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Abstract

The existence of modified group divisible designs with block size four is settled with a handful of possible exceptions. © 2000 Elsevier Science B.V. All rights reserved.

1. Introduction

A *group divisible design* (GDD) is a triple $(X,\mathscr{G},\mathscr{B})$ which satisfies the following properties:

(1) $\mathscr{G}$ is a partition of a set X (of *points*) into subsets called *groups*,

(2) $\mathscr{B}$ is a set of subsets of X (*blocks*) such that a group and a block contain at most one common point,

(3) every pair of points from distinct groups occurs in a unique block.

The *group-type* (*type*) of the GDD is the multiset $\{|G|: G\in\mathscr{G}\}$. We usually use an 'exponential' notation to describe group-type: group-type $g_1^{u_1}\cdots g_s^{u_s}$ indicates that there are u_i groups of size g_i for $1\leqslant i\leqslant s$. A *pairwise balanced design* (PBD) can be defined as a GDD whose groups all have size 1 (in this case, the groups need not be specified). See [6] for related definitions.

A *K-modified GDD* (K-MGDD) of type a^b is a set of ab points, equipped with a parallel class of blocks of size a, a parallel class of blocks of size b, and every block in the first parallel class meeting every block of the second; all other blocks having sizes in the set K, so that every unordered pair of points occurs together in exactly one block. As with GDDs, when $K=\{k\}$, we denote the K-MGDD by k-MGDD.

An *incomplete group divisible design* with block sizes from K is a quadruple $(V,\mathscr{G},\mathscr{H},\mathscr{B})$ where V is a finite set of cardinality v, $\mathscr{G}=(G_1,G_2,\ldots,G_s)$ is a partition of V, $\mathscr{H}=\{H_1,\ldots,H_t\}$ is a set of disjoint subsets of V (the G_is are *groups* and

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H_js are *holes*), and $\mathscr{B}$ is a family of subsets of V (*blocks*) with the properties:
(1) Any pair of distinct elements of V which occurs together in a group or a hole does not occur in any block.
(2) Each other pair of distinct elements from V occurs in exactly one block.
Let $H_{ij} = G_i \cap H_j$, and $h_{ij} = |H_{ij}|$. The IGDD has *type*

$$(g_1; h_{11}, h_{12}, \ldots, h_{1t})^{a_1}(g_2; h_{21}, h_{22}, \ldots, h_{2t})^{a_2} \cdots (g_r; h_{r1}, h_{r2}, \ldots, h_{rt})^{a_r}$$

when it has a_i groups of size g_i with sizes $h_{i1}, h_{i2}, \ldots, h_{it}$ of intersections with the t holes.

If we remove one or more subdesigns from a TD(k, v), we obtain a transversal design with holes. In the case of one hole, it is an *incomplete transversal design* (ITD). More formally, an ITD, denoted by TD$(k,m)-$TD(k,n), is a quadruple $(X, Y, \mathscr{G}, \mathscr{B})$, where X is a set of km points, $\mathscr{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k *groups* of m points each, $Y \subseteq X$ is a set of kn points such that $|Y \cap G_j| = n$ for $1 \leqslant j \leqslant k$, and $\mathscr{B}$ is a set of subsets (*blocks*) of X, each of which intersects each group in exactly one point, and such that every pair of points $\{x, y\}$ from distinct groups is either in Y or occurs in a unique block but not both. The set Y is a *hole*.

A k-HTD (*holey transversal design* with block size k) of type $\{u_i\colon 1 \leqslant i \leqslant r\}$ is a structure $(X, \{Y_i\}_{1 \leqslant i \leqslant r}, \mathscr{G}, \mathscr{B})$ where X is a km-set (of points), $\mathscr{G} = \{G_1, G_2, \ldots, G_k\}$ is a partition of X into k *groups* of m points each, $\{Y_1, Y_2, \ldots, Y_r\}$ is a partition of X into r *holes*, each hole $Y_i (1 \leqslant i \leqslant r)$ is a set of ku_i points such that $|Y_i \cap G_j| = u_i$ for $1 \leqslant j \leqslant k$, and $\mathscr{B}$ is a collection of subsets of X (*blocks*), each meeting each group in exactly one point, and such that no block contains two distinct points of any group or any hole, but any other pair of points of X is contained in exactly one block of $\mathscr{B}$.

The existence of modified group divisible designs has been studied by Assaf [3] and Assaf and Wei [4]. They have applications in constructing various types of combinatorial objects [2,8]. The existence of modified group divisible designs with block size three has been completely settled in [3]. In [4], the following result is proved. Let $E = \{\{10, 8\}, \{10, 15\}, \{10, 18\}, \{10, 23\}, \{19, 11\}, \{19, 12\}, \{19, 14\}, \{19, 15\}, \{19, 18\}, \{19, 23\}\}$.

Theorem 1.1. *If* $m, n \neq 6$, *then a* 4*-MGDD of type* m^n *exists if and only if* $(m-1)(n-1) \equiv 0 \pmod 3$ *with the possible exception of* $\{m, n\} \in E$.

The case when one of the m or n takes on the value six, except for some small cases, was left open, mainly due to the nonexistence of a 4-MGDD of type 6^4. We address the existence of 4-MGDDs of type 6^n. We develop some new constructions for MGDDs to settle this with few possible exceptions. We then settle the existence of 4-MGDDs with index greater than one completely.

2. Main constructions

Before we proceed, we need some direct constructions.

Lemma 2.1 (Asraf and Wei [4]). *There is a* 4-*MGDD of type* 6^7.

Proof. Let $V=\mathbb{Z}_{21}\times\{0,1\}$. A parallel class is $G_1=\{(3i,j)\colon i=0,1,\dots,6\}$ for $j=0,1$ and their translates. The second parallel class is $\{(7i,j)\colon i=0,1,2;\ j=0,1\}$ and its translates. The base blocks are:

$$\{(0,0),(1,0),(5,0),(2,1)\},\ \{(0,0),(6,1),(17,1),(19,1)\},$$
$$\{(0,0),(2,0),(10,1),(15,1)\},\ \{(0,0),(8,0),(11,1),(12,1)\},$$
$$\{(0,0),(10,0),(5,1),(9,1)\}.$$

Develop these under $\mathbb{Z}_{21}$ to obtain the blocks of the 4-MGDD. □

Lemma 2.2 (Asraf and Wei [4]). *There is a* 4-*MGDD of type* 6^{10}.

