The Convergence of T-Sum of Fuzzy Numbers on Banach Spaces

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Abstract—This paper presents the membership function of finite (or infinite) sum (defined by the sup-t-norm convolution) of fuzzy numbers on Banach spaces, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape function, which generalizes Hong and Hwang's results [1] of the real case. As applications, we calculate the membership function of the limit distribution of Yager's, Hamacher's and Dombi's sum.

Keywords—Fuzzy numbers on Banach spaces, Archimedean t-norm, Convergence of T-sum.

1. INTRODUCTION

In 1991, Fullér calculated the membership function of the product-sum of triangular fuzzy numbers, and he asked for conditions on which the product-sum on L-R fuzzy numbers has the same membership function. The answer for this question was given by Triesch [2] and Hong [3], which is the conditions that log L and log R are concave functions.

Recently, Hong and Hwang [1] determined the exact membership function of the t-norm-based sum of fuzzy numbers, in the case of Archimedean t-norm having convex additive generator function and fuzzy numbers with concave shape functions, which is the generalization of Fullér and Keresztfalvi's result [4].

The purpose of this paper is to study the membership function of the t-norm-based sum of fuzzy numbers on Banach spaces, which generalizes earlier results by Fullér [5] and Hong and Hwang [1]. The idea follows from Hong and Hwang's paper [1].

2. DEFINITIONS

A function $T : [0, 1] \times [0, 1] \rightarrow [0, 1]$ is said to be a triangular norm (t-norm) iff it is symmetric, associative, nondecreasing in each argument, and $T(x, 1) = x$, for all $x \in [0, 1]$. A t-norm $T$ is Archimedean iff $T$ is continuous and $T(x, x) < x$, for all $x \in (0, 1)$.

Every Archimedean t-norm $T$ is representable by a continuous and decreasing function $f : [0, 1] \rightarrow [0, \infty]$ with $f(1) = 0$ and

$$T(x, y) = f^{-1}(f(x) + f(y)),$$
where \( f^{-1} \) is the pseudo-inverse of \( f \), defined by

\[
  f^{-1}(y) := \begin{cases} 
    f^{-1}(y), & \text{if } y \in [0, f(0)], \\
    0, & \text{if } y \in [f(0), \infty].
  \end{cases}
\]

The function \( f \) is the additive generator of \( T \) [6].

The associativity of triangular norm \( T \) allows us to extend these mappings to an arbitrary finite number of arguments in a unique way, by means of a recursive definition. Moreover if \( T \) is Archimedean, then for \( x_i \in [0, 1], i = 1, 2, \ldots, n \),

\[
  T(x_1, x_2, \ldots, x_n) = f^{[-1]}(f(x_1) + \cdots + f(x_n)).
\]

Let \( E \) be a real Banach space. A fuzzy set in \( E \) is a function \( \xi : E \to [0, 1] \). Suppose that a sequence of fuzzy sets \( \xi_1, \xi_2, \ldots \) and a t-norm \( T \) are given. The \( T \)-sum \( \xi_1 + \xi_2 + \cdots + \xi_n \) is the fuzzy set defined by \( (\xi_1 + \xi_2 + \cdots + \xi_n)(z) = \sup_{x_1 + \cdots + x_n = z} T(\xi_1(x_1), \ldots, \xi_n(x_n)) \). If \( T \) is Archimedean, from the property that the additive generator \( f \) is continuous and decreasing, we have

\[
  (\xi_1 + \cdots + \xi_n)(z) = \sup_{x_1 + \cdots + x_n = z} f^{-1}\left( f(\xi_1(x_1)) + \cdots + f(\xi_n(x_n)) \right).
\]

A subset \( M \) of the Banach space \( E \) is said to be convex iff for all \( x, y \in M \), we have \( ax + (1 - a)y \in M \) for all real \( a \in [0, 1] \). Let \( M \) be a convex subset of \( E \), and let \( \phi : M \to R \). Then \( \phi \) is said to be convex in \( M \) if for every \( x_1, x_2 \in M \), \( \phi(\theta x_1 + (1 - \theta)x_2) \leq \theta \phi(x_1) + (1 - \theta)\phi(x_2) \) for every \( 0 \leq \theta \leq 1 \). If \( -\phi \) is convex, then \( \phi \) is said to be concave on \( M \).

The following lemma is well known.

