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A local classification of a class of (α, β) metrics with constant flag curvature

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ABSTRACT

We first compute Riemannian curvature and Ricci curvature of (α, β) metrics. Then we apply these formulae to discuss a special class (α, β) metrics $F = \alpha(1 + \frac{\beta}{\alpha})^p$ $(|p| \geqslant 1)$ which have constant flag curvature. We obtain the sufficient and necessary conditions that $F = \frac{(\alpha+\beta)^2}{\alpha}$ have constant flag curvature. Then we prove that such metrics must be locally projectively flat and complete their local classification. Using the same method we find a necessary condition that flag curvature of $F = \frac{\alpha^2}{\alpha+\beta}$ is constant and proved that there are no non-trivial Matsumoto metrics. Furthermore, we give a negative answer whether there are non-trivial metrics $F = \alpha(1 + \frac{\beta}{\alpha})^p$ $(|p| \geqslant 1)$ of constant flag curvature when β is closed.

1. Introduction

In Finsler geometry one important open problem is to classify those metrics of constant flag curvature, which is the generalization of sectional curvature in Riemannian geometry.

For the indicatrix of various Finsler metrics are quite different from each other, this problem is quite difficult. Obviously the simplest metrics are that the indicatrix are symmetric ellipses. This means that Finsler metrics are Riemannian. In the late 1920s a completely rigorous result as well-known is available: for every real number k there exists an uniqueness simple-connected complete Riemannian manifold of constant sectional curvature equal to k up to an isometry [4].

The next simpler case is Randers metrics. Its indicatrix can be viewed as a parallel translation of the indicatrix of a Riemannian metric along a vector. So they are non-symmetric metrics and quite close to Riemannian metrics. Yasuda and Shimada [15] and Matsumoto [7] first gave a characterization of Randers metrics with constant flag curvature in 1980s. Under the guidance of Yasuda–Shimada's theorem, Bao and Shen constructed a family of Randers metrics on Lie group S^3 to prove there really exists the examples stated in there theorem [3]. Later some other examples were constructed by Shen [13], Yasuda–Shimada's result was found to be some wrong. Soon Bao and Robles gave a corrected characterization by three conditions [1]. At the same time this was also done by Matsumoto and Shimada independently [9]. Another viewpoint of Randers metrics is Zermelo navigation problem. Roughly speaking, a Randers metric can be considered as a disturbance of a Riemannian metric by a vector field. By this navigation point, Bao, Robles and Shen simplified the three characterized conditions and gave a local classification [2].

Are there any other class Finsler metrics of constant flag curvature and how to classify them? In 1990s R. Bryant constructed a 2-parameter family of locally projectively flat Finsler metrics on S^2 with flag curvature $K = 1$ by using moving frame and complex geometry [6]. Bryant's metrics are not (α, β) metrics. However, its indicatrix are the quartic curves.

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In fact, in early 1920s L. Berwald gave a Finsler metric on unit ball B^n of zero flag curvature which belongs to the class $F=\alpha(1+\frac{\beta}{\alpha})^2$ and is projectively flat [5]. Recently Mo, Shen and Yang found more such kinds of metrics by deformation of Randers metrics [12]. Soon Shen and Yildirim proved that these are all the non-trivial metrics $F=\alpha(1+\frac{\beta}{\alpha})^2$ of constant flag curvature which are locally projectively flat [14].

Hence one question is whether there are any other metrics $F=\alpha(1+\frac{\beta}{\alpha})^2$ of constant flag curvature. In this paper, we proved that the answer is negative. In other words, we conclude that if $F=\alpha(1+\frac{\beta}{\alpha})^2$ have constant flag curvature, then F must be locally projectively flat. Thus we complete the local classification of such metrics.

The main idea of the proof is a classical method of separating irrational part and rational part of Riemannian curvature and Ricci curvature in Finsler geometry. We improve this method in Lemma 4.1 and it can simplify the proof of Bao and Robles in [1]. On the other hand We use Ricci curvature instead of Riemannian curvature to get some necessary conditions because Ricci curvature is simpler. By these thoughts we obtained the necessary and sufficient conditions that $F=\alpha(1+\frac{\beta}{\alpha})^2$ have constant flag curvature and found that these conditions can conclude that F are locally projectively flat.

Furthermore, we use the same methods to analyze a more general class (α,β) metrics $F=\alpha(1+\frac{\beta}{\alpha})^p$ $(|p|\geqslant 1)$ and get some results. Especially we proved that there are no non-trivial Matsumoto metrics with constant flag curvature which were called a slop of mountain in [10].

2. Notation and definitions

The (α,β) metrics were first introduced by Matsumoto [8]. They are Finsler metrics built from a Riemannian metric $\alpha=\sqrt{a_{ij}(x)y^iy^j}$, 1-form $\beta=b_i(x)y^i$ and a C^∞ function $\phi(s)$ on a manifold M. A Finsler function of (α,β) metrics is given by the form

$$F:=\alpha\phi(s),\quad s=\frac{\beta}{\alpha}.$$

To satisfy that F is positive and strongly convex on $TM\backslash 0$, it is known that if and only if

$$\phi(s)>0,\quad \phi(s)-s\phi'(s)+\big(B-s^2\big)\phi''(s)>0,\quad |s|^2\leqslant B<b_0$$

where $B:=b_ib_ja^{ij}=\|\beta\|_\alpha^2$.

There are many examples of (α,β) metrics. The most familiar such kind of metrics are Randers metrics $F=\alpha+\beta$. As we know that Randers metrics satisfy to be positive and strongly convex if and only if $B=\|\beta\|_\alpha^2<1$. Till now mathematician have mastered their curvatures quite well.

Another examples, we will discuss in this paper, are a more general class $F=\alpha(1+\frac{\beta}{\alpha})^p$ in (α,β) metrics which include Randers metrics and Matsumoto metrics [10]. It is clear that there exist a positive real number $\epsilon(p)>0$ depending on index p s.t. when $B<\epsilon$ and $|p|\geqslant 1$, F satisfy the positive and strongly convex conditions.

Let x denote points on the manifold M, $y\in T_xM$ denote tangent vectors at point x. The fundamental tensor of a Finsler metric F is formally analogous to the metric tensor in Riemannian geometry. In local coordinates it is defined by

$$g_{ij}:=\frac{1}{2}\big(F^2\big)_{y^iy^j}.$$

Spray coefficients G^i are defined by

$$G^i:=\frac{1}{4}g^{il}\left\{2\frac{\partial g_{jl}}{\partial x^k}-\frac{\partial g_{jk}}{\partial x^l}\right\}y^jy^k.$$

A Finsler metric is called locally projectively flat if there exists a local coordinate such that the geodesics can be parametrized as a straight line. This is equivalent to that under this local coordinate spray coefficients must satisfy

$$G^i=Py^i$$

where P is a smooth positively homogeneous function on $TM\backslash\{0\}$ and is called a projective factor.

As done in Randers metrics, there are two key quantities:

$$r_{ij}:=\frac{1}{2}(b_{i|j}+b_{j|i}),\qquad s_{ij}:=\frac{1}{2}(b_{i|j}-b_{j|i})$$

where $b_{i|j}$ means the coefficients of the covariant derivative of β with respect to α. Furthermore, we denote

$$r^i{}_j:=a^{ik}r_{kj},\qquad s^i{}_j:=a^{ik}s_{kj},$$
$$r_{00}:=r_{ij}y^iy^j,\qquad r_{i0}:=r_{ij}y^j,$$
$$s_i:=b_js^j{}_i,\qquad s_0:=s_iy^i,$$
$$s_{00}:=s_{ij}y^iy^j=0,\qquad s_{i0}:=s_{ij}y^j,$$
$$r:=r_{ij}b^ib^j,\qquad s_b:=s_{ij}b^ib^j=0.$$

3. The flag and Ricci curvature of (α,β) metrics

Now we recall the definition of Riemannian curvature. For a vector $y = y^i \frac{\partial}{\partial x^i}|_x \in T_xM$, define $R_y = R^i{}_j\, dx^j \otimes \frac{\partial}{\partial x^i}|_x : T_xM \to T_xM$ by

$$R^i{}_j := 2\big(G^i\big)_{x^j} - y^k\big(G^i\big)_{x^k y^j} + 2G^k\big(G^i\big)_{y^k y^j} - \big(G^i\big)_{y^k}\big(G^k\big)_{y^j}.$$

For any tangent plane $P = \operatorname{span}\{y,u\} \subset T_xM$ define

$$K(P,y) := \frac{g_y(R_y(u),u)}{g_y(y,y)g_y(u,u) - g_y(y,u)g_y(y,u)}$$

and K is called flag curvature. Usually, $K(P,y)$ depends on the direction $y \in P$. In Riemannian case, $K(P,y)$ is independent of $y \in P$. So flag curvature generalizes sectional curvature in Riemannian geometry. Later we will use a basic fact time and again that a Finsler metric F has constant flag curvature if and only if [1]

$$R^i{}_j = KF^2\left(\delta^i_j - \frac{y^i}{F}F_{y^j}\right).$$

As we know that the spray coefficients G^i of an (α,β) metric $F := \alpha\phi(s)$ and the spray coefficients ${}^\alpha G^i$ of the Riemannian metric α are related by [14]

$$G^i = {}^\alpha G^i + \Theta(-2\alpha Q s_0 + r_{00})l^i + \Psi(-2\alpha Q s_0 + r_{00})b^i + \alpha Q s^i{}_0$$

where

$$Q := \frac{\phi'}{\phi - s\phi'},$$
$$\Theta := \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (B - s^2)\phi'')},$$
$$\Psi := \frac{\phi''}{2((\phi - s\phi') + (B - s^2)\phi'')}$$

$l^i := \frac{y^i}{\alpha}$ and $s := \frac{\beta}{\alpha}$.

Denote $\zeta^i := \Theta(-2\alpha Q s_0 + r_{00})l^i + \Psi(-2\alpha Q s_0 + r_{00})b^i + \alpha Q s^i{}_0$. By Berwald's formula for Riemannian curvature:

$$R^i{}_j = {}^\alpha R^i{}_j + \big\{2\zeta^i{}_{|j} - y^k\big(\zeta^i{}_{|k}\big)_{y^j} - \big(\zeta^i\big)_{y^k}\big(\zeta^k\big)_{y^j} + 2\zeta^k\big(\zeta^i\big)_{y^j y^k}\big\},$$

we can compute Riemannian curvature of (α,β) metrics using Maple and have the following proposition:

Proposition 3.1. *For an (α,β) metric $F := \alpha\phi(s)$, its Riemannian curvature tensor $R^i{}_j$ is locally given by*

$$R^i{}_j = {}^\alpha R^i{}_j + T^i{}_j$$

where

$$\begin{aligned}
T^i{}_j = {} & \delta^i{}_j d + l^i l_j ll + l^i b_j lb + l^i s_j ls + l^i r_j lr + l^i s_{0j} ls_1 + l^i r_{j0} lr_1 - l^i s_{j|0}(u\alpha) + l^i s_{0|j}(2u\alpha) - l^i s_k s^k{}_j\big(u\alpha^2 Q\big) \\
& + l^i r_{00|j}(2f) - l^i r_{j0|0}(2f) - l^i r_{k0} s^k{}_j(2Q\alpha f) + l^i r_{jk} s^k{}_0(4Q\alpha f) \\
& + b^i l_j bl + b^i b_j bb + b^i s_j bs + b^i r_j br + b^i s_{0j} bs_1 + b^i r_{j0} br_1 - b^i s_{j|0}(v\alpha) + b^i s_{0|j}(2v\alpha) \\
& - b^i s_k s^k{}_j\big(vQ\alpha^2\big) + b^i r_{00|j}(2g) - b^i r_{j0|0}(2g) - b^i r_{k0} s^k{}_j(2Q\alpha g) + b^i r_{jk} s^k{}_0(4Q\alpha g) \\
& + r^i{}_j cr + s^i{}_j cs + s^i{}_0 l_j sl + s^i{}_0 b_j sb_1 + s^i{}_0 s_j ss + s^i{}_0 s_{0j} ss_1 + s^i{}_0 r_{j0} sr \\
& + s^i l_j sl_1 + s^i b_j sb_2 - s^i s_j\big(Qv\alpha^2\big) + s^i r_{j0}(2gQ\alpha) + r^i{}_0 l_j rl + r^i{}_0 b_j rb + r^i{}_0 s_j rs + r^i{}_0 r_{j0} rr \\
& + s^i{}_k s^k{}_0 l_j ssl + s^i{}_k s^k{}_0 b_j ssb - s^i{}_k s^k{}_j\big(Q^2\alpha^2\big) + s^i{}_{0|j}(2Q\alpha) - s^i{}_{j|0}(Q\alpha) + s^i{}_{0|0} l_j sl_2 - s^i{}_{0|0} b_j Q_s
\end{aligned}$$

and

$$\begin{aligned}
d = {} & \frac{r_{00}^2}{\alpha^2}\big(2f_s g\big(B - s^2\big) - f_s + f^2 - 2sfg\big) \\
& + \frac{r_{00}s_0}{\alpha}\big(2vf_s\big(B - s^2\big) + 2fu + 2gu_s\big(B - s^2\big) - 2vfs - u_s + 2Qf_s - 2f_B\big) \\
& + s_0^2\big(-2u_B + u^2 + 2u_s v\big(B - s^2\big) + 2Qu_s\big) + r_0 s_0(-2u_B + 4vf)
\end{aligned}$$

$$+\frac{r_{00}r_0}{\alpha}(-2f_B+4fg)-\frac{r_{00|0}}{\alpha}f+2\alpha s_k s^k{}_0 Qu+r_{k0}s^k{}_0(4Qf)-s_{0|0}u,$$

$$\begin{aligned}
ll={}&\frac{r_{00}^2}{\alpha^2}\big((sf_sg_s-4f_sg-2sf_{ss}g)\big(B-s^2\big)+6sfg+f^2+6s^2f_sg+sff_s+sf_{ss}+2f_s-s^2fg_s\big)\\
&+\frac{r_{00}s_0}{\alpha}\big((-2sgu_{ss}+sg_su_s+sf_sv_s-2sf_{ss}v-5f_sv)\big(B-s^2\big)+2sf_{Bs}\\
&+fu+sf_su+4s^2gu_s-5Qf_s+su_{ss}+sfu_s+sQ_sf_s-2Qf_{ss}s-s^2fv_s\\
&+6s^2f_sv-2gu_s\big(B-s^2\big)+7sfv+u_s+2f_B\big)+s_0^2\big((su_sv_s-3u_sv-2su_{ss}v)\big(B-s^2\big)\\
&+4s^2u_sv-2Qu_{ss}s+suu_s-3Qu_s+2u_{Bs}s+sQ_su_s\big)+r_0s_0(-4sf_sv+2sfv_s+2su_{Bs}-6fv)\\
&+\frac{r_{00}r_0}{\alpha}(2f_B+2sf_{Bs}+2sfg_s-4fg-4sf_sg)+\alpha s_ks^k{}_0(sQ_su-Qu-2sQu_s)+\frac{r_{00|0}}{\alpha}(f+sf_s)\\
&+s_{0|0}(su_s)+r_{k0}s^k{}_0(2sQ_sf-4sQf_s-6Qf),
\end{aligned}$$