Proof. Let $V=\mathbb{Z}_5\times\mathbb{Z}_{10}\cup H_{10}$, where $H_{10}=\{h_0,h_1,\dots,h_9\}$. The first parallel class is $\{(0,a)\colon a\in\mathbb{Z}_{10}\}$ and its translates together with H_{10}. The second parallel class is $\{(a,0)\colon a\in\mathbb{Z}_5\}\cup\{h_0\}$ and its translates. The base blocks are:

$$\{(3,0),(4,1),(6,2),(7,3)\},\ \{(4,0),(5,1),(7,3),(8,2)\},\ \{(5,0),(6,1),(8,2),(9,3)\},$$
$$\{(0,0),(6,1),(7,3),(9,2)\},\ \{(0,0),(1,1),(7,2),(8,3)\},\ \{(1,0),(2,1),(8,4),(9,2)\},$$
$$\{(0,0),(2,1),(3,2),(9,4)\},\ \{(1,0),(4,2),(6,4),(9,3)\},\ \{(0,0),(3,1),(5,3),(8,2)\},$$
$$\{(2,0),(4,2),(7,1),(9,4)\},\ \{(1,0),(3,3),(6,2),(8,1)\},\ \{(0,0),(2,2),(5,1),(7,4)\},$$
$$\{(0,0),(1,3),(3,4),(4,1)\},\ \{(2,0),(3,3),(5,2),(6,1)\},\ \{(1,0),(2,3),(4,4),(5,1)\},$$
$$\{(0,4),(3,6),(1,8),h_7\},\ \{(0,5),(4,7),(1,9),h_8\},\ \{(0,0),(4,6),(1,8),h_9\},$$
$$\{(0,1),(3,7),(4,9),h_0\},\ \{(0,0),(3,2),(4,8),h_1\},\ \{(0,1),(2,3),(1,9),h_2\},$$
$$\{(0,3),(4,8),(1,9),h_5\},\ \{(0,2),(4,7),(2,8),h_4\},\ \{(0,0),(4,4),(3,9),h_6\},$$
$$\{(0,3),(3,4),(2,8),h_0\},\ \{(0,4),(4,5),(3,9),h_1\},\ \{(0,5),(3,9),(2,5),h_3\},$$
$$\{(0,0),(3,6),(1,9),h_4\},\ \{(0,1),(4,2),(3,8),h_6\},\ \{(0,2),(4,3),(2,9),h_7\},$$
$$\{(0,4),(3,7),(1,8),h_2\},\ \{(0,3),(1,5),(2,7),h_6\},\ \{(0,1),(3,6),(2,7),h_3\},$$
$$\{(0,2),(2,3),(3,7),h_9\},\ \{(0,0),(2,1),(1,7),h_5\},\ \{(0,3),(1,6),(4,7),h_1\},$$
$$\{(0,0),(4,2),(3,4),h_3\},\ \{(0,1),(4,3),(2,5),h_4\},\ \{(0,0),(4,5),(2,6),h_2\},$$
$$\{(0,2),(3,4),(2,6),h_5\},\ \{(0,0),(4,1),(2,5),h_7\},\ \{(0,1),(2,2),(1,6),h_8\},$$
$$\{(0,0),(3,3),(2,4),h_8\},\ \{(0,1),(1,4),(4,5),h_9\},\ \{(0,2),(1,5),(3,6),h_0\}.$$

These base blocks under the group $\alpha\colon (x,y)\mapsto(x+1,y)$ and $\alpha\colon h_i\mapsto h_{i+1}$ generate the design. □

Lemma 2.3. *There is a* 4-*MGDD of type* 6^{13}.

Proof. Let $V=\mathbb{Z}_{78}$. A parallel class is $\{6i\colon i=0,1,\ldots,12\}$ and its translates. The second parallel class is $\{13i\colon i=0,1,\ldots,5\}$ and its translates. The base blocks are $\{0,1,3,10\},\{0,4,27,38\},\{0,5,25,33\},\{0,14,29,61\},\{0,16,35,57\}$. Develop these blocks over $\mathbb{Z}_{78}$. □

Lemma 2.4 (Asraf and Wei [4]). *There is a* 4-*MGDD of type* 6^{19}.

Proof. Let $V=\mathbb{Z}_{57}\times\{0,1\}$. The first parallel class is $\{(3i,j)\colon i=0,1,\ldots,18\}$ for $j=0,1$ and their translates. The second parallel class is $\{(19i,j)\colon i=0,1,2;\ j=0,1\}$ and its translates. Base blocks are

$$\{(0,0),(8,0),(28,0),(2,1)\},\ \{(0,0),(10,0),(26,0),(6,1)\},$$

$$\{(0,0),(1,1),(9,1),(35,1)\},\{(0,0),(10,1),(15,1),(32,1)\},$$

$$\{(0,0),(11,0),(25,0),(4,1)\},\ \{(0,0),(3,1),(5,1),(16,1)\},$$

$$\{(0,0),(1,0),(13,1),(56,1)\},\ \{(0,0),(2,0),(22,1),(42,1)\},$$

$$\{(0,0),(4,0),(28,1),(29,1)\},\{(0,0),(5,0),(44,1),(54,1)\},$$

$$\{(0,0),(7,0),(18,1),(34,1)\},\ \{(0,0),(13,0),(21,1),(46,1)\},$$

$$\{(0,0),(17,0),(43,1),(47,1)\},\ \{(0,0),(22,0),(17,1),(45,1)\},$$

$$\{(0,0),(23,0),(7,1),(14,1)\}.$$

Develop the blocks under $\mathbb{Z}_{57}$. □

Lemma 2.5. *There is a* 4-*MGDD of type* 6^{25}.

Proof. We construct the 4-MGDD on the points $((\{a,b,c,d\}\times\mathbb{Z}_6)\cup\{\infty\})\times\mathbb{Z}_6$. The first parallel class, containing blocks of size 25, consists of $((\{a,b,c,d\}\times\{i\}\times\mathbb{Z}_6)\cup(\{\infty\}\times\{i\})$ for $i\in\mathbb{Z}_6$. The second parallel class, containing blocks of size six, consists of $\{x\}\times\mathbb{Z}_6\times\{i\}$ for $x\in\{a,b,c,d\}$ and $i\in\mathbb{Z}_6$, and the block $\{\infty\}\times\mathbb{Z}_6$.