**Lemma 1.** (See [7].) Let \( \phi \) be convex [concave] in a convex set \( M \subset E \). Let \( \{x_i\}_{i=1}^n \) be points in \( M \) and \( \{\lambda_i\}_{i=1}^n \) satisfy \( \lambda_i \geq 0 \) and \( \sum_{i=1}^n \lambda_i = 1 \). Then,

\[
  \phi \left( \sum_{i=1}^n \lambda_i x_i \right) \leq \sum_{i=1}^n \lambda_i \phi(x_i).
\]

**3. T-NORM-BASED ADDITION OF FUZZY NUMBERS**

Let \( M \) be a convex subset of \( E \) whose interior contains the origin. Let \( g : M \to [0, 1] \) be a concave function on \( M \).

Define a fuzzy set \( \xi \) on \( E \) by

\[
  \xi(x) = \begin{cases} 
    g(x), & \text{if } x \in M, \\
    0, & \text{otherwise}.
  \end{cases}
\]

If we assume the concavity of shape function \( g \) of a fuzzy set \( \xi \) in Banach space instead of the concavity of functions \( L, R \) in the real case, we have the following result.

**Theorem 3.1.** Let \( T \) be an Archimedean t-norm with additive generator \( f \) and let \( \xi_i(i = 1, \ldots, n) \) be fuzzy sets such that \( \xi_i(x) = \xi(x - a_i) \), where \( a_i \in E, i = 1, \ldots, n \). If \( f \) is a convex function, then the membership function of \( T \)-sum \( \tilde{A}_n = \xi_1 + \cdots + \xi_n \) is given by

\[
  \tilde{A}_n(z) = \begin{cases} 
    f^{[-1]}\left( n f \left( g \left( \frac{z - A_n}{n} \right) \right) \right), & \text{if } z \in nM, \\
    0, & \text{otherwise},
  \end{cases}
\]

where \( A_n = a_1 + \cdots + a_n \) and \( nM = \{na \mid a \in M \} \).
Proof. Since $M$ is convex, $\overline{M + \cdots + M} = nM$. It is easy to see that the support of $\mathcal{A}_n$ is included in $nM$. Since $\xi_i(x) = \xi(x - a_i)$ for all $x$, we have by (*)

$$\mathcal{A}_n(z) = f^{-1} \left( \inf_{x_1 + \cdots + x_n = z} \left( \sum_{i=1}^{n} f(\xi_i(x_i)) \right) \right)$$

$$= f^{-1} \left( \inf_{y_1 + \cdots + y_n = z - A_n} \left( \sum_{i=1}^{n} f(\xi_i(y_i)) \right) \right)$$

$$\mathcal{A}_n(z) = \overline{B}_n(z - A_n),$$

where $\overline{B}_n$ denotes the $n$-fold sum $\xi + \cdots + \xi$. We note that if $x_1 + \cdots + x_n = z$ and some $x_i$ are outside of $M$, then $f^{-1}(\sum_{i=1}^{n} f(\xi(x_i))) = 0$. Hence by (*), we have that

$$\overline{B}_n(z) = f^{-1} \left( \inf_{x_1 + \cdots + x_n = z} \left( \sum_{i=1}^{n} f(\xi_i(x_i)) \right) \right)$$

$$= f^{-1} \left( \inf_{x_i \in M, i=1,\ldots,n} \left( \sum_{i=1}^{n} f(\xi_i(x_i)) \right) \right).$$

Suppose that $x_1 + \cdots + x_n = z$ and $x_i \in M, i = 1,\ldots,n$. Then we have

$$\frac{1}{n} \sum_{i=1}^{n} f(\xi_i(x_i)) = \frac{1}{n} \sum_{i=1}^{n} f(g(x_i))$$

$$\geq f \left( \frac{1}{n} \sum_{i=1}^{n} g(x_i) \right)$$

$$\geq f \left( g \left( \frac{1}{n} \sum_{i=1}^{n} x_i \right) \right) = f \left( g \left( \frac{z}{n} \right) \right),$$

where the first inequality comes from the convexity of $f$ and the second inequality comes from the decreasing property of $f$ and the concavity of $g$. By taking $x_i = z/n, i = 1,\ldots,n$, $\overline{B}_n(z) = f^{-1}(nf(g(z/n)))$, which implies that $\mathcal{A}_n(z) = f^{-1}(nf(g((z - A_n)/n))), z \in nM$.