$$\begin{aligned}
lb={}&\frac{r_{00}^2}{\alpha^2}\big((2f_{ss}g-f_sg_s)\big(B-s^2\big)-2fg-ff_s+sfg_s-6sf_sg-f_{ss}\big)\\
&+s_0^2\big((2u_{ss}v-u_sv_s)\big(B-s^2\big)-uu_s-Q_su_s-2u_{Bs}-4su_sv+2Qu_{ss}\big)\\
&+\frac{r_{00}s_0}{\alpha}\big((2f_{ss}v+2gu_{ss}-g_su_s-f_sv_s)\big(B-s^2\big)-Q_sf_s\\
&-f_su+2Qf_{ss}-2f_{Bs}-4sgu_s+sfv_s-u_{ss}-fu_s-2fv-6sf_sv\big)-r_0s_0(2u_{Bs}+2fv_s-4f_sv)\\
&+\frac{r_{00}r_0}{\alpha}(-2fg_s+4f_sg-2f_{Bs})+r_{k0}s^k{}_0(4Qf_s-2Q_sf)+\alpha s_ks^k{}_0(2Qu_s-Q_su)-s_{0|0}u_s-\frac{r_{00|0}}{\alpha}f_s,
\end{aligned}$$

$$\begin{aligned}
ls={}&r_{00}\big((2gu_s-f_sv)\big(B-s^2\big)-uf+4f_B+sfv-u_s-Qf_s\big)\\
&+\alpha s_0\big(2u_B-u^2+Qu_s+u_sv\big(B-s^2\big)\big)-\alpha r_0(2fv+2u_B),
\end{aligned}$$

$$lr=r_{00}(4f_B+4fg)+\alpha s_0(4u_B+4fv),$$

$$ls_1=\frac{r_{00}}{\alpha}(3Qf+3f_s+3sQf_s)+s_0(3sQu_s+3u_s),$$

$$\begin{aligned}
lr_1={}&\frac{r_{00}}{\alpha}\big(2f_sg\big(B-s^2\big)-2sfg-f_s-2f^2\big)\\
&+s_0\big((4f_sv-2gu_s)\big(B-s^2\big)+4Qf_s+u_s-2fu-4f_B-4sfv\big)-r_0(4f_B+4fg),
\end{aligned}$$

$$\begin{aligned}
bl={}&\frac{r_{00}^2}{\alpha^2}\big(\big(sg_s^2-2sgg_{ss}-2gg_s\big)\big(B-s^2\big)+sg_{ss}+g_s+4s^2gg_s\big)\\
&+\frac{r_{00}s_0}{\alpha}\big((2sg_sv_s-3g_sv-2sg_{ss}v-2sgv_{ss})\big(B-s^2\big)+sv_{ss}\\
&+2sg_{Bs}-2Qsg_{ss}+5s^2g_sv+Q_ssg_s+2s^2gv_s-2sgv-3Qg_s\big)+\frac{r_{00}r_0}{\alpha}(-2sgg_s+2sg_{Bs})\\
&+s_0^2\big(\big(sv_s^2-2svv_{ss}-vv_s\big)\big(B-s^2\big)+3s^2vv_s-2sQv_{ss}-3sv^2-Qv_s-2v_B+sQ_sv_s+2sv_{Bs}\big)\\
&+s_0r_0(2sv_{Bs}-2gv+2sgv_s-4sg_sv-2v_B)+\frac{r_{00|0}}{\alpha}(sg_s)+r_{k0}s^k{}_0(-2Qg+2sgQ_s-4sQg_s)\\
&+s_{0|0}(sv_s-v)+\alpha s_ks^k{}_0(sQ_sv+Qv-2sQv_s),
\end{aligned}$$

$$\begin{aligned}
bb={}&\frac{r_{00}^2}{\alpha^2}\big(\big(2gg_{ss}-g_s^2\big)\big(B-s^2\big)-g_{ss}-4sgg_s\big)\\
&+\frac{r_{00}s_0}{\alpha}\big((2g_{ss}v+2gv_{ss}-2g_sv_s)\big(B-s^2\big)-2g_{Bs}-v_{ss}+2vg-2sgv_s-5sg_sv+2Qg_{ss}-Q_sg_s\big)\\
&+s_0^2\big(\big(2vv_{ss}-v_s^2\big)\big(B-s^2\big)-Q_sv_s-2v_{Bs}+2v^2+2Qv_{ss}-3svv_s\big)+\frac{r_{00}r_0}{\alpha}(-2g_{Bs}+2gg_s)\\
&+s_0r_0(-2gv_s+4g_sv-2v_{Bs})-\frac{r_{00|0}}{\alpha}g_s-s_{0|0}v_s+\alpha s_ks^k{}_0(2Qv_s-Q_sv)+r_{k0}s^k{}_0(4Qg_s-2Q_sg),
\end{aligned}$$

$$\begin{aligned}
bs={}&r_{00}\big((-g_sv+2gv_s)\big(B-s^2\big)+4g_B-Qg_s-v_s+2sgv\big)\\
&+\alpha s_0\big(vv_s\big(B-s^2\big)+2v_B+Qv_s+sv^2\big)-\alpha r_0(2gv+2v_B),
\end{aligned}$$

$$br=r_{00}\big(4g^2+4g_B\big)+\alpha s_0(4v_B+4gv),$$

$$bs_1 = \frac{r_{00}}{\alpha}(3g_s + 3Q\,sg_s) + s_0\big(3v_s - 3(v - sv_s)Q\big),$$
$$br_1 = \frac{r_{00}}{\alpha}\big(2gg_s\big(B - s^2\big) - g_s\big) + s_0\big((4g_s v - 2gv_s)\big(B - s^2\big) - 2v_s - 2sgv + 4Q\,g_s - 4g_B\big) + r_0\big(-4g^2 - 4g_B\big),$$
$$cr = r_{00}(2g) + \alpha s_0(2v),$$
$$cs = \alpha s_0\big(2Q_s v\big(B - s^2\big) + 2Q\,Q_s + 2Q\,sv + 2v\big) + r_{00}\big(2g - Q_s + 2gQ_s\big(B - s^2\big) + 2Q\,sg\big),$$
$$sl = \frac{r_{00}}{\alpha}\big((Q_s sg_s - 2sgQ_{ss})\big(B - s^2\big) + 2Q\,s^2 g - 2Q\,sg + Q\,s^2 g_s + sQ_{ss} + sg_s\big)$$
$$+ s_0\big((sQ_s v_s - Q_s v - 2Q_{ss}sv)\big(B - s^2\big) + Q\,s^2 v_s - Q_s Q + sQ_s^2 - 3Q\,sv$$
$$- v + sv_s - 2Q\,Q_{ss}s + 2Q\,s^2 v\big),$$
$$sb_1 = \frac{r_{00}}{\alpha}\big((2Q_{ss}g - Q_s g_s)\big(B - s^2\big) + 2gQ - 2sgQ_s - Q\,sg_s - Q_{ss} - g_s\big)$$
$$+ s_0\big((2Q_{ss}v - Q_s v_s)\big(B - s^2\big) - v_s - Q_s^2 + 2Q\,v - 2Q_s sv + 2Q\,Q_{ss} - Q\,sv_s\big),$$
$$ss = \alpha\big(-Q_s v\big(B - s^2\big) - Q\,Q_s - v - Q\,sv\big),$$
$$ss_1 = -3Q^2 + 3Q\,Q_s s + 3Q_s,$$
$$sr = -2\big(Q_s\big(B - s^2\big) + sQ + 1\big)g + Q_s,$$
$$sl_1 = r_{00}(-2gQ + 2Q_s sg - Q\,sg_s) + \alpha s_0(Q\,v + Q\,sv_s - 2Q_s sv),$$
$$sb_2 = r_{00}(-2Q_s g + Q\,g_s) + \alpha s_0(2Q_s v - Q\,v_s),$$
$$ss_2 = -\alpha^2 Q\,v,$$
$$sr_1 = 2\alpha Q\,g,$$
$$rl = \frac{r_{00}}{\alpha}sg_s + s_0(sv_s - v),$$
$$rb = -\frac{r_{00}}{\alpha}g_s - s_0 v_s,$$
$$rs = -v\alpha,$$
$$rr = -2g,$$
$$ssl = \alpha(Q - sQ_s)Q,$$
$$ssb = \alpha Q\,Q_s,$$
$$sl_2 = -(Q - sQ_s).$$

Here we denote

$$f(s,B) := \Theta, \qquad g(s,B) := \Psi,$$
$$u(s,B) := -2\Theta Q, \qquad v(s,B) := 2\Psi Q.$$

Proof. As the calculation is tiresomely long and can be done by Maple programme, we write the procedure in Appendix A. □

Remark. For a Randers metric $F = \alpha + \beta$, its spray coefficients are given by

$$G^i = {}^{\alpha}G^i + \frac{1}{2(1+s)}(-2\alpha s_0 + r_{00})l^i + \alpha s^i{}_0.$$

So

$$f(s,B) = \frac{1}{2(1+s)}, \qquad g(s,B) = 0,$$
$$u(s,B) = -\frac{1}{1+s}, \qquad v(s,B) = 0, \qquad Q = 1.$$

According to Proposition 3.1 we can compute Riemannian curvature $R^i{}_j$:

$$R^i{}_j = {}^\alpha R^i{}_j + \delta^i_j d + l^i l_j ll + l^i b_j lb + l^i s_{0j}\left(\frac{3s_0}{1+s}\right) + \alpha l^i s_{j|0}\left(\frac{1}{1+s}\right) - \alpha l^i s_{0|j}\left(\frac{2}{1+s}\right) + \alpha^2 l^i s_k s^k{}_j\left(\frac{1}{1+s}\right)$$
$$+ l^i r_{00|j}\left(\frac{1}{1+s}\right) - l^i r_{j0|0}\left(\frac{1}{1+s}\right) - \alpha l^i r_{k0} s^k{}_j\left(\frac{1}{1+s}\right) + \alpha l^i r_{jk} s^k{}_0\left(\frac{2}{1+s}\right) - 3s^i{}_0 s_{0j}$$

where

$$d = r_{00}^2 \frac{3}{4\alpha^2(1+s)^2} - r_{00}s_0 \frac{3}{\alpha(1+s)^2} + s_0^2 \frac{3}{(1+s)^2} - r_{00|0}\frac{1}{2\alpha(1+s)} + r_{k0}s^k{}_0 \frac{2}{1+s} - s_k s^k{}_0 \frac{2\alpha}{1+s} + s_{0|0}\frac{1}{1+s},$$
$$ll = -r_{00}^2 \frac{3}{4\alpha^2(1+s)^3} + r_{00}s_0\frac{3}{\alpha(1+s)^3} - s_0^2\frac{3}{(1+s)^3}$$
$$+ \alpha s_k s^k{}_0 \frac{1-s}{(1+s)^2} - r_{k0}s^k{}_0\frac{s+3}{(1+s)^2} + r_{00|0}\frac{1}{2\alpha(1+s)^2} + s_{0|0}\frac{s}{(1+s)^2},$$
$$lb = -r_{00}^2 \frac{3}{4\alpha^2(1+s)^3} + r_{00}s_0\frac{3}{\alpha(1+s)^3} - s_0^2\frac{3}{(1+s)^3}$$
$$- r_{k0}s^k{}_0\frac{2}{(1+s)^2} + \alpha s_k s^k{}_0\frac{2}{(1+s)^2} + r_{00|0}\frac{1}{2\alpha(1+s)^2} - s_{0|0}\frac{1}{(1+s)^2}.$$

The above equation coincides with the formula of Riemannian curvature in [1].

Ricci curvature is the trace of Riemannian curvature. When we research flag curvature, Ricci curvature is often more convenient and simpler. In Proposition 3.1 we get a local expression of Riemannian curvature of an (α,β) metric. It is natural to compute Ricci curvature by Maple and we have a similar formula.

Proposition 3.2. *Under the same notation in Proposition 3.1 Ricci curvature $R^m{}_m$ of an (α,β) metric is given by*

$$R^m{}_m = {}^\alpha R^m{}_m + T^m{}_m$$

where

$$T^m{}_m := (n-1)\frac{r_{00}^2}{\alpha^2}c_1 + \frac{r_{00}^2}{\alpha^2}c_2 + (n-1)s_0^2 c_3 + s_0^2 c_4 + (n-1)\frac{r_{00}s_0}{\alpha}c_5 + \frac{r_{00}s_0}{\alpha}c_6 + (n-1)\frac{r_{00}r_0}{\alpha}c_7 + \frac{r_{00}r_0}{\alpha}c_8$$
$$+ (n-1)\left(r_0 s_0(4fv - 2u_B) + s_k s^k{}_0\alpha(2Qu) + r_{k0}s^k{}_0(4Qf) - \frac{r_{00|0}}{\alpha}f - us_{0|0}\right)$$
$$+ rr_{00}c_{10} + r^k{}_k r_{00}c_{11} + s_0 r_0 c_{12} + \alpha s_0 r c_{15} + \alpha s_0 r^k{}_k c_{16} + r_0^2 c_{17} + \alpha s_k s^k{}_0 c_{20} + s^k{}_0 s_{0k} c_{21} + s^k{}_0 r_{k0} c_{22}$$
$$+ \alpha s^k{}_0 r_k c_{23} + s_{0|0}c_{24} + \frac{r_{00|0}}{\alpha}c_{25} + \alpha r_{k0}s^k c_{26} + b^k r_{00|k}c_{27} + \alpha b^k s_{k|0}c_{28} + \alpha b^k s_{0|k}c_{29} + \alpha r^k{}_0 s_k c_{30}$$
$$+ r^k{}_0 r_{k0}c_{31} + b^k r_{k0|0}c_{32} - \alpha^2 s^i{}_k s^k{}_i Q^2 + 2\alpha s^k{}_{0|k}Q$$

and

$$c_1 = f^2 + 2gf_s(B-s^2) - f_s - 2sfg,$$
$$c_2 = (2gg_{ss} - g_s^2)(B-s^2)^2 + (-g_{ss} - 6sgg_s)(B-s^2) + 2sg_s,$$
$$c_3 = u^2 + 2u_s v(B-s^2) + 2Qu_s - 2u_B,$$
$$c_4 = (2vv_{ss} - v_s^2)(B-s^2)^2 + (2Qv_{ss} - 2v_{Bs} - 4svv_s + 2Q_{ss}v - 2Q_s v_s + 2v^2)(B-s^2)$$
$$+ 6Qv + 2Q_{ss}Q - 4sQv_s - 4sQ_s v - 4v_s - s^2v^2 - Q_s^2 - 2sv_B,$$
$$c_5 = 2Qf_s - 2f_B - u_s + 2fu + 2f_s v(B-s^2) + 2gu_s(B-s^2) - 2sfv,$$
$$c_6 = (2g_{ss}v + 2gv_{ss} - 2g_s v_s)(B-s^2)^2$$
$$+ (2Q_{ss}g - 2sgv_s - 2Q_s g_s - 8sg_s v - 2g_{Bs} + 2Qg_{ss} - v_{ss} + 2gv)(B-s^2) - 4g_s - 8Qsg_s - v + sv_s - Q_{ss},$$
$$c_7 = 4fg - 2f_B,$$
$$c_8 = (4gg_s - 2g_{Bs})(B-s^2) - 2g_s,$$
$$c_9 = 4g_B + (4gv_s - 2g_s v)(B-s^2) + 4sgv - v_s - 2Q_s g,$$
$$c_{10} = 4g^2 + 4g_B,$$
$$c_{11} = 2g,$$

$$c_{12} = (8g_s v - 2v_{Bs} - 4gv_s)(B - s^2) - 2sv_B - 4sgv - 4g_B + 4Qg_s - 3v_s,$$
$$c_{13} = 2(vv_s(B - s^2) + sv^2 + v_B + Q_s v),$$
$$c_{15} = 4gv + 4v_B, \qquad c_{16} = 2v,$$
$$c_{17} = -4g_B - 4g^2, \qquad c_{18} = -2v_B - 4gv,$$
$$c_{19} = -v^2, \qquad c_{20} = (-2Q_s v + 2Qv_s)(B - s^2) - v,$$
$$c_{21} = 2Q_s + 2sQQ_s - 2Q^2, \qquad c_{22} = (4Qg_s - 4Q_s g)(B - s^2) - 4Qsg + 2Q_s - 2g,$$
$$c_{23} = 4gQ, \qquad c_{24} = -v_s(B - s^2) - sv - Q_s,$$
$$c_{25} = -g_s(B - s^2), \qquad c_{26} = 4gQ,$$
$$c_{27} = 2g, \qquad c_{28} = -v,$$
$$c_{29} = 2v, \qquad c_{30} = -v,$$
$$c_{31} = -2g, \qquad c_{32} = -2g.$$

