) has a nonoscillatory solution $z(n) = \lambda^n > 0$. This means $y(n) = \frac{z(n+1)}{z(n)} = \lambda > 0$ is a nonoscillatory solution of (2.12). Noticing that $d_n > 0$ and $y(n) = \frac{d_n x(q^n)}{qx(q^{n+1})}$, we get that (2.10) has a nonoscillatory solution. Hence, we get that (2.10) is nonoscillatory.

Remark. As in the case $c > 2$, using Lemma 1.1, we can also prove that (2.10) is nonoscillatory, when $|b| \leq \frac{1}{q(q-1)}$, but we cannot use Theorem 4.1 in [2] to prove the oscillation of (2.10) when $|b| > \frac{1}{q(q-1)}$, since the graininess function of $q^{\mathbb{N}_0}$ is unbounded.

Finally we consider the q-difference equation for the case $c < 2$.

$$x^{\Delta\Delta}(t) + \frac{b(-1)^n}{t^c} x(qt) = 0 \tag{2.14}$$

where $t = q^n \in \mathbb{T} = q^{\mathbb{N}_0}$, $q > 1$, $b \neq 0$, $c < 2$.

To show that (2.14) is oscillatory, for all $c < 2$, we need the following useful comparison theorem [7].

Theorem 2.2. *Assume $a \in C^1_{rd}$, $a(t) \geq 1$, $\mu(t)a^{\Delta}(t) \geq 0$ and $a^{\Delta\Delta}(t) \leq 0$. Then (1.1) is oscillatory implies $x^{\Delta\Delta}(t) + a(t)p(t)x(\sigma(t)) = 0$ is oscillatory on $[t_0, \infty)$.*

Letting $b_0 := \frac{q+1}{q(q-1)^2} > \frac{1}{q(q-1)}$, we have by Theorem 2.1, that

$$x^{\Delta\Delta}(t) \pm b_0 \frac{(-1)^n}{t^2} x(qt) = 0$$

is oscillatory. Let $a(t) = At^{\alpha}$, $A > 0$, $0 < \alpha < 1$. We have $a(t) \geq 1$, for large t and $a^{\Delta}(t) \geq 0$. It is easy to get that

$$a^{\Delta\Delta}(t) = \frac{At^{\alpha}(q^{\alpha}-1)(q^{\alpha}-q)}{t^2 q(q-1)^2} \leq 0.$$

Repeated applications of Theorem 2.2, give us that

$$x^{\Delta\Delta}(t) \pm Bt^{\beta} b_0 \frac{(-1)^n}{t^2} x(qt) = 0$$

is oscillatory, for all $\beta > 0$, $B > 0$. So the equation

$$x^{\Delta\Delta}(t) \pm Bb_0 \frac{(-1)^n}{t^{2-\beta}} x(qt) = 0$$

is oscillatory, for all $\beta > 0$, $B > 0$. This means that the equation

$$x^{\Delta\Delta}(t) + b\frac{(-1)^n}{t^c} x(qt) = 0$$

is oscillatory, for $b \neq 0$, $c < 2$. □

References

[1] Jia Baoguo, Lynn Erbe, Allan Peterson, A Wong-type oscillation theorem for second order linear dynamic equations on time scales, J. Difference Equ. Appl. (in press).

[2] Lynn Erbe, Jia Baoguo, Allan Peterson, Oscillation and nonoscillation of solutions of second order linear dynamic equations with integrable coefficients on time scales (submitted for publication).

[3] Shaozhu Chen, Lynn Erbe, Riccati techniques and discrete oscillations, J. Math. Anal. Appl. 142 (1989) 468–487.

[4] J.S.W. Wong, Oscillation and nonoscillation of solutions of second order linear differential equations with integrable coefficients, Trans. Amer. Math. Soc. 144 (1969) 197–215.

[5] M. Bohner, A. Peterson, Dynamic Equations on Time Scales: An Introduction with Applications, Birkhäuser, Boston, 2001.

[6] M. Bohner, A. Peterson, Advances in Dynamic Equations on Time Scales, Birkhäuser, Boston, 2003.

[7] Lynn Erbe, Allan Peterson, Pavel Řehák, Comparison theorems for linear dynamic equations on time scales, J. Math. Anal. Appl. 275 (2002) 418–438.

[8] V. Kac, P. Cheung, Quantum Calculus, Universitext, Springer, New York, 2001.

[9] W. Kelley, A. Peterson, Difference Equations: An Introduction with Applications, Academic Press, 2001.

[10] D. Willett, On the oscillatory behavior of the solutions of second order linear differential equations, Ann. Polon. Math. 21 (1969) 175–194.