Remark 3.2. Theorem 3.1 is the generalization of Hong and Hwang's theorem [1, Theorem 2] in the real case. In fact, if $E = R$, then the convex set $M$ is a closed interval $[-\alpha, \beta]$ containing 0. And $\xi_i$ is the fuzzy number $(a_i, \alpha, \beta)_{L-R}$ of $L-R$ type. Let the shape function $g$ be satisfied with $g(0) = 1$ and $g(\partial M) = 0$, where $\partial M$ is the boundary of $M$. Define functions $L, R : [0, 1] \to [0, 1]$ by the following:

$$g(z) = \begin{cases} 
R \left( \frac{z}{\beta} \right), & \text{if } 0 \leq z \leq \beta, \\
L \left( \frac{-z}{\alpha} \right), & \text{if } -\alpha \leq z \leq 0, \\
0, & \text{otherwise.}
\end{cases}$$

Then $\xi_i$ is the fuzzy number $(a_i, \alpha, \beta)_{L-R}$ of $L-R$ type. Since

$$g \left( \frac{z - A_n}{n} \right) = \begin{cases} 
R \left( \frac{z - A_n}{n\beta} \right), & \text{if } A_n \leq z \leq A_n + n\beta, \\
L \left( \frac{A_n - z}{n\alpha} \right), & \text{if } A_n - n\alpha \leq z \leq A_n, \\
0, & \text{otherwise,}
\end{cases}$$
by Theorem 3.1, we obtain the following form:

\[ \tilde{A}_n(z) = \begin{cases} 
  f^{-1}\left[nf\left(R\left(\frac{z-A_n}{n\beta}\right)\right)\right], & \text{if } A_n \leq z \leq A_n + n\beta, \\
  f^{-1}\left[nf\left(L\left(\frac{A_n-z}{n\alpha}\right)\right)\right], & \text{if } A_n - n\alpha \leq z \leq A_n, \\
  0, & \text{otherwise,}
\end{cases} \]

which is exactly the formula of Theorem 2 [1] in the real case.

Now we restrict considerations to Euclidean n-space \( \mathbb{R}^n \). Suppose that \( g: \mathbb{R}^n \to \mathbb{R} \) is a concave function with \( g(0) = 1 \). Let \( \alpha(s) \) be a differentiable curve parametrized by arc length, such that \( \alpha(0) = 0 \). For sufficiently large \( s_0 \), if \( 0 \leq s_2 \leq s_1 < s_0 \), then by the concavity of \( g \),

\[
g(\alpha(s_2)) = g\left(1 - \frac{\alpha(s_2)}{\alpha(s_1)}\right) \cdot 0 + \frac{\alpha(s_2)}{\alpha(s_1)} \cdot \frac{\alpha(s_1)}{\alpha(s_1)}
\]

Hence,

\[
\lim_{n \to \infty} \tilde{A}_n(z) = f^{-1}\left(\frac{1}{|z-A|}f'_{\alpha(0^+)g}\right),
\]

where \( \alpha(s) \) is a differentiable curve parametrized by arc length, passing through \((z - A_n)/n\), and \( \alpha(0) = 0 \).

**Proof.** Suppose first that \( z \neq A \). Then for sufficiently large \( n \), \( z \neq A_n \). Assume that the generator function \( f \) is convex. If \( 0 \leq y < x < 1 \), then

\[
f(x) = f\left(\frac{y-x}{y-1}\cdot 1 + \frac{x-1}{y-1}y\right) \leq \frac{y-x}{y-1}f(1) + \frac{x-1}{y-1}f(y) = \frac{x-1}{y-1}f(y),
\]

hence, \( f(x)/(x-1) \) is increasing and \( \lim_{x \to 1^-} f(x)/(x-1) = f'_{\alpha}(1) \) exists. If we consider the following relation:

\[
nf\left(g\left(\frac{z-A_n}{n}\right)\right) = \frac{f\left(\frac{(z-A_n)/n}{g\left((z-A_n)/n\right)-1}\right)}{f\left((z-A_n)/n-1\right)(z-A_n)/n} \leq f^{-1}\left(\frac{1}{|z-A|}f'_{\alpha(0^+)g}\right),
\]

it converges to \( f'_{\alpha}(1)D_{\alpha(0^+)g}|z-A| \) as \( n \to \infty \). If \( z = A \), we have the following inequality:

\[
\tilde{A}_n(A) = f^{-1}\left(\inf_{x_1+\ldots+x_n=A} \sum_{i=1}^{n} f\left(\xi_i(x_i)\right)\right)
\]