Proof. In Proposition 3.1 contract the upper index and down index:

$$\begin{aligned}
T^m{}_m = {} & n(d) + ll + s(lb) + \frac{s_0}{\alpha} ls + \frac{r_0}{\alpha} lr + \frac{r_{00}}{\alpha} lr_1 - s_{0|0}u + s_{0|0}(2u) - s_k s^k{}_0(u\alpha Q) + \frac{r_{00|0}}{\alpha}(2f) - \frac{r_{00|0}}{\alpha}(2f) \\
& - r_{k0}s^k{}_0(2Qf) + r_{k0}s^k{}_0(4Qf) + s(bl) + B(bb) + r(br) - s_0(bs_1) + r_0 br_1 \\
& - b^k s_{k|0}(v\alpha) + b^k s_{0|k}(2v\alpha) + s^k s_k(vQ\alpha^2) + b^k r_{00|k}(2g) - b^k r_{k0|0}(2g) + r_{k0}s^k(2Q\alpha g) + r_k s^k{}_0(4Q\alpha g) \\
& + r^k{}_k cr + s^k{}_k cs + s_0 sb_1 + s^k{}_0 s_k ss + s^k{}_0 s_{0k} ss_1 + s^k{}_0 r_{k0} sr + \frac{s_0}{\alpha} sl_1 - s^k s_k(Qv\alpha^2) + s^k r_{k0}(2gQ\alpha) + \frac{r_{00}}{\alpha} rl \\
& + r_0 rb + r_{k0}s^k rs + r^k{}_0 r_{k0} rr + s_{0k}s^k{}_0 \frac{ssl}{\alpha} + s_k s^k{}_0 ssb - s^i{}_k s^k{}_i Q^2\alpha^2 + s^k{}_{0|k}(2Q\alpha) - s^k{}_{k|0} Q\alpha - s_{0|0}Q_s \\
& + s^k{}_0 r_{k0} Q_s - s^k{}_0 s_{0k} Q_s.
\end{aligned}$$

Substituting the coefficients in Proposition 3.1 into above equation and simplifying it, then we can get the result. □

One important case of (α, β) metrics is that 1-form β is closed i.e. $s_{ij} = 0$. Under this condition Riemannian curvature and Ricci curvature are much easier.

Corollary 3.3. *If* 1*-form* β *is closed, Riemannian curvature tensor* $R^i{}_j$ *and Ricci curvature* $R^m{}_m$ *of an* (α, β) *metric can be locally expressed by*:

$$R^i{}_j = {}^\alpha R^i{}_j + T^i{}_j,$$
$$R^m{}_m = {}^\alpha R^m{}_m + T^m{}_m$$

where

$$\begin{aligned}
T^i{}_j = {} & \delta^i{}_j d + l^i l_j ll + l^i b_j lb + l^i r_j lr + l^i r_{j0} lr_1 + l^i r_{00|j}(2f) - l^i r_{j0|0}(2f) + b^i l_j bl + b^i b_j bb + b^i r_j br + b^i r_{j0} br_1 \\
& + b^i r_{00|j}(2g) - b^i r_{j0|0}(2g) + r^i{}_j cr + r^i{}_0 l_j rl + r^i{}_0 b_j rb + r^i{}_0 r_{j0} rr,
\end{aligned}$$

$$\begin{aligned}
T^m{}_m := {} & \frac{r_{00}^2}{\alpha^2}(n-1)\big(f^2 + 2f_s g(B - s^2) - 2sfg - f_s\big) + \frac{r_0 r_{00}}{\alpha}(n-1)(4fg - 2f_B) - \frac{r_{00|0}}{\alpha}(n-1)f \\
& + \frac{r_{00}^2}{\alpha^2}\big((2gg_{ss} - g_s^2)(B - s^2)^2 - (6gg_s s + g_{ss})(B - s^2) + 2sg_s\big) \\
& + \frac{r_0 r_{00}}{\alpha}\big((4gg_s - 2g_{sB})(B - s^2) - 2g_s\big) - \frac{r_{00|0}}{\alpha} g_s(B - s^2) \\
& - r_0^2(4g^2 + 4g_B) + rr_{00}(4g^2 + 4g_B) + 2gr_{00|i}b^i - 2gr_{i0}b^i + 2gr_{00}r^i_i - 2gr_{i0}r^i{}_0,
\end{aligned}$$

and

$$
\begin{aligned}
d &= \frac{r_{00}^2}{\alpha^2}\big(2f_s g\big(B-s^2\big) - f_s + f^2 - 2sfg\big) + \frac{r_{00}r_0}{\alpha}(-2f_B + 4fg) - \frac{r_{00|0}}{\alpha}f,\\
ll &= \frac{r_{00}^2}{\alpha^2}\big((sf_s g_s - 4f_s g - 2sf_{ss}g)\big(B-s^2\big) + 6sfg + f^2 + 6s^2 f_s g + sff_s + sf_{ss} + 2f_s - s^2 fg_s\big)\\
&\quad + \frac{r_{00}r_0}{\alpha}(2f_B + 2sf_{Bs} + 2sfg_s - 4fg - 4sf_s g) + \frac{r_{00|0}}{\alpha}(f + sf_s),\\
lb &= \frac{r_{00}^2}{\alpha^2}\big((2f_{ss}g - f_s g_s)\big(B-s^2\big) - 2fg - ff_s + sfg_s - 6sf_s g - f_{ss}\big) + \frac{r_{00}r_0}{\alpha}(-2fg_s + 4f_s g - 2f_{Bs}) - \frac{r_{00|0}}{\alpha}f_s,\\
lr &= r_{00}(4f_B + 4fg),\\
lr_1 &= \frac{r_{00}}{\alpha}\big(2f_s g\big(B-s^2\big) - 2sfg - f_s - 2f^2\big) - r_0(4f_B + 4fg),\\
bl &= \frac{r_{00}^2}{\alpha^2}\big(\big(sg_s^2 - 2sgg_s - 2gg_s\big)\big(B-s^2\big) + sg_{ss} + g_s\big) + \frac{r_{00}r_0}{\alpha}(-2sgg_s + 2sg_{Bs}) + \frac{r_{00|0}}{\alpha}(sg_s),\\
bb &= \frac{r_{00}^2}{\alpha^2}\big(\big(2gg_{ss} - g_s^2\big)\big(B-s^2\big) - g_{ss} - 4sgg_s\big) + \frac{r_{00}r_0}{\alpha}(-2g_{Bs} + 2gg_s) - \frac{r_{00|0}}{\alpha}g_s,\\
br &= r_{00}\big(4g^2 + 4g_B\big),\\
br_1 &= \frac{r_{00}}{\alpha}\big(2gg_s\big(B-s^2\big) - g_s\big) + r_0\big(-4g^2 - 4g_B\big),\\
cr &= r_{00}(2g),\\
rl &= \frac{r_{00}}{\alpha}sg_s,\\
rb &= -\frac{r_{00}}{\alpha}g_s,\\
rr &= -2g.
\end{aligned}
$$

Proof. Substituting $s_{ij} = 0$ into Propositions 3.1 and 3.2 will get the result. □

4. A local classification of Finsler metrics $F := \frac{(\alpha+\beta)^2}{\alpha}$ with constant flag curvature

L. Berwald constructed a projectively flat (α, β) metric $F = \frac{(\alpha+\beta)^2}{\alpha}$ with zero flag curvature on $B^n \subset \mathbb{R}^n$. The detailed example is given by [5]

$$
F := \frac{(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle)^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}.
$$

A slight generalization of Berwald's example was worked out by Mo, Shen and Yang [12]. It can expressed as following form:

$$
F := \frac{[(1+\langle a, x\rangle)(\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)} + \langle x, y\rangle) + (1-|x|^2)\langle a, y\rangle]^2}{(1-|x|^2)^2\sqrt{|y|^2 - (|x|^2|y|^2 - \langle x, y\rangle^2)}}
$$

where $a \in \mathbb{R}^n$ is an arbitrary constant vector with $|a| < 1$. This metric is also projectively flat with zero flag curvature. Later Shen and Yildirim concluded that except the above examples there are no other non-trivial metrics $F = \frac{(\alpha+\beta)^2}{\alpha}$ which are both locally projectively flat and have constant flag curvature [14] up to a local isometry.

How about this kind of metrics if we get rid of the condition of projective flatness? This was asked by Shen in [11]. Now let us discuss this question.

Lemma 4.1. *Suppose r_{00} and s_0 of (α, β) metrics $F = \alpha(1 + \frac{\beta}{\alpha})^p$ ($|p| \geqslant 1$) on a manifold M satisfy*

$$
(r_{00} + c\alpha s_0)^2 \equiv 0 \mod(s + a),
$$

then

$$
r_{00} - \frac{c}{a}\alpha s s_0 = \sigma\alpha^2\big(s^2 - a^2\big)
$$

where $s = \frac{\beta}{\alpha}$ and σ is a smooth function on a manifold M.

Proof.

$$r_{00} + c\alpha s_0 = r_{00} - \frac{c}{a}\alpha s s_0 + \frac{c}{a}\alpha s_0(s+a). \tag{1}$$

Since

$$(r_{00} + c\alpha s_0)^2 \equiv 0 \quad \mathrm{mod}(s+a),$$

this means

$$r_{00} + c\alpha s_0 \equiv 0 \quad \mathrm{mod}(s+a).$$

From (1) we know that

$$r_{00} - \frac{c}{a}\alpha s s_0 \equiv 0 \quad \mathrm{mod}(s+a).$$

So we may assume

$$\left(r_{00} - \frac{c}{a}\alpha s s_0\right) = \alpha(s+a)D$$

where D is a polynomial of s. We separate the irrational part and rational part, then obtain

$$\left(r_{00} - \frac{c}{a}\alpha s s_0\right) + \alpha s \,\mathrm{Rat}(D) + \alpha a \,\mathrm{Irrat}(D) = 0, \tag{2}$$

$$\alpha s \,\mathrm{Irrat}(D) + \alpha a \,\mathrm{Rat}(D) = 0. \tag{3}$$

From (3) we can solve

$$\mathrm{Rat}(D) = -\frac{s}{a}\,\mathrm{Irrat}(D).$$

Substituting to (2) we obtain

$$\left(r_{00} - \frac{c}{a}\alpha s s_0\right) - \frac{\alpha}{a}\left(s^2 - a^2\right)\mathrm{Irrat}(D) = 0.$$

This means that

$$r_{00} - \frac{c}{a}\alpha s s_0 \equiv 0 \quad \mathrm{mod}\left(s^2 - a^2\right).$$

Thus

$$r_{00} - \frac{c}{a}\alpha s s_0 = \sigma\alpha^2\left(s^2 - a^2\right). \qquad \square$$

Remark. The above lemma can simplify the proof of Bao and Robles in [1] and can immediately conclude one condition: $r_{00} - 2\beta s_0 = \sigma\alpha^2(1 - s^2)$.

Lemma 4.2. *Suppose $F := \frac{(\alpha+\beta)^2}{\alpha}$ is a Finsler metric on an n-dimensional manifold M with constant flag curvature K, then F must satisfy*

(1) $r_{00} = \sigma(1 + 2B - 3s^2)\alpha^2$,
(2) $s_0 = 0$,
(3) $s^k{}_0 s_{0k} = 0$,

where σ is a smooth function on M.

Proof. The spray coefficients of F are given by

$$G^i = {}^{\alpha}G^i - \frac{2s-1}{1+2B-3s^2}\left(\frac{-4}{1-s}\alpha s_0 + r_{00}\right)l^i + \frac{1}{1+2B-3s^2}\left(\frac{-4}{1-s}\alpha s_0 + r_{00}\right)b^i + \frac{2}{1-s}\alpha s^i{}_0$$

i.e.

$$f(s,B)=-\frac{2s-1}{1+2B-3s^2},\qquad g(s,B)=\frac{1}{1+2B-3s^2},$$
$$u=\frac{4(2s-1)}{(1+2B-3s^2)(1-s)},\qquad v=\frac{-4}{(1+2B-3s^2)(1-s)},$$
$$Q=\frac{2}{1-s}.$$

Since F have constant flag curvature, F also have constant Ricci curvature. This means that

$$^{\alpha}R^m{}_m+T^m{}_m-K(n-1)F^2=0. \tag{4}$$

By Proposition 3.2 we can compute $T^m{}_m$ by Maple

$$\begin{aligned}
T^m{}_m={}&(n-1)\frac{r_{00}^2}{\alpha^2}\frac{\bar c_1}{A_1^3}+\frac{r_{00}^2}{\alpha^2}\frac{\bar c_2}{A_1^4}+(n-1)s_0^2\frac{\bar c_3}{A_1^3A_2^2}+s_0^2\frac{\bar c_4}{A_1^4A_2^3}+(n-1)\frac{r_{00}s_0}{\alpha}\frac{\bar c_5}{A_1^3A_2}\\
&+\frac{r_{00}s_0}{\alpha}\frac{\bar c_6}{A_1^4A_2}+(n-1)\frac{r_{00}r_0}{\alpha}\frac{\bar c_7}{A_1^2}+\frac{r_{00}r_0}{\alpha}\frac{\bar c_8}{A_1^3}\\
&+(n-1)\left(r_0s_0(4fv-2u_B)+s_ks^k{}_0\alpha(2Qu)+r_{k0}s^k{}_0(4Qf)-\frac{r_{00|0}}{\alpha}f-us_{0|0}\right)\\
&+rr_{00}\frac{\bar c_{10}}{A_1^2}+r^k{}_kr_{00}\frac{\bar c_{11}}{A_1}+s_0r_0\frac{\bar c_{12}}{A_1^3A_2^2}+\alpha s_0r\frac{\bar c_{15}}{A_1^2A_2}+\alpha s_0r^k{}_k\frac{\bar c_{16}}{A_1A_2}+r_0^2\frac{\bar c_{17}}{A_1^2}+\alpha s_ks^k{}_0\frac{\bar c_{20}}{A_1^2A_2^2}\\
&+s^k{}_0s_{0k}\frac{\bar c_{21}}{A_2^3}+s^k{}_0r_{k0}\frac{\bar c_{22}}{A_1^2A_2}+\alpha s^k{}_0r_k\frac{\bar c_{23}}{A_1A_2}+s_{0|0}\frac{\bar c_{24}}{A_1^2A_2}+\frac{r_{00|0}}{\alpha}\frac{\bar c_{25}}{A_1^2}+\alpha r_{k0}s^k\frac{\bar c_{26}}{A_1A_2}\\
&+b^kr_{00|k}\frac{\bar c_{27}}{A_1}+\alpha b^ks_{k|0}\frac{\bar c_{28}}{A_1A_2}+\alpha b^ks_{0|k}\frac{\bar c_{29}}{A_1A_2}+\alpha r^k{}_0s_k\frac{\bar c_{30}}{A_1A_2}+r^k{}_0r_{k0}\frac{\bar c_{31}}{A_1}\\
&+b^kr_{k0|0}\frac{\bar c_{32}}{A_1}-\alpha^2s^i{}_ks^k{}_i\frac{4}{A_2^2}+\alpha s^k{}_{0|k}\frac{4}{A_2}
\end{aligned}\tag{5}$$

where

$$A_1=1+2B-3s^2,\qquad A_2=1-s$$

and $\bar c_i$ $(i=1,2,\dots,33)$ are polynomials of variations s and B. Substitute $T^m{}_m$ to Eq. (4) and multiple it by A_1^4:

$$^{\alpha}R^m{}_m\big(1+2B-3s^2\big)^4+T^m{}_m\big(1+2B-3s^2\big)^4-K(n-1)F^2\big(1+2B-3s^2\big)^4=0.$$