There is a 4-IGDD of type $(36;6,6,6,6,6,6)^4$, which is a holey transversal design $\mathrm{TD}(4,36)-6\mathrm{TD}(4,6)$ [1]. Place this 4-IGDD on the points $\{a,b,c,d\}\times\mathbb{Z}_6\times\mathbb{Z}_6$, with holes on $\{a,b,c,d\}\times\{i\}\times\mathbb{Z}_6$ for $i\in\mathbb{Z}_6$ and groups on $\{x\}\times\mathbb{Z}_6\times\mathbb{Z}_6$ for $x\in\{a,b,c,d\}$. For $x\in\{a,b,c,d\}$, place a 4-MGDD of type 6^7 on $(\{x\}\times\mathbb{Z}_6\times\mathbb{Z}_6)\cup(\{\infty\}\times\mathbb{Z}_6)$, aligning the parallel class of blocks of size seven on $(\{x\}\times\{j\}\times\mathbb{Z}_6)\cup(\{\infty\}\times\{j\})$ for $j\in\mathbb{Z}_6$, and the parallel class of blocks of size six on $\{x\}\times\mathbb{Z}_6\times\{j\}$ for $j\in\mathbb{Z}_6$ together with the block $\{\infty\}\times\mathbb{Z}_6$. Omit the blocks of size seven in this placement (each appears within one of the final blocks of size 25). □

Lemma 2.6. *There is a* 4-*MGDD of type* 6^{31}.

Proof. Let $V = \mathbb{Z}_{93} \times \{0,1\}$. The first parallel class consists of the translates of $\{(0,0),(31,0),(62,0),(0,1),(31,1),(62,1)\}$. The second parallel class is $\{(3i,j)$: $i=0,1,\ldots,30\}$ for $j=0,1$ and their translates. Base blocks are

$$\{(0,0),(1,0),(8,0),(87,1)\},\ \{(0,1),(1,1),(8,1),(3,0)\},$$

$$\{(0,0),(5,0),(14,1),(27,1)\},\{(0,0),(10,0),(17,1),(67,1)\},$$

$$\{(0,0),(14,0),(43,1),(53,1)\}.$$

Multiply the first coordinate of each block by 16^i for $i=1,2,3,4$ to obtain 20 further blocks. Develop them over $\mathbb{Z}_{93}$. □

Lemma 2.7. *There is a* 4-*MGDD of type* 6^{37}.

Proof. Let $V=\mathbb{Z}_{222}$. The first parallel class is $\{37i$: $i=0,1,\ldots,5\}$ and its translates, and the second parallel class is $\{6i$: $i=0,1,\ldots,36\}$ and its translates. The base blocks are $\{0,1,8,21\},\{0,25,56,117\},\{0,43,128,28\},\{0,49,182,196\},\{0,67,129,70\}$. Multiply each of them by 211 and 121 to obtain 10 more blocks. Develop these 15 blocks over $\mathbb{Z}_{222}$. □

Here is the first recursive construction.

Lemma 2.8. *Suppose there exists a* 4-*MGDD of type* 6^r *and there exists a* 4-*IGDD of type* $(6r;r,r,\ldots,r)^h$, *then there is a* 4-*MGDD of type* 6^{rh}.

Proof. Align the h copies of 4-MGDD of type 6^r on the h groups of the IGDD so that the block of size r coincides with the hole. Use each hole to form a new block of size rh. □

Let $I_n=\{1,2,\ldots,n\}$ be an index set on n elements.

Lemma 2.9. *Suppose there exists a* TD$(7,m)$ *and a* 4-*MGDD of type* $(3a+1)^6$ *where* $0\leqslant a\leqslant m-1$. *Then there exists a* 4-*MGDD of type* $(6m+3a+1)^6$.

Proof. Let $G_1,\ldots,G_7$ be the groups of a TD$(7,m)$, and let $\mathscr{B}$ be its blocks. Let $V=\bigcup_{i=1}^{6} G_i$. Truncate G_7 to $a+1$ points, $s_0,s_1,\ldots s_a$. We construct a 4-MGDD of type $(6m+3a+1)^6$ on the point set $(V\times I_6)\cup(\{s_0\}\times I_6)\cup(\{s_i$: $i=1,2,\ldots,a\}\times I_3\times I_6)$. The first parallel class, consisting of blocks of size $6m+3a+1$, of the 4-MGDD contains $(G_j\times I_6)\cup(\{s_0\}\times\{j\})\cup(\{s_i : i=1,2,\ldots,a\}\times I_3\times\{j\})$, for $j\in I_6$. The second parallel class, consisting of blocks of size six, contains $\{x\}\times I_6$ for $x\in\{s_0\}\cup(\{s_i$: $i=1,2,\ldots,a\}\times I_3)$, and $(B\setminus\{s_0\})\times\{i\}$ for $i\in I_6$ and all $B\in\mathscr{B}$ with $s_0\in B$.

For every block B of size seven in the original TD$(7,m)$ containing the point s_0, we put a 4-MGDD of type 6^7 on $B\times I_6$ so that the blocks of size six align on

$(B \setminus \{s_0\}) \times \{i\}$ for $i \in I_6$, and the block $\{s_0\} \times I_6$, and the blocks of size seven align on $(\{x_j\} \times I_6) \cup (\{s_0\} \times \{j\})$ where $x_j \in B \cap G_j$ for $j \in I_6$. Omit the blocks of size seven in each placement, as each contains points of a block of size $6m+3a+1$.