\[
\geq f^{-1}\left(f\left(\xi_1(a_1 + A - A_n)\right) + \sum_{i=2}^{n} f\left(\xi_i(a_i)\right)\right)
\]

\[
= f^{-1}\left(f\left(\xi_1(a_1 + A - A_n)\right)\right),
\]

where the last equality holds because \( f(\xi_1(a_1)) = f(g(0)) = f(1) = 0 \). Since the shape function \( g \) of \( \xi_i \) is continuous, \( \tilde{A}_n(A) \) tends to 1. This completes the proof.
This theorem is also the extension of Hong and Hwang's convergence theorem [1, Theorem 3] in the real case, by the following meaning.

**REMARK 3.4.** In particular, if the curve $\alpha$ in Theorem 3.3 is a straight line, by referring to Remark 3.2, we find that

$$D_{\alpha'(0^+)}g = \begin{cases} \frac{R_+'(0)}{\beta}, & \text{if } z > A, \\ \frac{L_+'(0)}{\alpha}, & \text{if } z < A. \end{cases}$$

Hence, by Theorem 3.3 we have

$$\lim_{n \to \infty} \tilde{A}_n(z) = \begin{cases} f[-1]\left(\frac{z-A}{\beta}f'_-(1)R_+'(0)\right), & \text{if } z \geq A, \\ f[-1]\left(\frac{A-z}{\alpha}f'_-(1)L_+'(0)\right), & \text{if } z \leq A, \end{cases}$$

which is the exact formula of Hong and Hwang's theorem [1, Theorem 3] in the real case.

**REMARK 3.5.** As shown in Theorem 3.3, we note that the membership function of the limit of $T$-sum $\tilde{A}_n$ is independent of the shape in the outside of sufficiently small neighborhood of the origin.

4. APPLICATIONS

Now we apply Theorem 3.1 and Theorem 3.3 for Yager's, Hamacher's and Dombi's parametrized t-norms. To simplify the representations, let $M = \{(x, y) \in R^2 \mid x^2 + y^2 \leq 1\}$ and $a_i = (0, 0) \in R^2$. And let $g$ be the shape function defined by $g(x, y) = 1 - \sqrt{x^2 + y^2}$. For each parametrized t-norm of above, we compute the membership function of t-norm-based sum $\tilde{A}_n$ and its limit.

(i) Yager's t-norm with $r \geq 1$:

$$T(x, y) = 1 - \min\left\{1, \sqrt{1 - x}^r + (1 - y)^r\right\},$$

with the additive generator $f_r(x) = (1 - x)^r$. Then,

$$\tilde{A}_n(z) = \begin{cases} 1 - n^{(1/r) - 1}|z|, & \text{if } |z| > n^{1-(1/r)}, \\ 0, & \text{otherwise}. \end{cases}$$

For $r = 1$,

$$\lim \tilde{A}_n(z) = \begin{cases} 1 - |z|, & \text{if } |z| \leq 1, \\ 0, & \text{otherwise}, \end{cases}$$

and for $r > 1$, $\lim \tilde{A}_n(z) = 1$.

(ii) Hamacher's t-norm with $0 \leq r \leq 2$:

$$H_r(x, y) = \frac{xy}{r + (1 - r)(x + y - xy)},$$

with the additive generator $f_r(x) = \log(r + (1 - r)x)/x$. Then,

$$\tilde{A}_n(z) = \begin{cases} r \left(\frac{r+(1-r)(1-|z|/n)}{1-|z|/n}\right)^n, & \text{if } |z| \leq n, \\ 0, & \text{otherwise}, \end{cases}$$

and the limit is $\lim \tilde{A}_n(z) = r/(e^{|z|} - 1 + r)$. 
(iii) Dombi's $t$-norm with $r > 1$:

$$T(x, y) = \frac{1}{1 + \sqrt[1/r]{((1/x) - 1)^r + ((1/y) - 1)^r}},$$

with the additive generator $f_r(x) = (1 - x)^r / x^r$. Then,

$$\tilde{A}_n(z) = \begin{cases} 
\frac{n - |z|}{n + (n^{1/r} - 1)|z|}, & \text{if } |z| \leq n, \\
0, & \text{otherwise},
\end{cases}$$

and the limit is $\lim \tilde{A}_n(z) = 1$.

REFERENCES