Obviously

$$^{\alpha}R^m{}_m\big(1+2B-3s^2\big)^4-K(n-1)F^2\big(1+2B-3s^2\big)^4\equiv 0\mod\big(1+2B-3s^2\big).$$

So

$$T^m{}_m\big(1+2B-3s^2\big)^4\equiv 0\mod\big(1+2B-3s^2\big).$$

From (5) we can find out that

$$\frac{r_{00}^2}{\alpha^2}\bar c_2+s_0^2\frac{\bar c_4}{A_2^3}+\frac{r_{00}s_0}{\alpha}\frac{\bar c_6}{A_2}\equiv 0\mod\big(1+2B-3s^2\big).$$

For

$$1+2B-3s^2=-3\left(s-\sqrt{\frac{1+2B}{3}}\right)\left(s+\sqrt{\frac{1+2B}{3}}\right),$$

we have

$$\frac{r_{00}^2}{\alpha^2}\bar c_2+s_0^2\frac{\bar c_4}{A_2^3}+\frac{r_{00}s_0}{\alpha}\frac{\bar c_6}{A_2}\equiv 0\mod\left(s+\sqrt{\frac{1+2B}{3}}\right)$$

and

$$\frac{r_{00}^2}{\alpha^2}\bar c_2+s_0^2\frac{\bar c_4}{A_2^3}+\frac{r_{00}s_0}{\alpha}\frac{\bar c_6}{A_2}\equiv 0\mod\left(s-\sqrt{\frac{1+2B}{3}}\right).$$

Simplify above two equations by Maple and get

$$\left(\frac{r_{00}}{\alpha}-\frac{12(B+2+\sqrt{3+6B})}{(5+B)\sqrt{3+6B}+9+9B}s_0\right)^2\equiv 0 \mod\left(s+\sqrt{\frac{1+2B}{3}}\right) \tag{6}$$

and

$$\left(\frac{r_{00}}{\alpha}+\frac{12(B+2-\sqrt{3+6B})}{(5+B)\sqrt{3+6B}-9-9B}s_0\right)^2\equiv 0 \mod\left(s-\sqrt{\frac{1+2B}{3}}\right). \tag{7}$$

From (6) and Lemma 4.1 we know that

$$r_{00}+\frac{12\sqrt{3}(B+2+\sqrt{3+6B})}{((5+B)\sqrt{3+6B}+9+9B)\sqrt{1+2B}}\alpha s s_0=\sigma_1\alpha^2\left(1+2B-3s^2\right).$$

Similarly from (7) and Lemma 4.1 we can conclude that

$$r_{00}+\frac{12\sqrt{3}(B+2-\sqrt{3+6B})}{((5+B)\sqrt{3+6B}-9-9B)\sqrt{1+2B}}\alpha s s_0=\sigma_2\alpha^2\left(1+2B-3s^2\right).$$

So

$$s s_0\equiv 0 \mod\left(1+2B-3s^2\right).$$

This holds if and only if that

$$s_0=0.$$

Thus

$$r_{00}=\sigma\alpha^2\left(1+2B-3s^2\right).$$

Substituting $T^m{}_m$ and $s_0=0$ to Eq. (4) and multiplying it by A_2^3, we have

$$s^k{}_0 s_{0k}\bar{c}_{21}\equiv 0 \mod(1-s),$$

where $\bar{c}_{21}=6(3s-1)$. Hence

$$s^k{}_0 s_{0k}=0. \qquad \square$$

Theorem 4.3. *Suppose $F:=\frac{(\alpha+\beta)^2}{\alpha}$ is a Finsler metric on an n-dimensional manifold M. Then F has constant flag curvature K if and only if the following three conditions hold*:

(1) *β is closed and $K=0$,*
(2) *$r_{00}=\sigma(1+2B-3s^2)\alpha^2$ and $\sigma_0+2\sigma^2\beta=0$,*
(3) $^\alpha R_{kijl}=a_{ij}(6\sigma^2b_kb_l-(4B+5)\sigma^2a_{kl})+(a_{ik}a_{jl}+a_{il}a_{jk})\sigma^2(4B+5)-6(b_ja_{ik}+b_ia_{jk})b_l\sigma^2-6(b_ja_{il}+b_ia_{jl})b_k\sigma^2+6b_ib_ja_{kl}\sigma^2.$

Proof. "⟸": It can be proved by a direct calculation.

"⟹": From Lemma 4.2 we know that $r_{00}=\sigma(1+2B-3s^2)\alpha^2$, $s_0=0$ and $s^k{}_0s_{0k}=0$. Thus we have

$$r_{ij}=\sigma\left((1+2B)a_{ij}-3b_ib_j\right),\quad s_i=0,\quad s^i{}_k s^k{}_j=0.$$

So

$$\begin{aligned}
&r_0=\sigma\beta(1-B),\qquad r=\sigma\left(B-B^2\right),\\
&r_{00|0}=\sigma_0\alpha^2\left(1+2B-3s^2\right)+\sigma\left(4r_0\alpha^2-6r_{00}\beta\right)\quad\text{where } \sigma_0=\sigma_{x^i}y^i,\\
&r_{00|j}b^j=\sigma_b\alpha^2\left(1+2B-3s^2\right)+\sigma\left(4r\alpha^2-6r_0\beta\right)\quad\text{where } \sigma_b=\sigma_{x^i}b^i,\\
&r_{j0|0}b^j=\sigma_0\beta(1-B)+\sigma(r_0\beta-3Br_{00}),\\
&r^i_i=\sigma\left((1+2B)n-3B\right),\\
&r_{i0}r^i{}_0=\sigma^2\left((1+2B)^2-(6+3B)s^2\right),\\
&r_{j0}=\sigma\alpha\left((1+2B)l_j-3sb_j\right),\\
&r_j=\sigma(1-B)b_j,
\end{aligned}$$

$$r_{j0|0}=\sigma_0\alpha\big((1+2B)l_j-3sb_j\big)+\sigma\alpha\left(4r_0l_j-3\frac{r_{00}}{\alpha}b_j-3sr_{j0}+3ss_{0j}\right),$$
$$r_{00|j}=\sigma\alpha(4\alpha r_j-6sr_{j0}-6ss_{0j})+\sigma_j\alpha^2\big((1+2B)-3s^2\big)\quad\text{where }\sigma_j=\sigma_{x^j},$$
$$r^i{}_0=\sigma\alpha\big((1+2B)l^i-3sb^i\big),$$
$$r^i{}_j=\sigma\big((1+2B)\delta^i_j-3b^ib_j\big).$$

Substitute the above equations to the expression of $T^i{}_j$ in Proposition 3.1 and simplify it by Maple:

$$\begin{aligned}T^i{}_j={}&-\alpha\delta^i_j\big(\sigma^2\alpha\big(2s^2+2s-4B-5\big)+\sigma_0(1-2s)\big)-\alpha l^il_j\big(\sigma^2\alpha(2s+4B+5)+\sigma_0\big)\\&+\alpha l^ib_j\big(\sigma^2\alpha(2s+4)+2\sigma_0\big)-\alpha^2l^i\sigma_j(4s-2)-\alpha b^il_j\big(2\sigma_0-2\sigma^2\alpha s\big)-\alpha^2b^ib_j\big(2\sigma^2\big)+2\alpha^2b^i\sigma_j\\&+l^is_{0j}\sigma\alpha\frac{6(s-2)}{A_2}-b^is_{0j}\sigma\alpha\frac{12}{A_2}-s^i{}_0l_j\sigma\alpha\frac{2(2s-9)}{A_2^2}-s^i{}_0b_j\sigma\alpha\frac{18s}{A_2^2}+s^i{}_0s_{0j}\frac{6(3s-1)}{A_2^3}+s^i{}_{0|j}\alpha\frac{4}{A_2}\\&-s^i{}_{j|0}\alpha\frac{2}{A_2}+s^i{}_{0|0}l_j\frac{2(2s-1)}{A_2^2}-s^i{}_{0|0}b_j\frac{2}{A_2^2}\end{aligned}$$

where

$$A_1=1+2B-3s^2,\qquad A_2=1-s.$$

For F have constant flag curvature, this is equivalent to

$$^{\alpha}R^i{}_j+T^i{}_j=KF^2\left(\delta^i_j-\frac{y^i}{F}F_{y^j}\right).\tag{8}$$

For the same reason in Lemma 4.2 we know that

$$s^i{}_0s_{0j}\big[6(3s-1)\big]\equiv 0\mod(1-s).$$

This means that $s^i{}_0=0$ or $s_{0j}=0$, i.e. β is closed. On the other hand, under above conditions Ricci curvature is given by

$$\begin{aligned}T^m{}_m={}&-2\beta^2\sigma^2(n-2)-\beta\big(4\sigma_0-2\sigma_0n+2\sigma^2\alpha(n-1)\big)\\&-\alpha^2\big(-2\sigma_b+3\sigma^2(1+2B)+2\sigma^2-3\sigma^2n-2\sigma^2(1+2B)n\big)-\alpha\sigma_0(n-1).\end{aligned}\tag{9}$$

Then we can separate the rational part and the irrational part from (8)

$$^{\alpha}R^m{}_m+\mathrm{Rat}\big(T^m{}_m\big)=K(n-1)\alpha^2\big(1+6s^2+s^4\big),$$
$$\mathrm{Irrat}\big(T^m{}_m\big)=\alpha^2K(n-1)\big(4s+4s^3\big).$$

From (9) we have:

$$\mathrm{Irrat}\big(T^m{}_m\big)=-2\beta\sigma^2\alpha(n-1)-\alpha\sigma_0(n-1).$$

Therefore

$$K=-\frac{n-1}{4\beta(1+s^2)}\big(2\sigma^2\beta+\sigma_0\big).$$

Notice K is a constant number, the above equation holds if and only if

$$2\sigma^2\beta+\sigma_0=0\quad\text{and}\quad K=0.$$

So we can express Riemannian curvature of α

$$^{\alpha}R^i{}_j=-T^i{}_j=\delta^i_j\sigma^2\big(6\beta^2-(5+4B)\alpha^2\big)+l^il_j(5+4B)\sigma^2\alpha^2-6\sigma^2\alpha\beta\big(l^ib_j+b^il_j\big)+6b^ib_j\sigma^2\alpha^2.$$

Thus we have

$$\begin{aligned}^{\alpha}R_{kijl}={}&a_{ij}\big(6\sigma^2b_kb_l-(4B+5)\sigma^2a_{kl}\big)+(a_{ik}a_{jl}+a_{il}a_{jk})\sigma^2(4B+5)\\&-6(b_ja_{ik}+b_ia_{jk})b_l\sigma^2-6(b_ja_{il}+b_ia_{jl})b_k\sigma^2+6b_ib_ja_{kl}\sigma^2.\qquad\Box\end{aligned}$$

Furthermore, we find the metrics in Theorem 4.3 are locally projectively flat. To prove this we need several lemmas.

Lemma 4.4. *Suppose α and $\tilde{\alpha}$ are two conformal Riemannian metrics on M, that is, there exists a smooth function ρ such that $\tilde{a}_{ij}=e^{2\rho}a_{ij}$. Their curvature tensors are related by*

$$\tilde{R}^i_{kjl}=R^i_{kjl}+\rho_{kj}\delta^i_l-\rho_{kl}\delta^i_j+a_{kj}a^{ip}\rho_{pl}-a_{kl}a^{ip}\rho_{pj}$$

where

$$\rho_{ij}=\frac{\partial^2\rho}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\rho}{\partial x^k}-\frac{\partial\rho}{\partial x^i}\frac{\partial\rho}{\partial x^j}+\frac{1}{2}a_{ij}a^{kl}\frac{\partial\rho}{\partial x^k}\frac{\partial\rho}{\partial x^l}.$$

Proof. It can be proved by a direct computation. □

Using the notation in Theorem 4.3 we have the following lemma:

Lemma 4.5. *Suppose $F:=\frac{(\alpha+\beta)^2}{\alpha}$ satisfy the three conditions in Theorem 4.3 and $\sigma\neq 0$. Let $\tilde{\alpha}^2=\sigma^2\alpha^2$ and $\tilde{K}$ be sectional curvature of $\tilde{\alpha}$, then $\tilde{K}=-1$.*

Proof. From Lemma 4.1 we have the relationship of Riemannian curvature between α and $\tilde{\alpha}$:

$$^{\tilde{\alpha}}\tilde{R}^i{}_j={}^{\alpha}R^i{}_j+\rho_{j0}y^i-\rho_{00}\delta^i_j+\alpha l_ja^{ip}\rho_{p0}-\alpha^2a^{ip}\rho_{pj}$$

where $\rho_{j0}:=\rho_{jk}y^k$, $\rho_{00}:=\rho_{ij}y^iy^j$.

For $\tilde{\alpha}^2=\sigma^2\alpha^2$ we have

$$\rho=\ln|\sigma|.$$

Hence

$$\frac{\partial\rho}{\partial x^i}=\frac{\partial\ln\sigma}{\partial x^i}=\frac{1}{\sigma}\sigma_{x^i},$$
$$\frac{\partial^2\rho}{\partial x^i\partial x^j}=-\frac{\sigma_{x^i}\sigma_{x^j}}{\sigma^2}+\frac{\sigma_{x^ix^j}}{\sigma}.$$

Because σ satisfies

$$\sigma_{x^i}+2\sigma^2b_i=0,$$

we can obtain

$$\begin{aligned}\rho_{ij}&=\frac{\partial^2\rho}{\partial x^i\partial x^j}-\Gamma^k_{ij}\frac{\partial\rho}{\partial x^k}-\frac{\partial\rho}{\partial x^i}\frac{\partial\rho}{\partial x^j}+\frac{1}{2}a_{ij}a^{kl}\frac{\partial\rho}{\partial x^k}\frac{\partial\rho}{\partial x^l}\\&=-\frac{2\sigma_{x^i}\sigma_{x^j}}{\sigma^2}+\frac{\sigma_{x^ix^j}}{\sigma}-\frac{\Gamma^k_{ij}\sigma_{x^k}}{\sigma}+\frac{a_{ij}a^{kl}\sigma_{x^k}\sigma_{x^l}}{2\sigma^2}\\&=-\frac{2}{\sigma^2}\left(-2\sigma^2b_i\right)\left(-2\sigma^2b_j\right)+\frac{(2\sigma^2b_i)_{|j}}{\sigma}+\frac{a_{ij}a^{kl}(-2\sigma^2b_k)(-2\sigma^2b_l)}{2\sigma^2}\\&=-8\sigma^2b_ib_j-4\sigma_{x^j}b_i-2\sigma b_{i|j}+2\sigma^2Ba_{ij}\\&=-8\sigma^2b_ib_j+8\sigma^2b_ib_j-2\sigma^2\left((1+2B)a_{ij}-3b_ib_j\right)+2\sigma^2Ba_{ij}\\&=-2\sigma^2(1+B)a_{ij}+6\sigma^2b_ib_j.\end{aligned}$$

So

$$\rho_{i0}=-2\sigma^2\alpha l_i(1+B)+6\sigma^2\beta b_i,$$
$$a^{ip}\rho_{p0}=-2\sigma^2\alpha(1+B)l^i+6\sigma^2\beta b^i,$$
$$a^{ip}\rho_{pj}=-2\sigma^2(1+B)\delta^i_j+6\sigma^2b^ib_j.$$