For every other block $B \in \mathscr{B}$ of size seven in the truncated TD$(7,m)$, put a 4-IGDD of type $(9,3)^6$ on $((B \setminus \{s_i\}) \times I_6) \cup (\{s_i\} \times I_3 \times I_6)$ so that the hole aligns on $\{s_i\} \times I_3 \times I_6$ and the groups align on $(\{a_i\} \times I_6) \cup (\{s_i\} \times I_3 \times \{i\})$ where $a_i = B \cap G_i$, with G_i being the ith group in the original design. For every block B of size six, put a 4-GDD of type 6^6 on the set $B \times I_6$, aligning the groups on $\{x\} \times I_6$ for $x \in B$. Finally, put a 4-MGDD of type $(3a+1)^6$ on the set $(\{s_0\} \times I_6) \cup (\{s_i\colon i = 1,2,\ldots,a\} \times I_3 \times I_6)$, to get a 4-MGDD of type $(6m+3a+1)^6$. □

With the two recursions, we are now in a position to close the spectrum of 4-MGDDs of type 6^r.

Lemma 2.10. *If* $g \equiv 1 \pmod 6$, $g \geqslant 43$, *there exists a* 4-*MGDD of type* 6^g.

Proof. When m is odd and $m \geqslant 7$, there exists a TD$(7,m)$ with the possible exceptions of $m = 15, 39$ [1]. Apply Lemma 2.9 with $a = 0,2,4,6$ to obtain a 4-MGDD of type $(6m+1)^6$, $(6m+7)^6$, $(6m+13)^6$ and $(6m+19)^6$. □

Combining Lemmas 2.1, 2.3–2.7 and 2.10, we obtain:

Lemma 2.11. *If* $g \equiv 1 \pmod 6$, *there exists a* 4-*MGDD of type* 6^g.

Lemma 2.12. *There are* 4-*MGDDs of type* 6^{28} *and* 6^{40}.

Proof. There exist 4-HTDs of type 7^6 and 10^6 [1]; these are 4-IGDDs of types $(42;7,7,7,7,7,7)^4$ and $(60;10,10,10,10,10,10)^4$, respectively. Apply Lemma 2.8. □

Lemma 2.13. *There exists a* 4-*MGDD of type* 6^{34}.

Proof. Start with a 3-GDD of type 6^6, whose blocks can be partitioned into frame parallel classes [10]. Give weight 4 using a resolvable 3-MGDD of type 3^4, and extend the resulting parallel classes to get a 4-IGDD of type $(6^4 9^1)^6$. Use 4-MGDDs of types 6^7 and 6^{10} to fill groups. □

Lemma 2.14. *If* $m \geqslant 63$, *there exists a* 4-*MGDD of type* $(6m+10)^6$.

Proof. A TD$(7,m)$ exists for all $m \geqslant 63$ [1]. Apply Lemma 2.9 with $a = 3$ to obtain a 4-MGDD of type $(6m+10)^6$, using the 4-MGDD of type 6^{10} from Lemma 2.2. □

Lemma 2.15. *Let* $g \equiv 4 \pmod 6$. *If* $g \notin \{70, 94, 100, 118, 130, 142, 166, 190, 214, 238, 244, 286, 334, 370, 382\}$ *and* $g \geqslant 52$, *then there exists a* 4-*MGDD of type* g^6.

Proof. Lemma 2.14 handles all cases when $g > 382$. Now apply Lemma 2.9 with $a=3$ and values of $m \leqslant 62$ for which a TD(7,m) exists [1]. □

Lemma 2.16. *If* $g \geqslant 52$ *and* $g \neq 70, 118$, *then there is a* 4-*MGDD of type* g^6.

Proof. First apply Lemma 2.15. Then use Lemma 2.9 with $a = 9$ and values of $m = 11, 12, 17, 19, 23, 27, 31, 35, 36, 43, 51, 57$, and 59. The 4-MGDD of type 6^{28} exists by Lemma 2.12. □

Lemma 2.17. *There is a* 4-*MGDD of type* 6^{46}.

Proof. Give weight nine to all points in a block of a TD(6,7), and give weight six to all other points. Append a new column of six points. Take a parallel class of blocks of size six including the block in which all points have weight nine. For every block in the parallel class, put a 4-MGDD of type $(k+1)^6$ $(k = 6, 9)$ on the corresponding points together with the new adjoined points. For every other block, put a 4-GDD of type 6^6 or $6^5 9^1$ [9]. This gives a 4-MGDD of type 6^{46}. □

Lemma 2.18. *There exists a* 4-*MGDD of type* 6^{70}.

Proof. Take a 4-MGDD of type 7^6 (Lemma 2.1) and give every point weight 10. For every block of size six, put a 4-MGDD of type 10^6 (Lemma 2.2) on the 60 points. For every block of size four, put a 4-GDD of type 10^4. This gives a 4-MGDD of type 6^{70}. □

Lemma 2.19. *There exists a* 4-*MGDD of type* 6^{118}.

Proof. Take a 4-MGDD of type 13^6 (Lemma 2.3). Give every point weight nine and append a new column of six points. For every block of size 6, employ a 4-MGDD of type 10^6 (Lemma 2.2). For every other block of size four, employ with a 4-GDD of type 9^4 [9]. This gives a 4-MGDD of type 6^{118}. □

Combining Lemmas 2.12, 2.15–2.19, we have the following result.

Lemma 2.20. *If* $g \equiv 4 \pmod 6$, $g \neq 4, 16, 22$, *there exists a* 4-*MGDD of type* 6^g.

Finally, we combine Lemmas 2.11 and 2.20 to yield:

Theorem 2.21. *There is a* 4-*MGDD of type* 6^n *for all* $n \in \{16, 22\}$, $n \equiv 1 \pmod 3$ *and* $n \geqslant 7$.

In addition, we update the theorem of Assaf and Wei [4].

Lemma 2.22. *There is a* 4-*MGDD of type* 10^8.

Proof. Let $V=\mathbb{Z}_{10}\times(\mathbb{Z}_7\cup\{\infty\})$. The first parallel class is $\{\{i\}\times(\mathbb{Z}_7\cup\{\infty\}: i\in\mathbb{Z}_{10}\}$. The second parallel class is $\{\mathbb{Z}_{10}\times\{j\}: j\in\mathbb{Z}_7\cup\{\infty\}\}$. Base blocks are:

$$\{(0,0),(1,1),(3,3),(9,2\},\{(0,0),(4,4),(5,1),(8,6\},$$

$$\{(0,0),(5,5),(7,3),(1,6\},\{(0,\infty),(1,1),(7,3),(8,6\},$$

$$\{(0,\infty),(2,2),(5,1),(3,4\},\{(0,\infty),(4,4),(9,1),(6,0\}. \qquad \square$$

Lemma 2.23. *There is a* 4-*MGDD of type* 10^{23}.