Then

$$\begin{aligned}{}^{\tilde{\alpha}}\tilde{R}^i_j&={}^{\alpha}R^i{}_j-2\sigma^2\alpha^2(1+B)l^il_j+6\sigma^2\alpha\beta l^ib_j-\left(-2\sigma^2\alpha^2(1+B)+6\sigma^2\beta^2\right)\delta^i_j\\&\quad-2\sigma^2\alpha^2(1+B)l^il_j+6\sigma^2\alpha\beta b^il_j+2\sigma^2\alpha^2(1+2B)\delta^i_j-6\sigma^2\alpha^2b^ib_j\\&={}^{\alpha}R^i{}_j+\delta^i_j\left(4\sigma^2\alpha^2(1+B)-6\sigma^2\beta^2\right)-4(1+2B)\sigma^2\alpha^2l^il_j+6\sigma^2\alpha\beta l^ib_j+6\sigma^2\alpha\beta b^il_j-6\sigma^2\alpha^2b^ib_j.\end{aligned}$$

Furthermore ${}^{\alpha}R^i{}_j$ satisfies:

$$ {}^{\alpha}R^i{}_j = \delta^i_j\sigma^2\big(6\beta^2 - (5+4B)\alpha^2\big) + l^i l_j(5+4B)\sigma^2\alpha^2 - 6\sigma^2\alpha\beta\big(l^i b_j + b^i l_j\big) + 6b^i b_j\sigma^2\alpha^2. $$

Thus we have

$$ \begin{aligned} {}^{\tilde{\alpha}}\tilde{R}^i{}_j &= \sigma^2\delta^i_j\big(-\alpha^2\big) + l^i l_j\sigma^2\alpha^2 \\ &= -\tilde{\alpha}^2\big(\delta^i_j - \tilde{l}^i\tilde{l}_j\big). \end{aligned} $$

This means that $\tilde{\alpha}$ have constant sectional curvature -1. □

Theorem 4.6. *Suppose $F := \frac{(\alpha+\beta)^2}{\alpha}$ is a Finsler metric on an n-dimensional manifold M. If F has constant flag curvature K, then F must be locally projectively flat.*

Proof. From Theorem 4.3 we know that

$$ \begin{aligned} & s_{ij} = 0, \\ & r_{00} = \sigma\big(1 + 2B - 3s^2\big)\alpha^2 \end{aligned} $$

and

$$ \sigma_{x^i} + 2\sigma^2 b_i = 0. $$

Thus

$$ G^i = {}^{\alpha}G^i - \sigma\alpha(2s-1)y^i + \sigma\alpha^2 b^i. $$

If $\sigma = 0$, then $r_{00} = 0$ and sectional curvature of α is zero. This means β is parallel with respect to α and α is flat. Thus F is locally Minkowskian [14]. Hence it is locally projectively flat.

If $\sigma \neq 0$, from Lemma 4.5 we know $\tilde{\alpha}$ is projectively flat. Then there is a local coordinate system $\{x^i\}$ in which, the spray coefficients of $\tilde{\alpha}$ can be expressed in the following form

$$ {}^{\tilde{\alpha}}G^i = \tilde{P}y^i. $$

Notice that

$$ \begin{aligned} {}^{\alpha}G^i &= {}^{\tilde{\alpha}}G^i - \frac{\sigma_0 y^i}{\sigma} + \frac{\sigma_{x^k}g^{ki}\alpha^2}{2\sigma} \\ &= {}^{\tilde{\alpha}}G^i - \frac{\sigma_0 y^i}{\sigma} - \sigma\alpha^2 b^i. \end{aligned} $$

So

$$ G^i = {}^{\tilde{\alpha}}G^i - \frac{\sigma_0 y^i}{\sigma} - \sigma\alpha(2s-1)y^i. $$

In the same local coordinate system, the spray coefficients G^i of F can be expressed in the form

$$ G^i = Py^i. $$

Thus F is locally projectively flat. □

Remark. Since the locally projectively flat Finsler metrics $F = \alpha(1 + \frac{\beta}{\alpha})^2$ with constant flag curvature have been classified [14], thus we complete the local classification of such metrics with constant flag curvature.

5. Some results on more general case

Now we discuss a more general class (α, β) metrics $F = \alpha(1 + \frac{\alpha}{\beta})^p$ $(|p| \geqslant 1)$ which include Randers metrics and Matsumoto metrics. The spray coefficients of $F = \alpha(1 + \frac{\beta}{\alpha})^p$ are given by

$$ G^i = {}^{\alpha}G^i + f(s,B)r_{00}l^i + g(s,B)r_{00}b^i + u(s,B)\alpha s_0 l^i + v(s,B)\alpha s_0 b^i + Q\alpha s^i{}_0 $$

where

$$f(s,B)=\frac{2(p-1)ps-p}{2((p^2-1)s^2+(p-2)s-Bp^2+Bp-1)},$$
$$g(s,B)=\frac{-(p-1)p}{2((p^2-1)s^2+(p-2)s-Bp^2+Bp-1)},$$
$$u(s,B)=\frac{(2(p-1)s-1)p^2}{((p^2-1)s^2+(p-2)s-Bp^2+Bp-1)((p-1)s-1)},$$
$$v(s,B)=\frac{(p-1)p^2}{((p^2-1)s^2+(p-2)s-Bp^2+Bp-1)((p-1)s-1)}.$$

Here we called an (α,β) metric F is trivial if

$$r_{ij}=s_{ij}=0.$$

So if an (α,β) metric F is trivial, we have

$$G^i={}^\alpha G^i.$$

In Section 4 we have discussed the case of $p=2$.

When $p^2=1$ i.e. $p=1$ or -1, we can see that the denominator of $f(s,B)$ and $g(s,B)$ is 1 degree polynomial of s. These are quite different from other case.

5.1. The case of $p=1$

When $p=1$, F is a Randers metric and we have already knew the answer [1,2].

5.2. The case of $p=-1$

When $p=-1$, $F=\frac{\alpha^2}{(\alpha+\beta)}$ including the famous Matsumoto metrics [10]. By a similarly analysis, we obtain a necessary condition that such kind of metrics have constant flag curvature.

Theorem 5.1. *Suppose a Finsler metric $F:=\frac{\alpha^2}{(\alpha+\beta)}$ has constant flag curvature K, then F must satisfy*

$$r_{00}+\frac{18}{(4B-1)(2B+1)}\alpha s s_0=\sigma\alpha^2\big((1+2B)^2-9s^2\big).$$

Proof. The spray coefficients of F are given by

$$G^i={}^\alpha G^i+\big(f(s,B)r_{00}+u(s,B)\alpha s_0\big)l^i+\big(g(s,B)r_{00}+v(s,B)\alpha s_0\big)b^i+Q\alpha s^i{}_0$$

where

$$f(s,B)=-\frac{1+4s}{2(1+2B+3s)},$$
$$g(s,B)=\frac{1}{1+2B+3s},$$
$$u(s,B)=-\frac{1+4s}{(1+2B+3s)(1+2s)},$$
$$v(s,B)=\frac{2}{(1+2B+3s)(1+2s)}.$$

Because F has constant flag curvature, it has constant Ricci curvature. Hence

$${}^\alpha R^m{}_m+T^m{}_m=(n-1)KF^2.$$

By Proposition 3.2 we can compute Ricci curvature $R^m{}_m$ and find out that

$$\left(\frac{r_{00}}{\alpha}-\frac{6}{4B-1}s_0\right)^2\equiv 0 \mod(1+2B+3s).$$

According Lemma 4.1 we have

$$r_{00}+\frac{18}{(4B-1)(2B+1)}\alpha s s_0=\sigma\alpha^2\big((1+2B)^2-9s^2\big). \qquad \square$$

Corollary 5.2. *If* 1*-form* β *of* $F := \frac{\alpha^2}{(\alpha+\beta)}$ *is closed, then there is no non-trivial metric which has constant flag curvature.*

Proof. Assume that F has constant flag curvature. By Theorem 5.1 we know that

$$r_{00} + \frac{18}{(4B-1)(2B+1)}\alpha s s_0 = \sigma\alpha^2\big((1+2B)^2 - 9s^2\big).$$

Since β is closed i.e. $s_{ij} = 0$, then

$$r_{00} = \sigma\alpha^2\big((1+2B)^2 - 9s^2\big).$$

Hence

$$\begin{aligned}
&r_0 = \frac{1}{2}\sigma\alpha s\big(2(1+2B)^2 - 18B\big),\\
&r = \sigma B\big((1+2B)^2 - 9B\big),\\
&r_{00|0} = 2\sigma\alpha\big(4(1+2B)\alpha r_0 - 9r_{00}s\big) + \sigma_0\big((1+2B)^2 - 9s^2\big)\alpha^2,\\
&r_{00|i}b^i = 2\sigma\alpha\big(4r\alpha(1+2B) - 9r_0 s\big) + \sigma_b\big((1+2B)^2 - 9s^2\big)\alpha^2,\\
&r_{i0|0}b^i = \sigma(18B-1)s\alpha r_0 - 9\sigma B r_{00} + \sigma_0\alpha s\big((1+2B)^2 - 9B\big),\\
&r^i_i = \sigma\big(n(1+2B)^2 - 9B\big),\\
&r_{i0}r^i{}_0 = \sigma^2\big((1+2B)^4 - 18(1+2B)^2 s^2 + 81Bs^2\big)\alpha^2.
\end{aligned}$$

Substitute these equations to $T^m{}_m$ and obtain

$$\begin{aligned}
T^m{}_m = {}& 9(8n-11)\sigma^2\alpha^2 s^4 + 18\big((5-4B)n + 6B - 7\big)\sigma^2\alpha^2 s^3\\
&+ 3\left[\left(\left(16B^2 - 50B + \frac{1}{4}\right)n - 32B^2 + 82B + \frac{7}{4}\right)\sigma^2\alpha - (2n-3)\sigma_0\right]\alpha s^2\\
&\times\left[\frac{3}{2}\big((8B^2 - 36B - 11)n - 32B^2 + 60B + 11\big)\sigma^2\alpha - 6\sigma_b\alpha + \sigma_0\left(\left(\frac{1}{2} + 4B\right)n - 8B - \frac{5}{2}\right)\right]\alpha s\\
&+ \left[\left(16B^3 + 33B^2 + 18B + \frac{11}{4}\right)n + 32B^3 - 102B^2 - 24B - \frac{11}{4}\right]\sigma^2\alpha^2 + 2(1+2B)\sigma_b\alpha^2\\
&+ \left(\left(B + \frac{1}{2}\right)n + 2B - \frac{1}{2}\right)\sigma_0\alpha.
\end{aligned}$$

So its rational part and irrational part are given by

$$\begin{aligned}
\mathrm{Rat}\big(T^m{}_m\big) = {}& 9(8n-11)\sigma^2\alpha^2 s^4 + 3\left(\left(16B^2 - 50B + \frac{1}{4}\right)n - 32B^2 + 82B + \frac{7}{4}\right)\sigma^2\alpha^2 s^2\\
&+ \sigma_0\left(\left(\frac{1}{2} + 4B\right)n - 8B - \frac{5}{2}\right)\alpha s\\
&+ \left[\left(16B^3 + 33B^2 + 18B + \frac{11}{4}\right)n + 32B^3 - 102B^2 - 24B - \frac{11}{4}\right]\sigma^2\alpha^2 + 2(1+2B)\sigma_b\alpha^2,\\
\mathrm{Irrat}\big(T^m{}_m\big) = {}& 18\big((5-4B)n + 6B - 7\big)\sigma^2\alpha^2 s^3 - 3(2n-3)\sigma_0\alpha s^2\\
&+ \left[\frac{3}{2}\big((8B^2 - 36B - 11)n - 32B^2 + 60B + 11\big)\sigma^2 - 6\sigma_b\right]\alpha^2 s + \left(\left(B + \frac{1}{2}\right)n + 2B - \frac{1}{2}\right)\sigma_0\alpha.
\end{aligned}$$

Because F has constant flag curvature, it also have constant Ricci curvature. This means that

$$^{\alpha}R^m{}_m + T^m{}_m = (n-1)KF^2 = (n-1)K\alpha^2\frac{1}{(1+s)^2}$$

i.e.

$$\big(1 + 2s + s^2\big)^{\alpha}R^m{}_m + T^m{}_m = (n-1)K\alpha^2.$$

Again we separate its rational part and irrational part and obtain:

$$\begin{aligned}
&\big(1+s^2\big)^{\alpha}R^m{}_m + \big(1+s^2\big)\,\mathrm{Rat}\big(T^m{}_m\big) + 2s\,\mathrm{Irrat}\big(T^m{}_m\big) = K\alpha^2(n-1),\\
&2s^{\alpha}R^m{}_m + 2s\,\mathrm{Rat}\big(T^m{}_m\big) + \big(1+s^2\big)\,\mathrm{Irrat}\big(T^m{}_m\big) = 0.
\end{aligned}$$

We can solve K and ${}^{\alpha}R^m{}_m$

$$K = \frac{-(1-s^2)^2}{2(n-1)s\alpha^2}\,\mathrm{Irrat}\big(R^m{}_m\big),$$

$${}^{\alpha}R^m{}_m = -\,\mathrm{Rat}\big(R^m{}_m\big) - \frac{1+s^2}{2s}\,\mathrm{Irrat}\big(R^m{}_m\big).$$

Hence

$$K = \frac{-(1-s^2)^2}{2(n-1)s\alpha^2}\sigma^2\Big[18\big((5-4B)n+6B-7\big)\sigma^2\alpha^2s^3 - 3(2n-3)\sigma_0\alpha s^2 + \Big[\frac{3}{2}\big((8B^2-36B-11\big)n - 32B^2+60B+11\big)\sigma^2 - 6\sigma_b\Big]\alpha^2 s + \Big(\Big(B+\frac{1}{2}\Big)n + 2B - \frac{1}{2}\Big)\sigma_0\alpha\Big]. \tag{10}$$

Since K is constant, (10) holds if and only if

$$\sigma = 0.$$

This means that

$$r_{ij} = s_{ij} = 0.$$

Thus complete the proof. □

Remark. From this corollary we know that there are no non-trivial Matsumoto metrics with constant flag curvature.

5.3. 1-form β is closed

Next we discuss the other case i.e. $p \neq -1, 1, 2$. We mainly concerns the metrics $F = \alpha(1+\frac{\beta}{\alpha})^p$ when 1-form β is closed.