Proof. Let $V=\mathbb{Z}_5\times\{0,1\}\times\mathbb{Z}_{23}$. The two parallel classes are $\{(0,0,i),(0,1,i): i\in\mathbb{Z}_{23}\}$ and $\{(i,0,0),(i,1,0): i\in\mathbb{Z}_5\}$ and its translates. The base blocks are

$$\{(0,0,0),(1,0,1),(4,0,2),(0,1,3)\},\ \{(0,0,0),(0,1,5),(2,1,1),(3,1,2)\},$$

$$\{(1,0,0),(4,0,5),(2,1,10),(3,1,15)\}.$$

Multiply each block by $(-,-,2^i)$ for $i=1,2,\ldots,10$ to obtain the remaining base blocks. $\square$

Lemma 2.24. *There is a* 4-*MGDD of type* 19^{11}.

Proof. Let $V=\mathbb{Z}_{19}\times\mathbb{Z}_{11}$. The two parallel classes are $\{(0,i): i\in\mathbb{Z}_{19}\}$ and $\{(i,0): i\in\mathbb{Z}_{11}\}$ together with their translates over $\mathbb{Z}_{19}\times\mathbb{Z}_{11}$. The base blocks are

$$\{(0,0),(1,1),(3,2),(12,3)\},\{(0,0),(1,2),(5,1),(13,8)\},$$

$$\{(0,0),(4,1),(6,7),(9,8)\}.$$

Multiply each block by $(1,4)^i$ (i.e., multiply the first component by 1^i and the second by 4^i) for $i=1,2,3,4$ to obtain 12 more blocks. Develop these blocks over $\mathbb{Z}_{19}\times\mathbb{Z}_{11}$. $\square$

Lemma 2.25. *There is a* 4-*MGDD of type* 19^{12}.

Proof. Take a 5-MGDD of type 6^{13} [10] and remove a group of size six to obtain a $\{4,5\}$-MGDD of type 6^{12}. Give weight three to each point and append a new column of 12 points. Employ 4-GDDs of type 3^4 and 3^5 and a 4-MGDD of type 4^{12}. $\square$

Lemma 2.26. *There is a* 4-*MGDD of type* 19^{14}.

Proof. Let $V=\mathbb{Z}_{19}\times(\mathbb{Z}_{13}\cup\{\infty\})$. The first parallel class is $\{\{i\}\times(\mathbb{Z}_{13}\cup\{\infty\}): i\in\mathbb{Z}_{19}\}$. The second parallel class is $\{\mathbb{Z}_{19}\times\{j\}: j\in\mathbb{Z}_{13}\cup\{\infty\}\}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\{(0,0),(5,5),(14,1),(11,4)\},$$

$$\{(0,0),(8,8),(18,5),(16,9)\},\{(0,0),(11,11),(15,8),(7,10)\},$$

$$\{(0,0),(15,2),(9,7),(3,5)\},\{(0,\infty),(1,1),(15,8),(12,10)\},$$

$$\{(0,\infty),(2,2),(16,8),(4,1)\}$$

and multiply each by $(11,1)^i$ for $i=0,1,2$ to obtain 21 base blocks. Develop these under the action of the group. □

Lemma 2.27. *There is a* 4-*MGDD of type* 19^{15}.

Proof. Let $V=\mathbb{Z}_{19}\times\mathbb{Z}_{15}$. The two parallel classes are $\{(i,0): i\in\mathbb{Z}_{19}\}$ and $\{(0,i): i\in\mathbb{Z}_{15}\}$ together with their translates over $\mathbb{Z}_{19}\times\mathbb{Z}_{15}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\{(0,0),(5,5),(14,14),(6,10)\},$$

$$\{(0,0),(10,10),(3,7),(1,9)\},\{(0,0),(13,13),(12,1),(16,9)\},$$

$$\{(0,0),(9,13),(8,9),(11,2)\},\{(0,0),(15,4),(3,12),(18,5)\},$$

$$\{(0,0),(17,6),(15,1),(4,3)\}.$$

and multiply each by $(11,1)^i$ for $i=0,1,2$ to obtain 21 base blocks. Develop these under the action of the group. □

Lemma 2.28. *There is a* 4-*MGDD of type* 19^{18}.

Proof. Let $V=\mathbb{Z}_{19}\times(\mathbb{Z}_{17}\cup\{\infty\})$. The first parallel class is $\{\{i\}\times(\mathbb{Z}_{17}\cup\{\infty\}): i\in\mathbb{Z}_{19}\}$. The second parallel class is $\{\mathbb{Z}_{19}\times\{j\}: j\in\mathbb{Z}_{17}\cup\{\infty\}\}$. Take the blocks

$$\{(0,0),(1,1),(3,3),(7,7)\},\{(0,0),(5,5),(14,14),(6,8)\},$$

$$\{(0,0),(8,8),(18,1),(11,13)\},\{(0,0),(15,15),(17,2),(13,4)\},$$

$$\{(0,0),(16,16),(2,6),(4,14)\},\{(0,0),(9,11),(12,7),(15,16)\},$$

$$\{(0,0),(14,16),(18,9),(7,12)\},\{(0,\infty),(1,1),(4,8),(8,7)\},$$

$$\{(0,\infty),(2,2),(13,4),(17,16)\}$$

and multiply each by $(7,1)^i$ for $i=0,1,2$ to obtain 27 base blocks. Develop these under the action of the group. □

Lemma 2.29. *There is a* 4-*MGDD of type* 19^{23}.

Proof. Let $V=\mathbb{Z}_{19}\times\mathbb{Z}_{23}$. The two parallel classes are $\{(0,i): i\in\mathbb{Z}_{23}\}$ and $\{(i,0): i\in\mathbb{Z}_{19}\}$ together with their translates over $\mathbb{Z}_{19}\times\mathbb{Z}_{23}$. The base blocks are

$$\{(0,0),(1,1),(3,2),(12,3)\},\{(0,0),(1,5),(5,1),(13,2)\},$$

$$\{(0,0),(4,1),(6,6),(9,11)\}.$$

Multiply each block by $(1,2)^i$ for $i=1,2,\ldots,10$ to obtain 30 more blocks. Develop these blocks over $\mathbb{Z}_{19}\times\mathbb{Z}_{23}$. □

With these lemmas, we can restate the theorem.

Let $F=\{\{6,16\},\{6,22\},\{10,15\},\{10,18\}\}$.

Theorem 2.30. *If* $\{m,n\} \neq \{6,4\}$, *then there exists a* 4-*MGDD of type* m^n *if and only if* $(m-1)(n-1) \equiv 0 \pmod 3$ *with the possible exceptions of* $\{m,n\} \in F$.

3. Index greater than one

The definitions at the outset can all be generalized to require each pair not in a group together (or not in a hole together, or not in a block of one of the distinguished parallel classes together) to appear in exactly λ blocks. In this case, we obtain various classes of designs with *index* λ. When $\lambda = 1$, we recover the definitions of the preceding sections.

In this section, we examine the existence of 4-MGDDs with index greater than one. Simple counting establishes that for a 4-MGDD of type m^n and index λ to exist, one requires that $\lambda(m-1)(n-1) \equiv 0 \pmod 3$ and $m,n \geqslant 4$. Hence when $\lambda \equiv 0 \pmod 3$, the basic necessary condition reduces to $m,n \geqslant 4$. When $\lambda \not\equiv 0 \pmod 3$, the basic necessary condition is the same as for index one. Now the union of two 4-MGDDs of type m^n, one of index λ_1 and the other of index λ_2, is a 4-MGDD of type m^n and index $\lambda_1 + \lambda_2$. Hence it suffices to examine cases with $\lambda \in \{2,3\}$ when the 4-MGDD of index one and type m^n is nonexistent or unknown although the basic necessary condition is met, and cases with $\lambda = 3$ when $m,n \equiv 0,2 \pmod 3$ and $m,n \geqslant 4$.

First, we treat the cases with $\lambda = 3$.

Lemma 3.1. *If whenever* $n,m \in S = \{4,5,6,7,8,9,10,11,12,14,15,18,19,23\}$ *there is a* 4-*MGDD of type* n^m *and index* 3, *then whenever* $n,m \geqslant 4$, *there is a* 4-*MGDD of type* n^m *and index* 3.

Proof. There exist PBDs with block sizes from S of order n and m [5]. Let $(V,\mathscr{B})$ be such a PBD of order m, and $(W,\mathscr{D})$ be such a PBD of order n. We form the required 4-MGDD on the point set $V \times W$. For $B \in \mathscr{B}$ and $D \in \mathscr{D}$, place a 4-MGDD of index 3 on $B \times D$, omitting the parallel classes on $\{b\} \times D$ for $b \in B$, and on $B \times \{d\}$ for $d \in D$. □

Lemma 3.2. *Let* $K \subseteq \{4,7,10,13,19\}$. *If a* K-*PBD of order* m *and index* 3 *exists, and* $n \in S$, *then a* 4-*MGDD of type* n^m *and index* 3 *exists except possibly when* $4 \in K$ *and* $n = 6$, *or when* $10 \in K$ and $n \in \{15,18\}$.

Proof. Let $(V,\mathscr{B})$ be the K-PBD of order m and index 3. Let W be an n-set. We form the required 4-MGDD on the point set $V \times W$. For $B \in \mathscr{B}$, place a 4-MGDD of index 1 on $B \times W$, omitting the parallel classes on $\{b\} \times W$ for $b \in B$, and on $B \times \{w\}$ for $w \in W$. □

In view of Lemma 3.1, useful ingredients for Lemma 3.2 have $m \in S$.

Lemma 3.3. *There is a* $\{4\}$*-PBD of index* 3 *and order* m *whenever* $m \equiv 0, 1 \pmod 4$. *There is a* $\{7\}$*-PBD of index* 3 *and order* 15. *There is a* $\{4, 10\}$*-PBD of index* 3 *and order* 11. *There are* $\{4, 7\}$*-PBDs of index* 3 *and orders* 14, 18, *and* 23.

Proof. For the first two statements, see [7]. For order 11, employ base blocks $\{0, 1, 5, 7\}$ and $\{\infty, 0, 1, 3\}$ over $\mathbb{Z}_{10} \cup \{\infty\}$, together with $\mathbb{Z}_{10}$ as a block of size 10. For order 14, on $\mathbb{Z}_7 \times \{0, 1\}$, take base blocks

$$\{(0,0),(1,0),(0,1),(3,1)\}, \{(0,0),(2,0),(0,1),(6,1)\},$$

$$\{(0,0),(4,0),(0,1),(5,1)\}, \{(0,0),(1,1),(2,1),(4,1)\},$$

$$\{(0,1),(1,0),(2,0),(4,0)\}, \{(0,1),(3,0),(5,0),(6,0)\},$$

together with the single block $\mathbb{Z}_7 \times \{1\}$ of size 7.

For order 18, on $\mathbb{Z}_9 \times \{0, 1\}$, form the base blocks

$$\{(0,0),(1,0),(2,0),(4,0),(0,1),(1,1),(3,1)\}, \{(0,0),(1,0),(4,0),(4,1)\},$$

$$\{(0,0),(2,0),(5,0),(7,1)\}, \{(0,0),(1,1),(4,1),(5,1)\}, \{(0,0),(2,1),(4,1),(6,1)\},$$

$$\{(0,0),(3,1),(6,1),(7,1)\}.$$

For order 23, on $\mathbb{Z}_{16} \cup \{\infty_i: 0 \leqslant i \leqslant 6\}$, form the base blocks

$$\{\infty_0, 0, 1, 3\}, \{\infty_1, 0, 1, 5\}, \{\infty_2, 0, 1, 8\}, \{\infty_3, 0, 2, 7\}, \{\infty_4, 0, 2, 5\},$$

$$\{\infty_5, 0, 3, 9\}, \{\infty_6, 0, 4, 10\}$$

with the short orbit $\{0, 4, 8, 12\}$, and a block of size 7 on the infinite points included three times. □

We must treat cases when $n = 6$ and $m \in \{4, 5, 6, 8, 9, 11, 12, 14, 18, 23\}$ to complete the solution for index 3.