Theorem 5.3. *There are no non-trivial Finsler metrics $F = \alpha(1+\frac{\beta}{\alpha})^p$ $(|p| \geqslant 1)$ which have constant flag curvature when β is closed expect $p = 1, 2$.*

Proof. When $p = -1$, we have proved in Corollary 5.2. When $p \neq -1, 1, 2$ and β is closed, we can compute $T^m{}_m$

$$T^m{}_m = \frac{r_{00}^2}{\alpha^2}(n-1)\frac{\bar{c}_1}{A_1^3} + \frac{r_0r_{00}}{\alpha}(n-1)\frac{\bar{c}_2}{A_1^2} + \frac{r_{00|0}}{\alpha}(n-1)\frac{\bar{c}_3}{A_1} + \frac{r_{00}^2}{\alpha^2}\frac{\bar{c}_4}{A_1^4} + \frac{r_0r_{00}}{\alpha}\frac{\bar{c}_5}{A_1^3} + \frac{r_{00|0}}{\alpha}\frac{\bar{c}_6}{A_1^2} + \big(r_0^2 - rr_{00}\big)\frac{\bar{c}_7}{A_1^2} + \big(r_{00|i}b^i - r_{i0}b^i + r_{00}r^i_i - r_{i0}r^i{}_0\big)\frac{\bar{c}_8}{A_1}$$

where $A_1 := (p^2-1)s^2 + (p-2)s - Bp^2 + Bp - 1$ and $\bar{c}_i$ $(i = 1, 2, \ldots, 8)$ are polynomials of s and B. From the equation of constant Ricci curvature

$$R^m{}_m + T^m{}_m = K(n-1)F^2,$$

we know that

$$r_{00}^2 \equiv 0 \mod\big((p^2-1)s^2 + (p-2)s - Bp^2 + Bp - 1\big).$$

Because $p \neq -1, 0, 1$, $A_1 = (p^2-1)s^2 + (p-2)s - Bp^2 + Bp - 1$ is a 2 degree polynomial of s. Thus

$$r_{00} = \sigma\alpha^2\big((p^2-1)s^2 + (p-2)s - Bp^2 + Bp - 1\big). \tag{11}$$

Notice that $p \neq 2$, the right side includes an irrational part. Hence (11) holds if and only if

$$\sigma = 0.$$

This means that there are no non-trivial such metrics which have constant flag curvature. □

Remark. Because the calculation is too tedious and too long, we do not continue to discuss this kind of metrics without the condition β is closed. But we conjecture that above theorem also holds if we get rid of the restriction on 1-form β.

6. Explanation & Acknowledgement

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Finally the author would like to thank for communicating editor in the discussion preceding the revision.

Furthermore, the author gave many corrections in this second revised version. To distinguish them, the paper's title is also slightly changed. But the main results are still correct and remained to be unchanged.

Appendix A. How to compute the flag and Ricci curvature of (α,β) metric

As we know that the spray coefficients G^i of an (α,β) metric $F:=\alpha\phi(s)$ and the spray coefficients ${}^{\alpha}G^i$ of a Riemannian metric α are related by:

$$G^i = {}^{\alpha}G^i + \Theta(-2\alpha Q s_0 + r_{00})l^i + \Psi(-2\alpha Q s_0 + r_{00})b^i + \alpha Q s^i{}_0$$

where

$$Q = \frac{-2\phi'}{\phi - s\phi'},$$
$$\Theta = \frac{\phi\phi' - s(\phi\phi'' + \phi'\phi')}{2\phi((\phi - s\phi') + (B - s^2)\phi'')},$$
$$\Psi = \frac{\phi''}{2((\phi - s\phi') + (B - s^2)\phi'')}$$

and $l^i := \frac{y^i}{\alpha}$, $s = \frac{\beta}{\alpha}$, $s^i{}_0 = s^i{}_j y^j$, $s_0 = s_i y^i$, $r_{00} = r_{ij}y^i y^j$, $B = a^{ij}b_i b_j$.

To simplify our computation, we separate the spray coefficients into three parts. Let

$$f(s,B) := \Theta, \qquad g(s,B) := \Psi,$$
$$u(s,B) := -2\Theta Q, \qquad v(s,B) := -2\Psi Q$$

and

$$P := f(s,B)r_{00}, \qquad L := g(s,B)r_{00},$$
$$M := u(s,B)\alpha s_0, \qquad W := Q\alpha, \qquad N := v(s,B)\alpha s_0.$$

Then

$$G^i = {}^{\alpha}G^i + \zeta_1^i + \zeta_2^i \tag{12}$$

if we denote

$$\zeta_1^i := Pl^i + Lb^i,$$
$$\zeta_2^i := Ml^i + Ws^i{}_0 + Nb^i.$$

By Berwald's formula for Riemannian curvature and (12) we have

$$R^i{}_j = {}^{\alpha}R^i{}_j + T^i_{1\ j} + T^i_{2\ j} + T^i_{3\ j}$$

where

$$T^i_{p\ j} := 2\zeta^i_{p\ |j} - y^k(\zeta^i_{p\ |k})_{y^j} - (\zeta^i_p)_{y^k}(\zeta^k_p)_{y^j} + 2\zeta^k_p(\zeta^i_p)_{y^j y^k} \quad (p=1,2),$$
$$T^i_{3\ j} := -(\zeta^i_1)_{y^k}(\zeta^k_2)_{y^j} - (\zeta^i_2)_{y^k}(\zeta^k_1)y^j + 2\zeta^k_1(\zeta^i_2)_{y^j y^k} + 2\zeta^k_2(\zeta^i_1)_{y^j y^k}.$$

Before calculation we need some basic preparation:

$$
\begin{aligned}
&\left(l^i\right)_{y^j}=\left(\frac{y^i}{\alpha}\right)_{y^j}=\frac{1}{\alpha}\left(\delta^i_j-l^il_j\right),\\
&\left(l^i\right)_{y^k}\left(l^k\right)_{y^j}=\left(l^i_{y^k}l^k\right)_{y^j}-l^k\left(l^i\right)_{y^ky^j}=\frac{1}{\alpha}\left(l^i\right)_{y^j}=\frac{1}{\alpha^2}\left(\delta^i_j-l^il_j\right),\\
&\left(l^i\right)_{y^jy^k}=\left(\frac{1}{\alpha}\left(\delta^i_j-l^il_j\right)\right)_{y^k}=\frac{1}{\alpha^2}\left(-\delta^i_jl_k-\delta^i_kl_j+3l^il_jl_k-l^ia_{jk}\right),\\
&(l_i)_{y^j}=\left(l^ka_{ki}\right)_{y^j}=\frac{1}{\alpha}(a_{ij}-l_il_j),\\
&s_{y^j}=\left(\frac{\beta}{\alpha}\right)_{y^j}=\frac{1}{\alpha}(b_j-sl_j),\\
&s_{y^iy^j}=\frac{1}{\alpha^2}(-b_il_j-b_jl_i+3sl_il_j-sa_{ij}),\\
&B_{|i}=\left(b_jb^j\right)_{|i}=2b_{j|i}b^j=2(r_i+s_i),\\
&s_{|i}=\left(\frac{\beta}{\alpha}\right)_{|i}=\frac{1}{\alpha}b_{j|i}y^j=\frac{1}{\alpha}(r_{i0}+s_{0i}),\\
&l^i{}_{|j}=0,\\
&b^i{}_{|j}=r^i{}_j+s^i{}_j.
\end{aligned}
$$

Let us first compute $T^i_{1\ j}$:

We can express $(\zeta^i_1)_{y^j}$ and $(\zeta^i_1)_{y^jy^k}$ by

$$
\begin{aligned}
&\left(\zeta^i_1\right)_{y^j}=P_{y^j}l^i+P\left(l^i\right)_{y^j}+L_{y^j}b^i,\\
&\left(\zeta^i_1\right)_{y^jy^k}=P_{y^jy^k}l^i+P_{y^j}\left(l^i\right)_{y^k}+P_{y^k}\left(l^i\right)_{y^j}+P\left(l^i\right)_{y^jy^k}+L_{y^jy^k}b^i.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left(\zeta^i_1\right)_{y^k}\left(\zeta^k_1\right)_{y^j}&=\left(P_{y^k}l^i+P\left(l^i\right)_{y^k}+L_{y^k}b^i\right)\left(P_{y^j}l^k+P\left(l^k\right)_{y^j}+L_{y^j}b^k\right)\\
&=\frac{2P}{\alpha}P_{y^j}l^i+\frac{2L}{\alpha}P_{y^j}b^i+PP_{y^k}\left(l^k\right)_{y^j}l^i+P^2\left(l^i\right)_{y^k}\left(l^k\right)_{y^j}\\
&\quad+PL_{y^k}\left(l^k\right)_{y^j}b^i+\left(P_{y^k}b^k\right)l^iL_{y^j}+PL_{y^j}\left(l^i\right)_{y^k}b^k+\left(L_{y^k}b^k\right)b^iL_{y^j}\\
&=\frac{2P}{\alpha}P_{y^j}l^i+\frac{2L}{\alpha}P_{y^j}b^i+PP_{y^k}\frac{1}{\alpha}\left(\delta^k_j-l^kl_j\right)l^i+P^2\frac{1}{\alpha^2}\left(\delta^i_j-l^il_j\right)\\
&\quad+PL_{y^k}\frac{1}{\alpha}\left(\delta^k_j-l^kl_j\right)b^i+\left(P_{y^k}b^k\right)l^iL_{y^j}+PL_{y^j}\frac{1}{\alpha}\left(\delta^i_k-l^il_k\right)b^k+\left(L_{y^k}b^k\right)b^iL_{y^j}\\
&=\frac{P^2}{\alpha^2}\delta^i_j+\frac{3P}{\alpha}P_{y^j}l^i-\frac{3P^2}{\alpha^2}l^il_j+\left(P_{y^k}b^k-\frac{Ps}{\alpha}\right)l^iL_{y^j}+\frac{2L}{\alpha}P_{y^j}b^i\\
&\quad+\frac{2P}{\alpha}L_{y^j}b^i-\frac{2PL}{\alpha^2}b^il_j+\left(L_{y^k}b^k\right)b^iL_{y^j},\\
2\zeta^k_1\left(\zeta^i_1\right)_{y^ky^j}&=2\left(Pl^k+Lb^k\right)\left(P_{y^jy^k}l^i+P_{y^j}\left(l^i\right)_{y^k}+P_{y^k}\left(l^i\right)_{y^j}+P\left(l^i\right)_{y^jy^k}+L_{y^jy^k}b^i\right)\\
&=2\biggl(\frac{P}{\alpha}P_{y^j}l^i+2\frac{P^2}{\alpha}\left(l^i\right)_{y^j}-\frac{P^2}{\alpha}\left(l^i\right)_{y^j}+\frac{P}{\alpha}L_{y^j}b^i+L\left(P_{y^jy^k}b^k\right)l^i+LP_{y^j}\left(l^i\right)_{y^k}b^k\\
&\quad+L\left(P_{y^k}b^k\right)\left(l^i\right)_{y^j}+LP\left(l^i\right)_{y^jy^k}b^k+L\left(L_{y^jy^k}b^k\right)b^i\biggr)\\
&=2\biggl(\frac{P}{\alpha}P_{y^j}l^i+\frac{P^2}{\alpha^2}\left(\delta^i_j-l^il_j\right)+\frac{P}{\alpha}L_{y^j}b^i+L\left(P_{y^jy^k}b^k\right)l^i+LP_{y^j}\frac{1}{\alpha}\left(\delta^i_k-l^il_k\right)b^k\\
&\quad+L\left(P_{y^k}b^k\right)\frac{1}{\alpha}\left(\delta^i_j-l^il_j\right)+LP\frac{1}{\alpha^2}\left(-\delta^i_jl_k-\delta^i_kl_j+3l^il_jl_k-l^ia_{jk}\right)b^k+L\left(L_{y^jy^k}b^k\right)b^i\biggr)\\
&=2\biggl(\left(\frac{P^2}{\alpha^2}+\frac{L}{\alpha}\left(P_{y^k}b^k\right)-\frac{PLs}{\alpha^2}\right)\delta^i_j+\left(-\frac{P^2}{\alpha^2}-\frac{L}{\alpha}\left(P_{y^k}b^k\right)+\frac{3PLs}{\alpha^2}\right)l^il_j
\end{aligned}
$$

$$+\left(-\frac{Ls}{\alpha}+\frac{P}{\alpha}\right)P_{y^j}l^i+L\left(P_{y^jy^k}b^k\right)l^i-\frac{PL}{\alpha^2}l^ib_j-\frac{PL}{\alpha^2}b^il_j+\frac{P}{\alpha}L_{y^j}b^i+\frac{L}{\alpha}P_{y^j}b^i+L\left(L_{y^jy^k}b^k\right)b^i\bigg).$$

Similarly we can write $\zeta^i_{1\ |j}$ and $y^k(\zeta^i_{1\ |k})_{y^j}$ by

$$\begin{aligned}
\zeta^i_{1\ |j}&=P_{\ |j}l^i+L_{\ |j}b^i+L\left(r^i{}_j+s^i{}_j\right),\\
y^k\left(\zeta^i_{1\ |k}\right)_{y^j}&=\left(\zeta^i_{1\ |k}y^k\right)_{y^j}-\zeta^i_{1\ |j}\\
&=\left(\left(P_{\ |k}y^k\right)l^i+\left(L_{\ |k}y^k\right)b^i+L\left(r^i{}_0+s^i{}_0\right)\right)_{y^j}-\left(P_{\ |j}l^i+L_{\ |j}b^i+L\left(r^i{}_j+s^i{}_j\right)\right)\\
&=\left(P_{\ |k}y^k\right)_{y^j}l^i+\left(P_{\ |k}y^k\right)\frac{1}{\alpha}\left(\delta^i_j-l^il_j\right)+\left(L_{\ |k}y^k\right)_{y^j}b^i+L_{y^j}r^i{}_0+L_{y^j}s^i{}_0\\
&\quad+Lr^i{}_j+Ls^i{}_j-P_{\ |j}l^i-L_{\ |j}b^i-Lr^i{}_j-Ls^i{}_j\\
&=\delta^i_j\left(\frac{1}{\alpha}P_{\ |k}y^k\right)-l^il_j\frac{1}{\alpha}\left(P_{\ |k}y^k\right)+l^i\left(P_{\ |k}y^k\right)_{y^j}-l^iP_{\ |j}+b^i\left(L_{\ |k}y^k\right)_{y^j}-L_{\ |j}b^i+L_{y^j}r^i{}_0+L_{y^j}s^i{}_0.
\end{aligned}$$

Therefore

$$\begin{aligned}
T^i_{1\ j}&=2\zeta^k_1\left(\zeta^i_1\right)_{y^ky^j}-\left(\zeta^i_1\right)_{y^k}\left(\zeta^k_1\right)_{y^j}+2\zeta^i_{1\ |j}-y^k\left(\zeta^i_{1\ |k}\right)_{y^j}\\
&=\left(\frac{P^2}{\alpha^2}+\frac{2L}{\alpha}\left(P_{y^k}b^k\right)-\frac{2PLs}{\alpha^2}-\frac{P_{\ |k}y^k}{\alpha}\right)\delta^i_j+\left(\frac{P^2}{\alpha^2}-\frac{2L}{\alpha}\left(P_{y^k}b^k\right)+\frac{6PLs}{\alpha^2}+\frac{P_{\ |k}y^k}{\alpha}\right)l^il_j\\
&\quad-\left(\frac{2Ls}{\alpha}+\frac{P}{\alpha}\right)l^iP_{y^j}+\left(\frac{Ps}{\alpha}-P_{y^k}b^k\right)l^iL_{y^j}+2L\left(P_{y^jy^k}b^k\right)l^i-\frac{2PL}{\alpha^2}l^ib_j+2L\left(L_{y^jy^k}b^k\right)b^i-b^iL_{y^j}\left(L_{y^k}b^k\right)\\
&\quad+3l^iP_{\ |j}-l^i\left(P_{\ |k}y^k\right)_{y^j}+3L_{\ |j}b^i-b^i\left(L_{\ |k}y^k\right)_{y^j}+2Lr^i{}_j+2Ls^i{}_j-L_{y^j}r^i{}_0-L_{y^j}s^i{}_0.
\end{aligned}$$

Then we compute $T^i_{2\ j}$:

$$\begin{aligned}
\left(\zeta^i_2\right)_{y^j}&=M_{y^j}l^i+M\left(l^i\right)_{y^j}+W_{y^j}s^i{}_0+Ws^i{}_j+N_{y^j}b^i,\\
\left(\zeta^i_2\right)_{y^jy^k}&=M_{y^jy^k}l^i+M_{y^j}\left(l^i\right)_{y^k}+M_{y^k}\left(l^i\right)_{y^j}+M\left(l^i\right)_{y^jy^k}+W_{y^jy^k}s^i{}_0+W_{y^j}s^i{}_k+W_{y^k}s^i{}_j+N_{y^jy^k}b^i.
\end{aligned}$$

So

$$\begin{aligned}
\left(\zeta^i_2\right)_{y^k}\left(\zeta^k_2\right)_{y^j}&=\left(M_{y^k}l^i+M\left(l^i\right)_{y^k}+W_{y^k}s^i{}_0+Ws^i{}_k+N_{y^k}b^i\right)\left(M_{y^j}l^k+M\left(l^k\right)_{y^j}+W_{y^j}s^k{}_0+Ws^k{}_j+N_{y^j}b^k\right)\\
&=\frac{3M}{\alpha}l^iM_{y^j}+\frac{2W}{\alpha}s^i{}_0M_{y^j}+\frac{2N}{\alpha}b^iM_{y^j}-\frac{3M^2}{\alpha^2}l^il_j+\frac{M^2}{\alpha^2}\delta^i_j\\
&\quad+\left(\frac{2M}{\alpha}+W_{y^k}s^k{}_0\right)s^i{}_0W_{y^j}-\frac{2MW}{\alpha^2}s^i{}_0l_j+\frac{2MW}{\alpha}s^i{}_j+\left(\frac{2M}{\alpha}+N_{y^k}b^k\right)b^iN_{y^j}\\
&\quad-\frac{2MN}{\alpha^2}b^il_j+\left(M_{y^k}s^k{}_0\right)l^iW_{y^j}+W\left(s^i{}_ks^k{}_0\right)W_{y^j}+\left(N_{y^k}s^k{}_0\right)b^iW_{y^j}\\
&\quad+Wl^iM_{y^k}s^k{}_j-\frac{MW}{\alpha^2}l^is_{0j}+Ws^i{}_0W_{y^k}s^k{}_j+W^2s^i{}_ks^k{}_j+Wb^is^k{}_jN_{y^k}\\
&\quad+\left(b^kM_{y^k}-\frac{Ms}{\alpha}\right)l^iN_{y^j}+\left(W_{y^k}b^k\right)s^i{}_0N_{y^j}+Ws^iN_{y^j},\\
2\zeta^k_2\left(\zeta^i_2\right)_{y^jy^k}&=2\left(Ml^k+Ws^k{}_0+Nb^k\right)\left(M_{y^jy^k}l^i+M_{y^j}\left(l^i\right)_{y^k}+M_{y^k}\left(l^i\right)_{y^j}+M\left(l^i\right)_{y^jy^k}\right.\\
&\quad\left.+W_{y^jy^k}s^i{}_0+W_{y^j}s^i{}_k+W_{y^k}s^i{}_j+N_{y^jy^k}b^i\right)\\
&=2\left(\frac{M-Ns}{\alpha}l^iM_{y^j}+\left(\frac{M^2-MNs}{\alpha^2}+\frac{W(M_{y^k}s^k{}_0)+N(M_{y^k}b^k)}{\alpha}\right)\delta^i_j\right.\\
&\quad+\left(\frac{-M^2+3MNs}{\alpha^2}-\frac{W(M_{y^k}s^k{}_0)+N(M_{y^k}b^k)}{\alpha}\right)l^il_j+\frac{M}{\alpha}W_{y^j}s^i{}_0\\
&\quad+\left(\frac{MW}{\alpha}+WW_{y^k}s^k{}_0+NW_{y^k}b^k\right)s^i{}_j+\frac{M}{\alpha}N_{y^j}b^i+Wl^i\left(M_{y^jy^k}s^k{}_0\right)\\
&\quad+\frac{W}{\alpha}M_{y^j}s^i{}_0-\frac{MW}{\alpha^2}s^i{}_0l_j-\frac{MW}{\alpha^2}l^is_{j0}+WW_{y^jy^k}s^k{}_0s^i{}_0+WW_{y^j}s^k{}_0s^i{}_k
\end{aligned}$$

$$+ WN_{y^j y^k} s^k{}_0 b^i + NM_{y^j y^k} b^k l^i + \frac{N}{\alpha} M_{y^j} b^i - \frac{MN}{\alpha^2} b^i l_j - \frac{MN}{\alpha^2} l^i b_j$$
$$+ NW_{y^j y^k} b^k s^i{}_0 + NW_{y^j} s^i + N\big(N_{y^j y^k} b^k\big) b^i \Big),$$

$$\zeta_2{}^i{}_{|j} = M_{|j} l^i + N_{|j} b^i + N\big(r^i{}_j + s^i{}_j\big) + W_{|j} s^i{}_0 + W s^i{}_{0|j},$$
$$y^k\big(\zeta_2{}^i{}_{|k}\big)_{y^j} = y^k\big(M_{|k} l^i + N_{|k} b^i + N\big(s^i{}_k + r^i{}_k\big) + W_{|k} s^i{}_0 + W s^i{}_{0|k}\big)_{y^j}$$
$$= y^k (M_{|k})_{y^j} l^i + y^k M_{|k} \frac{1}{\alpha}\big(\delta^i_j - l^i l_j\big) + (N_{|k})_{y^j} y^k b^i + N_{y^j}\big(s^i{}_0 + r^i{}_0\big)$$
$$+ (W_{|k})_{y^j} y^k s^i{}_0 + y^k W_{|k} s^i{}_j + W_{y^j} s^i{}_{0|0} + W s^i{}_{j|0}$$
$$= \delta^i_j\left(\frac{y^k M_{|k}}{\alpha}\right) - l^i l_j\left(\frac{y^k M_{|k}}{\alpha}\right) + l^i y^k (M_{|k})_{y^j} + b^i (N_{|k})_{y^j} y^k$$
$$+ s^i{}_0\big(N_{y^j} + (W_{|k})_{y^j} y^k\big) + s^i{}_j W_{|0} + s^i{}_{0|0} W_{y^j} + s^i{}_{j|0} W + r^i{}_0 N_{y^j}.$$

Thus

$$T_2{}^i{}_j = 2\zeta_2^k\big(\zeta_2^i\big)_{y^k y^j} - \big(\zeta_2^i\big)_{y^k}\big(\zeta_2^k\big)_{y^j} + 2\zeta_2{}^i{}_{|j} - y^k\big(\zeta_2{}^i{}_{|k}\big)_{y^j}$$
$$= \delta^i_j\left(\frac{M^2}{\alpha^2} + \frac{2W}{\alpha}\big(M_{y^k} s^k{}_0\big) + \frac{2N}{\alpha}\big(M_{y^k} b^k\big) - \frac{2MNs}{\alpha^2} - \frac{y^k M_{|k}}{\alpha}\right)$$
$$+ l^i l_j\left(\frac{M^2}{\alpha^2} - \frac{2W}{\alpha}\big(M_{y^k} s^k{}_0\big) - \frac{2N}{\alpha}\big(M_{y^k} b^k\big) + \frac{6MNs}{\alpha^2} + \frac{y^k M_{|k}}{\alpha}\right)$$
$$- l^i M_{y^j}\frac{M + 2Ns}{\alpha} + l^i N_{y^j}\left(-b^k M_{y^k} + \frac{Ms}{\alpha}\right) + l^i W_{y^j}\big(-M_{y^k} s^k{}_0\big) + 2W l^i\big(M_{y^j y^k} s^k{}_0\big) + \frac{3MW}{\alpha^2} l^i s_{0j}$$
$$+ 2N l^i\big(M_{y^j y^k} b^k\big) - W l^i (M_{y^k} s^k{}_j) - \frac{2MN}{\alpha^2} l^i b_j + l^i\big(2M_{|j} - y^k (M_{|k})_{y^j}\big) - b^i N_{y^j}\big(N_{y^k} b^k\big)$$
$$- b^i W_{y^j}\big(N_{y^k} s^k{}_0\big) + 2W b^i N_{y^j y^k} s^k{}_0 + 2N b^i\big(N_{y^j y^k} b^k\big) - W b^i s^k{}_j N_{y^k} + b^i\big(2N_{|j} - (N_{|k})_{y^j} y^k\big)$$
$$+ 2 s^i{}_j\big(W W_{y^k} s^k{}_0 + N W_{y^k} b^k\big) - s^i{}_0 W_{y^j}\big(W_{y^k} s^k{}_0\big) + s^i{}_k s^k{}_0 W_{y^j} W - s^i{}_0 W_{y^k} s^k{}_j W - W^2 s^i{}_k s^k{}_j$$
$$- s^i{}_0 N_{y^j}\big(W_{y^k} b^k\big) - s^i N_{y^j} W + 2N s^i W_{y^j} + 2W s^k{}_0 s^i{}_0 W_{y^j y^k} + 2N s^i{}_0\big(W_{y^j y^k} b^k\big)$$
$$+ s^i{}_0\big(2W_{|j} - N_{y^j} - (W_{|k})_{y^j} y^k\big) + s^i{}_j (2N - W_{|0}) + 2W s^i{}_{0|j} - s^i{}_{0|0} W_{y^j} - s^i{}_{j|0} W + 2N r^i{}_j - r^i{}_0 N_{y^j}.$$

Finally we compute $T_3{}^i{}_j$:

$$\big(\zeta_1^i\big)_{y^k}\big(\zeta_2^k\big)_{y^j} = \big(P_{y^k} l^i + P\big(l^i\big)_{y^k} + L_{y^k} b^i\big)\big(M_{y^j} l^k + M\big(l^k\big)_{y^j} + W_{y^j} s^k{}_0 + W s^k{}_j + N_{y^j} b^k\big)$$
$$= \frac{2P}{\alpha} l^i M_{y^j} + \frac{2L}{\alpha} b^i M_{y^j} + \frac{M}{\alpha} P_{y^j} l^i - l^i l_j \frac{2PM}{\alpha^2} + \frac{MP}{\alpha^2}\big(\delta^i_j - l^i l_j\big)$$
$$+ \frac{M}{\alpha} b^i L_{y^j} - \frac{2LM}{\alpha^2} b^i l_j + \big(P_{y^k} s^k{}_0\big) l^i W_{y^j} + \frac{P}{\alpha} s^i{}_0 W_{y^j} + \big(L_{y^k} s^k{}_0\big) b^i W_{y^j}$$
$$+ W P_{y^k} s^k{}_j l^i + \frac{WP}{\alpha} s^i{}_j - \frac{WP}{\alpha^2} l^i s_{0j} + W\big(L_{y^k} s^k{}_j\big) b^i + \big(P_{y^k} b^k\big) l^i N_{y^j}$$
$$+ \frac{P}{\alpha} b^i N_{y^j} - \frac{Ps}{\alpha} l^i N_{y^j} + \big(L_{y^k} b^k\big) b^i N_{y^j},$$

$$\big(\zeta_2^i\big)_{y^k}\big(\zeta_1^k\big)_{y^j} = \big(M_{y^k} l^i + M\big(l^i\big)_{y^k} + W_{y^k} s^i{}_0 + W s^i{}_k + N_{y^k} b^i\big)\big(P_{y^j} l^k + P\big(l^k\big)_{y^j} + L_{y^j} b^k\big)$$
$$= \frac{2M}{\alpha} l^i P_{y^j} + \frac{W}{\alpha} s^i{}_0 P_{y^j} + \frac{W}{\alpha} s^i{}_0 P_{y^j} + \frac{2N}{\alpha} b^i P_{y^j} + \frac{P}{\alpha} l^i M_{y^j} - \frac{2PM}{\alpha^2} l^i l_j$$
$$+ \frac{MP}{\alpha^2}\big(\delta^i_j - l^i l_j\big) + \frac{P}{\alpha} s^i{}_0 W_{y^j} - \frac{P}{\alpha^2} W s^i{}_0 l_j + \frac{P}{\alpha} b^i N_{y^j} - \frac{2PN}{\alpha^2} b^i l_j + \frac{WP}{\alpha} s^i{}_j - \frac{WP}{\alpha^2} s^i{}_0 l_j$$
$$+ \big(M_{y^k} b^k\big) l^i L_{y^j} + \frac{M}{\alpha} b^i L_{y^j} - \frac{Ms}{\alpha} l^i L_{y^j} + \big(W_{y^k} b^k\big) s^i{}_0 L_{y^j} + W\big(s^i{}_k b^k\big) L_{y^j} + \big(N_{y^k} b^k\big) b^i L_{y^j},$$

$$
\begin{aligned}
\zeta_1^k(\zeta_2^i)_{y^jy^k} &= (Pl^k + Lb^k)\big(M_{y^jy^k}l^i + M_{y^j}(l^i)_{y^k} + M_{y^k}(l^i)_{y^j} + M(l^i)_{y^jy^k} + W_{y^jy^k}s^i{}_0 + W_{y^j}s^i{}_k + W_{y^k}s^i{}_j + N_{y^jy^k}b^i\big) \\
&= \frac{P}{\alpha}l^iM_{y^j} + \frac{PM}{\alpha^2}(\delta^i_j - l^il_j) + \frac{P}{\alpha}s^i{}_0W_{y^j} + \frac{PW}{\alpha}s^i{}_j + \frac{P}{\alpha}b^iN_{y^j} + Ll^i(M_{y^jy^k}b^k) + \frac{L}{\alpha}b^iM_{y^j} \\
&\quad - \frac{Ls}{\alpha}l^iM_{y^j} + \frac{L}{\alpha}M_{y^k}b^k(\delta^i_j - l^il_j) - \frac{LMs}{\alpha^2}\delta^i_j - \frac{LM}{\alpha^2}b^il_j + \frac{3LMs}{\alpha^2}l^il_j - \frac{LM}{\alpha^2}l^ib_j \\
&\quad + Ls^i{}_0(W_{y^jy^k}b^k) - Ls^iW_{y^j} + L(W_{y^k}b^k)s^i{}_j + Lb^i(N_{y^jy^k}b^k),
\end{aligned}
$$

$$
\begin{aligned}
\zeta_2^k(\zeta_1^i)_{y^jy^k} &= (Ml^k + Ws^k{}_0 + Nb^k)\big(P_{y^jy^k}l^i + P_{y^j}(l^i)_{y^k} + P_{y^k}(l^i)_{y^j} + P(l^i)_{y^jy^k} + L_{y^jy^k}b^i\big) \\
&= \frac{M}{\alpha}P_{y^j}l^i + \frac{MP}{\alpha^2}(\delta^i_j - l^il_j) + \frac{M}{\alpha}b^iL_{y^j} + Wl^i(P_{y^jy^k}s^k{}_0) \\
&\quad + \frac{W}{\alpha}P_{y^j}s^i{}_0 + W(P_{y^k}s^k{}_0)\frac{1}{\alpha}(\delta^i_j - l^il_j) + \frac{WP}{\alpha^2}(-l_js^i{}_0 - l^is_{j0}) + Wb^i(L_{y^jy^k}s^k{}_0) + Nl^i(P_{y^jy^k}b^k) \\
&\quad + \frac{N}{\alpha}(b^i - sl^i)P_{y^j} + N(P_{y^k}b^k)\frac{1}{\alpha}(\delta^i_j - l^il_j) + \frac{NP}{\alpha^2}(-s\delta^i_j - b^il_j + 3sl^il_j - l^ib_j) + Nb^i(L_{y^jy^k}b^k).
\end{aligned}
$$