In the constructions of the next lemma, whenever the point set has the form $X \times Y$, parallel classes are obtained as $\{\{(x, y): x \in X\}: y \in Y\}$ and $\{\{(x, y): y \in Y\}: x \in X\}$.

Lemma 3.4. *Whenever* $m \in \{4, 5, 6, 8, 9, 11, 12, 14, 18, 23\}$, *a* 4*-MGDD of index three and type* 6^m *exists.*

Proof. For $m = 4$, the point set is $(\mathbb{Z}_5 \cup \{\infty\}) \times \{0, 1, 2, 3\}$. Base blocks are:

$$\{(0,0),(i,1),(2i,2),(3i,3)\}, \{(\infty,0),(i,1),(2i,2),(3i,3)\},$$

$$\{(0,0),(\infty,1),(2i,2),(3i,3)\}, \{(0,0),(i,1),(\infty,2),(3i,3)\},$$

$$\{(0,0),(i,1),(2i,2),(\infty,3)\}$$

for $i = 1, 2, 3$, and three copies of the base block $\{(0,0),(4,1),(3,2),(2,3)\}$.

For $m = 5$, the point set is $\mathbb{Z}_{30}$, parallel classes are equivalence classes modulo 5 and modulo 6, and base blocks are

$$\{0, 1, 2, 3\}, \{0, 2, 9, 16\}, \{0, 3, 7, 16\}, \{0, 3, 11, 22\}, \{0, 4, 8, 17\}.$$

For $m=6$, the point set is $(\mathbb{Z}_5 \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,3),(3,2)\},\{(3,0),(4,1),(1,3),(0,\infty)\},$$
$$\{(2,0),(3,1),(4,3),(0,\infty)\},\{(0,3),(1,1),(2,4),(\infty,0)\},$$
$$\{(0,4),(2,3),(4,2),(\infty,0)\},\{(4,2),(1,3),(0,\infty),(\infty,0)\},$$
$$\{(2,1),(3,4),(0,\infty),(\infty,0)\},\{(1,2),(2,1),(0,\infty),(\infty,0)\},$$
$$\{(0,0),(1,4),(3,1),(\infty,\infty)\}.$$

For $m=8$, the point set is $(\mathbb{Z}_7 \cup \{\infty\}) \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(3,3),(5,2)\},\{(0,0),(4,4),(6,3),(1,2)\},$$
$$\{(\infty,\infty),(0,0),(6,1),(5,2)\},\{(0,\infty),(\infty,0),(1,1),(2,2)\},$$
$$\{(0,\infty),(\infty,0),(3,2),(5,4)\},\{(0,\infty),(\infty,0),(4,3),(1,4)\},$$
$$\{(0,\infty),(4,0),(5,3),(6,1)\},\{(0,\infty),(1,0),(4,3),(6,4)\},$$
$$\{(0,\infty),(2,2),(3,4),(6,0)\},\{(0,\infty),(2,4),(3,2),(5,0)\},$$
$$\{(\infty,0),(0,1),(6,2),(3,3)\},\{(\infty,0),(0,3),(1,4),(3,1)\}.$$

For $m=9$, the point set is $\mathbb{Z}_9 \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\},\{(0,0),(2,2),(6,1),(5,4)\},$$
$$\{(0,\infty),(1,1),(4,4),(3,2)\},\{(0,\infty),(1,1),(5,0),(6,3)\},$$
$$\{(0,\infty),(1,1),(7,2),(3,0)\},\{(0,\infty),(2,2),(4,3),(7,4)\}.$$

Multiply each by $(8,1)^i$ for $i=0,1$ to obtain 12 base blocks, and develop over the group.

For $m=11$, the point set is $\mathbb{Z}_{11} \times (\mathbb{Z}_5 \cup \{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\},\{(0,\infty),(1,1),(5,0),(9,4)\},$$
$$\{(0,\infty),(2,2),(8,0),(6,4)\}.$$

Multiply each by $(4,1)^i$ for $i=0,1,2,3,4$ to obtain 15 base blocks, and develop over the group.

For $m=12$, there is a 5-MGDD of type 6^{13} [11] and hence a $\{4,5\}$-MGDD of type 6^{12}. Triplicate each block of size 4, and replace each 5-block by a $\{4\}$-PBD of order 5 and index 3.

For $m=14$, the point set is $(\mathbb{Z}_{13}\cup\{\infty\})\times(\mathbb{Z}_5\cup\{\infty\})$. Base blocks are:

$$\{(0,0),(6,1),(1,4),(10,3)\},\{(0,0),(7,2),(6,4),(10,1)\},$$
$$\{(0,0),(11,1),(5,3),(6,2)\},\{(0,0),(3,3),(6,4),(1,2)\},$$
$$\{(0,0),(9,2),(2,3),(5,1)\},\{(\infty,\infty),(0,0),(2,2),(11,1)\},$$
$$\{(0,\infty),(\infty,0),(3,1),(11,4)\},\{(0,\infty),(\infty,0),(5,3),(9,2)\},$$
$$\{(0,\infty),(\infty,0),(4,1),(8,3)\},\{(\infty,0),(0,2),(1,3),(2,4)\},$$
$$\{(\infty,0),(0,4),(12,1),(2,2)\},\{(0,\infty),(1,0),(12,1),(7,2)\},$$
$$\{(0,\infty),(1,0),(2,4),(7,2)\},\{(0,\infty),(1,0),(4,1),(9,4)\},$$
$$\{(0,\infty),(3,0),(2,2),(11,4)\},\{(0,\infty),(2,0),(6,2),(12,1)\},$$
$$\{(0,\infty),(8,0),(5,3),(6,4)\},\{(0,\infty),(10,0),(4,2),(12,3)\},$$
$$\{(0,\infty),(6,0),(10,4),(7,2)\},\{(0,\infty),(11,0),(8,2),(10,3)\},$$
$$\{(0,\infty),(3,0),(5,2),(9,1)\}.$$