Hence

$$
\begin{aligned}
T^i_{3\ j} &= -(\zeta_1^i)_{y^k}(\zeta_2^k)_{y^j} - (\zeta_2^i)_{y^k}(\zeta_1^k)y^j + 2\zeta_1^k(\zeta_2^i)_{y^jy^k} + 2\zeta_2^k(\zeta_1^i)_{y^jy^k} \\
&= \delta^i_j\bigg(\frac{2PM}{\alpha^2} - \frac{2LMs}{\alpha^2} + \frac{2W}{\alpha}(P_{y^k}s^k{}_0) + \frac{2N}{\alpha}(P_{y^k}b^k) - \frac{2NPs}{\alpha^2} + 2\frac{L}{\alpha}M_{y^k}b^k\bigg) - \frac{P}{\alpha}l^iM_{y^j} \\
&\quad - \frac{M}{\alpha}l^iP_{y^j} + l^il_j\bigg(\frac{2PM}{\alpha^2} + \frac{6LMs}{\alpha^2} - \frac{2W}{\alpha}(P_{y^k}s^k{}_0) - \frac{2N}{\alpha}(P_{y^k}b^k) + \frac{6NPs}{\alpha^2} - 2\frac{L}{\alpha}M_{y^k}b^k\bigg) \\
&\quad - (P_{y^k}s^k{}_0)l^iW_{y^j} - Wl^i(P_{y^k}s^k{}_j) + \frac{3WP}{\alpha^2}l^is_{0j} - P_{y^k}b^kl^iN_{y^j} + \frac{Ps}{\alpha}l^iN_{y^j} \\
&\quad - (M_{y^k}b^k)l^iL_{y^j} + \frac{Ms}{\alpha}l^iL_{y^j} + 2Ll^i(M_{y^jy^k}b^k) - \frac{2Ls}{\alpha}l^iM_{y^j} - \frac{2LM}{\alpha^2}l^ib_j \\
&\quad + 2Wl^i(P_{y^jy^k}s^k{}_0) + 2Nl^i(P_{y^jy^k}b^k) - \frac{2Ns}{\alpha}l^iP_{y^j} - \frac{2NP}{\alpha^2}l^ib_j \\
&\quad - (L_{y^k}s^k{}_0)b^iW_{y^j} - Wb^i(L_{y^k}s^k{}_j) - (L_{y^k}b^k)b^iN_{y^j} - (N_{y^k}b^k)b^iL_{y^j} \\
&\quad + 2Lb^i(N_{y^jy^k}b^k) + 2Wb^i(L_{y^jy^k}s^k{}_0) + 2Nb^i(L_{y^jy^k}b^k) \\
&\quad + s^i{}_j(2LW_{y^k}b^k) + s^i{}_0\big(2L(W_{y^jy^k}b^k) - (W_{y^kb^k})L_{y^j}\big) + s^i(-2LW_{y^j} + WL_{y^j}).
\end{aligned}
$$

Thus we need to compute

$$
\begin{aligned}
P_{y^j} &= (fr_{00})_{y^j} = f_ss_{y^j}r_{00} + 2fr_{j0} = f_s\frac{1}{\alpha}(b_j - sl_j)r_{00} + 2fr_{j0}, \\
P_{y^k}b^k &= f_s\frac{1}{\alpha}(B - s^2)r_{00} + 2fr_0, \\
P_{y^k}s^k{}_0 &= \frac{r_{00}s_0}{\alpha}f_s + 2fr_{k0}s^k{}_0, \\
P_{y^k}s^k{}_j &= \frac{r_{00}}{\alpha}f_ss_j - \frac{r_{00}}{\alpha^2}sf_ss_{0j} + 2fr_{k0}s^k{}_j, \\
P_{y^jy^k}b^k &= (P_{y^k}b^k)_{y^j} = \bigg(f_s\frac{1}{\alpha}(B - s^2)r_{00} + 2fr_0\bigg)_{y^j} \\
&= l_j\bigg(\frac{r_{00}}{\alpha^2}\big(-sf_{ss}(B - s^2) - f_s(B - s^2) + 2s^2f_s\big) - 2\frac{sf_s}{\alpha}r_0\bigg) \\
&\quad + b_j\bigg(\frac{r_{00}}{\alpha^2}\big(f_{ss}(B - s^2) - 2sf_s\big) + 2\frac{f_s}{\alpha}r_0\bigg) + 2f_s(B - s^2)\frac{r_{j0}}{\alpha} + 2fr_j, \\
P_{y^jy^k}s^k{}_0 &= l_j\bigg(-\frac{r_{00}s_0}{\alpha^2}(f_s + sf_{ss}) - 2\frac{r_{k0}s^k{}_0}{\alpha}sf_s\bigg) + b_j\bigg(\frac{r_{00}s_0}{\alpha^2}f_{ss} + 2\frac{r_{k0}s^k{}_0}{\alpha}f_s\bigg) \\
&\quad + 2r_{jk}s^k{}_0f + \frac{s_{0j}r_{00}}{\alpha^2}sf_s + 2\frac{r_{j0}s_0}{\alpha}f_s,
\end{aligned}
$$

$$P_{|j} = f_{|j} r_{00} + f r_{00|j} = \frac{f_s}{\alpha}(r_{j0} + s_{0j}) r_{00} + 2 f_B (r_j + s_j) r_{00} + f r_{00|j},$$

$$P_{|k} y^k = \frac{f_s}{\alpha} r_{00}^2 + 2 f_B (r_0 + s_0) r_{00} + f r_{00|0},$$

$$\begin{aligned}
(P_{|k} y^k)_{y^j} &= \frac{f_{ss}}{\alpha^2}(b_j - s l_j) r_{00}^2 - \frac{f_s}{\alpha^2} r_{00}^2 l_j + \frac{4 f_s}{\alpha} r_{00} r_{j0} + \frac{2 f_{Bs}}{\alpha}(b_j - s l_j)(r_0 + s_0) r_{00} \\
&\quad + 2 f_B (r_j + s_j) r_{00} + 4 f_B (r_0 + s_0) r_{j0} + \frac{f_s}{\alpha}(b_j - s l_j) r_{00|0} + 2 f r_{j0|0} + f r_{00|j} \\
&= l_j \left(-\frac{r_{00}}{\alpha^2}(f_{ss} s + f_s) - \frac{2 f_{Bs} s}{\alpha}(r_0 + s_0) r_{00} - \frac{s f_s}{\alpha} r_{00|0} \right) + b_j \left(\frac{r_{00}^2}{\alpha^2} f_{ss} + \frac{2 f_{Bs}}{\alpha}(r_0 + s_0) r_{00} \right. \\
&\quad \left. + \frac{f_s}{\alpha} r_{00|0} \right) + \frac{4 f_s}{\alpha} r_{00} r_{j0} + 2 f_B r_{00}(r_j + s_j) + 4 f_B (r_0 + s_0) r_{j0} + 2 f r_{j0|0} + f r_{00|j}.
\end{aligned}$$

By a similar calculation we can get

$$L_{y^j} = g_s \frac{1}{\alpha}(b_j - s l_j) r_{00} + 2 g r_{j0},$$

$$L_{y^j} b^j = g_s \frac{1}{\alpha}(B - s^2) r_{00} + 2 g r_0,$$

$$L_{y^k} s^k{}_0 = \frac{r_{00} s_0}{\alpha} g_s + 2 g r_{k0} s^k{}_0,$$

$$L_{y^k} s^k{}_j = \frac{r_{00}}{\alpha} g_s s_j - \frac{r_{00}}{\alpha^2} s g_s s_{0j} + 2 g r_{k0} s^k{}_j,$$

$$\begin{aligned}
L_{y^j y^k} b^k &= l_j \left(\frac{r_{00}}{\alpha^2}\left(-s g_{ss}(B - s^2) - g_s (B - s^2) + 2 s^2 g_s\right) - 2 \frac{s g_s}{\alpha} r_0 \right) \\
&\quad + b_j \left(\frac{r_{00}}{\alpha^2}\left(g_{ss}(B - s^2) - 2 s g_s\right) + 2 \frac{g_s}{\alpha} r_0 \right) + 2 g_s (B - s^2) \frac{r_{j0}}{\alpha} + 2 g r_j,
\end{aligned}$$

$$L_{y^j y^k} s^k{}_0 = l_j \left(-\frac{r_{00} s_0}{\alpha^2}(g_s + s g_{ss}) - 2 \frac{r_{k0} s^k{}_0}{\alpha} s g_s \right) + b_j \left(\frac{r_{00} s_0}{\alpha^2} g_{ss} + 2 \frac{r_{k0} s^k{}_0}{\alpha} g_s \right) + 2 r_{jk} s^k{}_0 g + \frac{s_{0j} r_{00}}{\alpha^2} s g_s + 2 \frac{r_{j0} s_0}{\alpha} g_s,$$

$$L_{|j} = \frac{g_s}{\alpha}(r_{j0} + s_{0j}) r_{00} + 2 g_B (r_j + s_j) r_{00} + g r_{00|j},$$

$$L_{|k} y^k = \frac{g_s}{\alpha} r_{00}^2 + 2 g_B (r_0 + s_0) r_{00} + g r_{00|0},$$

$$\begin{aligned}
(L_{|k} y^k)_{y^j} &= l_j \left(-\frac{r_{00}}{\alpha^2}(g_{ss} s + g_s) - \frac{2 g_{Bs} s}{\alpha}(r_0 + s_0) r_{00} - \frac{s g_s}{\alpha} r_{00|0} \right) + b_j \left(\frac{r_{00}^2}{\alpha^2} g_{ss} + \frac{2 g_{Bs}}{\alpha}(r_0 + s_0) r_{00} + \frac{g_s}{\alpha} r_{00|0} \right) \\
&\quad + \frac{4 g_s}{\alpha} r_{00} r_{j0} + 2 g_B r_{00}(r_j + s_j) + 4 g_B (r_0 + s_0) r_{j0} + 2 g r_{j0|0} + g r_{00|j},
\end{aligned}$$

$$M_{y^j} = (u \alpha s_0)_{y^j} = u_s s_0 b_j + s_0 l_j (u - s u_s) + u \alpha s_j,$$

$$M_{y^k} b^k = s_0 \left(B u_s + (u - s u_s) s\right),$$

$$M_{y^k} s^k{}_0 = u_s s_0^2 + u \alpha s_k s^k{}_0,$$

$$M_{y^k} s^k{}_j = u_s s_0 s_j + \frac{s_0}{\alpha}(u - s u_s) s_{0j} + u \alpha s_k s^k{}_j,$$

$$M_{y^j y^k} b^k = l_j \left(\frac{s_0}{\alpha}\left(-u_{ss} s (B - s^2) + u_s s^2 - u s\right) \right) + b_j \left(\frac{s_0}{\alpha}\left(u_{ss}(B - s^2) + u - s u_s\right) \right) + s_j \left(u_s (B - s^2) + u s\right),$$

$$M_{y^j y^k} s^k{}_0 = l_j \left(-\frac{s_0^2}{\alpha} u_{ss} s + s_k s^k{}_0 (u - s u_s) \right) + b_j \left(\frac{s_0^2}{\alpha} u_{ss} + s_k s^k{}_0 u_s \right) + s_j (s_0 u_s) + s_{j0} \frac{s_0}{\alpha}(u - s u_s),$$

$$M_{|j} = u_s s_0 (s_{0j} + r_{j0}) + 2 u_B \alpha s_0 (s_j + r_j) + u \alpha s_{0|j},$$

$$M_{|k} y^k = 2 u_B s_0^2 \alpha + u \alpha s_{0|0} + 2 r_0 u_B \alpha s_0 + u_s s_0 r_{00},$$

$$\begin{aligned}
(M_{|k} y^k)_{y^j} &= l_j \left(s_0^2 (2 u_B - 2 u_{Bs} s) + s_{0|0}(u - s u_s) + r_0 s_0 (2 u_B - 2 u_{Bs} s) - \frac{s_0 r_{00}}{\alpha} s u_{ss} \right) \\
&\quad + b_j \left(2 u_{Bs} s_0^2 + u_s s_{0|0} + 2 u_{Bs} r_0 s_0 + \frac{r_{00} s_0}{\alpha} u_{ss} \right) + s_j (4 u_B s_0 \alpha + 2 r_0 u_B \alpha + u_s r_{00})
\end{aligned}$$

$$+u\alpha s_{j|0}+u\alpha s_{0|j}+r_j(2u_B\alpha s_0)+r_{j0}(2u_s s_0),$$

$$N_{y^j}=v_s s_0 b_j+s_0 l_j(v-sv_s)+v\alpha s_j,$$

$$N_{y^k}b^k=s_0\big(Bv_s+(v-sv_s)s\big),$$

$$N_{y^k}s^k{}_0=v_s s_0^2+v\alpha s_k s^k{}_0,$$

$$N_{y^k}s^k{}_j=v_s s_0 s_j+\frac{s_0}{\alpha}(v-sv_s)s_{0j}+v\alpha s_k s^k{}_j,$$

$$N_{y^j y^k}b^k=l_j\left(\frac{s_0}{\alpha}\big(-v_{ss}s\big(B-s^2\big)+v_s s^2-vs\big)\right)+b_j\left(\frac{s_0}{\alpha}\big(v_{ss}\big(B-s^2\big)+v-sv_s\big)\right)+s_j\big(v_s\big(B-s^2\big)+vs\big),$$

$$N_{y^j y^k}s^k{}_0=l_j\left(-\frac{s_0^2}{\alpha}v_{ss}s+s_k s^k{}_0(v-sv_s)\right)+b_j\left(\frac{s_0^2}{\alpha}v_{ss}+s_k s^k{}_0 v_s\right)+s_j(s_0 v_s)+s_{j0}\frac{s_0}{\alpha}(v-sv_s),$$

$$N_{|j}=v_s s_0(s_{0j}+r_{j0})+2v_B\alpha s_0(s_j+r_j)+v\alpha s_{0|j},$$

$$N_{|k}y^k=2v_B s_0^2\alpha+v\alpha s_{0|0}+2r_0 v_B\alpha s_0+v_s s_0 r_{00},$$

$$\big(N_{|k}y^k\big)_{y^j}=l_j\left(s_0^2(2v_B-2v_{Bs}s)+s_{0|0}(v-sv_s)+r_0 s_0(2v_B-2v_{Bs}s)-\frac{s_0 r_{00}}{\alpha}sv_{ss}\right)$$
$$+b_j\left(2v_{Bs}s_0^2+v_s s_{0|0}+2v_{Bs}r_0 s_0+\frac{r_{00}s_0}{\alpha}v_{ss}\right)+s_j(4v_B s_0\alpha+2r_0 v_B\alpha+v_s r_{00})$$
$$+v\alpha s_{j|0}+v\alpha s_{0|j}+r_j(2v_B\alpha s_0)+r_{j0}(2v_s s_0),$$

$$W_{y^j}=(Q\alpha)_{y^j}=b_jQ_s+l_j(Q-sQ_s),$$

$$W_{y^k}b^k=BQ_s+s(Q-sQ_s),$$

$$W_{y^k}s^k{}_0=Q_s s_0,$$

$$W_{y^k}s^k{}_j=s_jQs+\frac{s_{0j}}{\alpha}(Q-sQ_s),$$

$$W_{y^j y^k}b^k=\frac{l_j}{\alpha}s\big(sQ-Q-Q_{ss}\big(B-s^2\big)\big)+\frac{b_j}{\alpha}\big(Q_{ss}\big(B-s^2\big)+Q-sQ_s\big),$$

$$W_{y^j y^k}s^k{}_0=\frac{l_j}{\alpha}(-sQ_{ss}s_0)+\frac{b_j}{\alpha}(Q_{ss}s_0)+\frac{s_{j0}}{\alpha}(Q-sQ_s),$$

$$W_{|j}=(Q\alpha)_{|k}=Q_s(r_{j0}+s_{0j}),$$

$$W_{|k}y^k=Q_s r_{00},$$

$$\big(W_{|k}y^k\big)_{y^j}=-l_j\frac{r_{00}}{\alpha}sQ_{ss}+b_j\frac{r_{00}}{\alpha}Q_{ss}+2Q_s r_{j0}.$$

Substituting these equations into the formula of Riemannian curvature $R^i{}_j$ and computing by Maple we obtain Proposition 3.1.

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