For $m=18$, the point set is $(\mathbb{Z}_{17}\cup\{\infty\})\times(\mathbb{Z}_5\cup\{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\},\{(0,0),(4,4),(13,3),(10,2)\},$$
$$\{(0,0),(6,1),(12,4),(4,3)\},\{(0,0),(7,2),(1,3),(14,1)\},$$
$$\{(0,0),(8,3),(2,4),(7,1)\},\{(0,0),(9,4),(4,1),(3,2)\},$$
$$\{(0,0),(6,3),(2,1),(8,2)\},\{(\infty,\infty),(0,0),(7,4),(5,1)\},$$
$$\{(0,\infty),(\infty,0),(5,4),(10,1)\},\{(0,\infty),(\infty,0),(1,2),(8,1)\},$$
$$\{(0,\infty),(\infty,0),(7,2),(2,3)\},\{(\infty,0),(0,1),(12,3),(14,4)\},$$
$$\{(\infty,0),(0,4),(1,2),(15,3)\},\{(0,\infty),(1,0),(3,2),(13,1)\},$$
$$\{(0,\infty),(6,0),(9,3),(8,4)\},\{(0,\infty),(2,0),(5,3),(9,1)\},$$
$$\{(0,\infty),(10,0),(14,4),(11,2)\},\{(0,\infty),(12,0),(8,1),(15,4)\},$$
$$\{(0,\infty),(1,0),(7,1),(11,2)\},\{(0,\infty),(16,0),(2,1),(6,2)\},$$
$$\{(0,\infty),(4,0),(11,2),(16,3)\},\{(0,\infty),(4,0),(12,3),(16,1)\},$$
$$\{(0,\infty),(14,0),(5,3),(13,2)\},\{(0,\infty),(13,0),(7,1),(15,3)\},$$
$$\{(0,\infty),(15,0),(9,1),(12,3)\},\{(0,\infty),(3,0),(12,1),(4,2)\},$$
$$\{(0,\infty),(6,0),(14,2),(3,4)\}.$$

For $m=23$, the point set is $\mathbb{Z}_{23}\times(\mathbb{Z}_5\cup\{\infty\})$. Base blocks are:

$$\{(0,0),(1,1),(2,2),(3,3)\},\{(0,\infty),(1,1),(3,3),(10,0)\},$$
$$\{(0,\infty),(1,1),(5,0),(14,4)\}.$$

Multiply each by $(2,1)^i$ for $i=0,1,\ldots,10$ to obtain 33 base blocks, and develop over the group. □

Theorem 3.5. *A* 4*-MGDD of index* 3 *and type* n^m *exists whenever* $n,m\geqslant 4$.

Proof. If $m,n\in S$, apply Lemma 3.1. If $m\in S\setminus\{6\}$, apply Lemma 3.2 using the PBDs from Lemma 3.3. This handles all cases except when $n=6$, or $m\in\{10,11\}$ and $n\in\{15,18\}$. When $n\in\{15,18\}$ and $m\in\{6,10,11\}$, but $(n,m)\neq(18,6)$, the cases are treated by using $m\in\{15,18\}$ in Lemma 3.2. When $m=6$ and $n\in\{7,10,19\}$, triplicate a 4-MGDD of index one. The remaining cases arise when $m=6$, and these are treated in Lemma 3.4. □

Now we turn to index 2. The only cases to treat are those missing when $\lambda=1$. For types 10^{15} and 10^{18}, employ a $\{4\}$-PBD of order 10 and index 2 together with a 4-MGDD of type 4^{15} or 4^{18}.

For 6^4, the point set is $(\mathbb{Z}_5\cup\{\infty\})\times\{0,1,2,3\}$. Base blocks are:

$$\{(\infty,0),(i,1),(2i,2),(3i,3)\},\{(0,0),(\infty,1),(2i,2),(3i,3)\},$$
$$\{(0,0),(i,1),(\infty,2),(3i,3)\},\{(0,0),(i,1),(2i,2),(\infty,3)\}.$$

for $i=1,2$, and two copies of the base blocks $\{(0,0),(4,1),(3,2),(2,3)\}$ and $\{(0,0),(3,1),(1,2),(4,3)\}$. Since $\{4,7\}$-PBDs of order 16, 22, 25, and 34 all exist, this settles the remaining cases for index 2.

Putting the pieces together, we obtain:

Theorem 3.6. *A* 4*-MGDD of type* n^m *and index* λ *exists whenever* $\lambda(m-1)(n-1)\equiv 0\pmod 3$ *and* $m,n\geqslant 4$, *except when* $\lambda=1$ *and* $\{m,n\}=\{6,4\}$, *and possibly when* $\lambda=1$ *and* $\{m,n\}\in\{\{6,16\},\{6,22\},\{10,15\},\{10,18\}\}$.

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References

[1] R.J.R. Abel, A.E. Brouwer, C.J. Colbourn, J.H. Dinitz, Mutually orthogonal latin squares (MOLS), in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton FL, 1996, pp. 111–142.

[2] A. Assaf, An application of modified group divisible designs, J. Combin. Theory A 68 (1994) 152–168.

[3] A. Assaf, Modified group divisible designs, Ars Combin. 29 (1990) 13–20.

[4] A. Assaf, R. Wei, Modified group divisible designs with block size 4 and $\lambda = 1$, preprint.

[5] F.E. Bennett, H.D.O.F. Gronau, A.C.H. Ling, R.C. Mullin, PBD-closure, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Ruton, FL, 1996, pp. 203–213.

[6] T. Beth, D. Jungnickel, H. Lenz, Design Theory, Cambridge University Press, Cambridge, England, 1986.

[7] H. Hanani, Balanced incomplete block designs and related designs, Discrete Math. 11 (1975) 255–369.

[8] A.C.H. Ling, Pairwise balanced designs and related codes, Ph.D. Thesis, University of Waterloo, 1996.

[9] R.C. Mullin, H-D.O.F. Gronau, PBDs: Recursive Constructions, in: C.J. Colbourn, J.H. Dinitz (Eds.), CRC Handbook of Combinatorial Designs, CRC Press, Boca Raton, FL, 1996, pp. 193–203.

[10] R. Rees, D.R. Stinson, Frames with block size four, Canadian J. Math. 44 (1992) 1030–1049.

[11] J.X. Yin, A.C.H. Ling, C.J. Colbourn, R.J.R. Abel, The existence of uniform 5-GDDs, J. Combin. Desi. 5 (1997) 275–299.