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On a Calderón–Zygmund commutator-type estimate

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ABSTRACT

In this paper we present a Calderón–Zygmund commutator-type estimate. This estimate enables us to prove an embedding result concerning weighted function spaces.

1. Introduction and statement of results

The theory of commutators has been extensively studied. They were introduced in a general form by A.P. Calderón [3] and [4], in which the author showed that these kinds of operators are bounded on L^2 and, under certain condition, they are of Calderón–Zygmund-type (see also his survey paper [5] and the paper of R. Coifman and Y. Meyer [7]). Successively, in the paper [8], R. Coifman, R. Rochberg and G. Weiss proved that, given a singular integral T with standard kernel (we refer to [9] and [20] for more details), the operator $[b,T]=bT-Tb$ is bounded in L^p, $1<p<\infty$ if b is a BMO function; the converse implication is due to S. Janson, who stated it in [15]. We refer also to the remarkable paper of C. Pérez [16], in which he underlined that $[b,T]$ is more singular than the operator T, proving that this commutator does not satisfy the corresponding $(1,1)$ estimate, but a weaker one, and to the paper of A. Uchiyama [23] in which is proved that the commutator of T and b is a compact operator provided that $b\in VMO$.

On the other hand there are an important class of commutator estimates called Kato–Ponce estimate, they are of the following general form

$$\big\|[P,f]g\big\|_{H^{s,p}}\leqslant C\|f\|_{\mathrm{Lip}(1)}\|g\|_{H^{s-1+\rho,p}}+\|f\|_{H^{s+\rho,p}}\|g\|_{L^\infty}, \tag{1}$$

for any Schwartz functions f,g whenever $1<p<\infty$, $\rho\geqslant 0$ and with $[P,f]g=P(fg)-fP(g)$. Here P is a pseudodifferential operator in the class $S^s_{1,0}$, $s>0$ (we refer to [22] and references therein).

In this paper we show a Calderón–Zygmund commutator-type estimate, similar to the inequality (1), introducing a homogeneous differential operator of fractional order s belonging to a bounded interval of $\mathbf{R}$. This is connected to the fact

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that the structure of the commutator operator is very rich, and that we can use the good properties of the kernels of such operators coupled with some cancelation involving terms in its integral representation. On the other hand, we can also exploit the relation between differential operators of negative fractional order and L^p $(1 < p < \infty)$ bounded singular integral operators with homogeneous kernels.

Our main result in this direction is:

Theorem 1.1. *Given a function $P \in L^1(\mathbf{R}^n)$ which satisfies, for $\varepsilon > 0$, $0 < s < 1$, and $j \in \mathbf{N} \setminus \{0\}$,*

$$\int\limits_{2^{j-1} \leqslant |x| \leqslant 2^j} |P(x)|\,dx \lesssim 2^{-j(\varepsilon+s)}, \qquad \int\limits_{|x| \leqslant 1} |P(x)|\,dx \lesssim 1, \tag{2}$$

if we set $P_k = 2^{nk} P(2^k x)$, for all $k \in \mathbf{Z}$, we have, for any $f, g \in \mathcal{S}(\mathbf{R}^n)$,

$$\left\| \int P_k(y)\big(f(x) - f(x-y)\big) g(x-y)\,dy \right\|_{L^r} \lesssim 2^{-ks} \big\| |D|^s f \big\|_{L^p} \|g\|_{L^q}, \tag{3}$$

provided that $1 \leqslant p, q, r \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$.

As an immediate consequence we get

Corollary 1.1. *Consider the operator $P_k^\delta = P_k D^\delta$. Then the estimate*

$$\big\| [P_k^\delta, f] g \big\|_{L^r} \lesssim 2^{(\delta - s)k} \big\| D^s f \big\|_{L^p} \|g\|_{L^q} \tag{4}$$

holds for all $k \in \mathbf{Z}$, all functions $f, g \in \mathcal{S}(\mathbf{R}^n)$ and all $1 \leqslant p, q, r \leqslant \infty$ such that $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $0 < s - \delta < 1$.

We apply these commutator estimates to problems of well-posedness and local and global existence for a class of evolution equations. In particular, we are interested in certain function spaces that come up in obtaining the relevant a priori estimates for these equations. These equations can be written in the general form,

$$\begin{cases} iu_t(t,x) + P(D)u(t,x) = F, & t \geqslant 0,\ x \in \mathbf{R}^n, \\ u(0,x) = u_0(x), \end{cases} \tag{5}$$

where the symbol $P(\xi)$ (it could be either a scalar or a real matrix) has some regularity properties. Some of the most important a priori estimates on solutions of problems of this type have the form

$$\big\| |D_x|^\alpha v \big\|_{B'_w L_t^2} \lesssim \|f\|_{\mathcal{H}} + \big\| |D_x|^\beta F \big\|_{B_w L_t^2}, \tag{6}$$

with $\alpha, \beta \in \mathbf{R}$ satisfying some relations, B_w is a suitable weighted Banach space (and B'_w is its dual space), and $\mathcal{H}$ is a Hilbert space. In fact, for the case of the Schrödinger equation (i.e., with $P(D) = -\Delta$ in (5)), C. Kenig, G. Ponce and L. Vega established the so-called smoothing estimates in [13] (see also the papers of A. Ruiz and L. Vega [19], B. Perthame and L. Vega [17], and V. Georgiev and M. Tarulli [11]). These were later extended to more general second order Schrödinger equations in [14]. The smoothing estimate reads:

$$\sup_m 2^{-m/2} \big\| |D_x|^{1/2} u \big\|_{L_t^2 L^2(|x| \sim 2^m)} \leqslant C \|u_0\|_{L^2} + \sum_m 2^m \big\| |D_x|^{-1/2} F \big\|_{L_t^2 L^2(|x| \sim 2^m)}, \tag{7}$$

and corresponds to (6) if one chooses $\alpha = 1/2$, $\beta = -1/2$ and $B_w = L^1(\mathbf{Z}, 2^{ks} dk, L^2(|x| \sim 2^m))$. Chihara proved in [6] that similar estimates can be obtained for the solutions of a larger class of partial differential equations of the form (6). More precisely, if we take $\alpha = d/2$, $\beta = -d/2$ and $B_w = L^2(\mathbf{R}^n, \langle x \rangle^{\delta/2} dx) := L_\delta^2$ the estimates are

$$\big\| |D_x|^{d/2} u \big\|_{L_t^2 L_{-\delta}^2} \lesssim \|u_0\|_{L^2} + \big\| |D_x|^{-d/2} F \big\|_{L_t^2 L_\delta^2}, \tag{8}$$

with $\langle x \rangle = \sqrt{1 + |x|^2}$, $d > 1$, $\delta > 0$, where the assumptions on $P(D)$ reflect the dispersive nature of Eq. (5); or in other words, its symbol $P(\xi)$ behaves like $|\xi|^m$, $m > 1$, with $d = m - 1$. We point out that this estimate is also valid if one uses some rotation invariant norms, just seen for the case of the Schrödinger equation, and which give scale-invariant estimates. There are analogous estimates, transposed to the framework of the wave equation, which correspond to the case $P(D) = \left(\begin{smallmatrix} 0 & -I \\ -\Delta & 0 \end{smallmatrix}\right)$, $u_0 = (f, g)$, in (5), as showed by N. Burq in [2] and M. Tarulli in [21]. These estimates have the form

$$\big\| |D_x|^\gamma \big(u, |D_x|^{-1} \partial_t u\big) \big\|_{L_t^2(\mathbf{R}, (L_{-\delta}^2 \times L_{-\delta}^2)} \lesssim \big\| (f, g) \big\|_{\dot{H}^\gamma \times \dot{H}^{\gamma-1}} + \big\| |D_x|^{\gamma-1} F \big\|_{L_t^2 L_\delta^2}, \tag{9}$$

which are valid as well if one uses norms that give scale-invariant estimates, instead of weighted Sobolev norms.

On the other hand, many authors (see, e.g., [18] or [12]) underline the necessity of using Besov (weighted) spaces instead of Sobolev (weighted) ones, because they allow one to better control some nonlinear terms arising in the evolution problem (5). (An example is the case of the magnetic potential perturbation $F = -A(t,x)\cdot\nabla u$, with some decay property assumed on the function $A(t,x)$ which is time and space depending; for more detail we refer again to [10].) The Besov norms give not only natural scale-invariant estimates, important for many problems in partial differential equations, but also enable us to exploit the Littlewood–Paley theory on which they are built. There is a vast literature on this subject. We have found many results on equivalences of Besov norms and Sobolev norms (we refer to [1]), but very few regarding the relation between weighted Besov norms and weighted Sobolev norms. A first attempt to do this was due to V. Georgiev, A. Stefanov and M. Tarulli [10]; we generalize those results here. Motivated by this, we shall show that weighted L^2-based Sobolev spaces, defined as the closure of Schwartz functions ϕ with respect to the norms

$$\sum_m 2^{m/2}\left\|D_x^{\gamma-d/2}\phi(x)\right\|_{L^2(|x|\sim 2^m)} \tag{10}$$

are embedded in suitable weighted Besov spaces $Y^{\gamma,d}$, defined as the closure of the functions

$$\phi(x)\in C_0^\infty\big(\mathbf{R}^n\setminus\{0\}\big)$$

with respect to the norm

$$\|\phi\|_{Y^{\gamma,d}} := \left(\sum_k 2^{2\gamma k}\|\phi\|^2_{Y_{d,k}}\right)^{1/2}. \tag{11}$$

In (11) the spaces $Y_{d,k}$ are defined by the norms[2]

$$\|\phi\|_{Y_{d,k}} = 2^{-dk/2}\sum_m 2^{m/2}\left\|\varphi\big(2^{-m}\cdot\big)\phi_k\right\|_{L^2_x}, \tag{12}$$

where $\phi_k = P_k\phi$ (see again Section 2). Similarly, we have the Banach space $\overline{Y}^{d,\gamma}$, with norm

$$\|\phi\|_{\overline{Y}^{\gamma,d}} := \left(\sum_k 2^{2\gamma k}\|\phi\|^2_{Y'_{d,k}}\right)^{1/2}, \tag{13}$$

where the spaces $Y'_{d,k}$, equipped with the norm

$$\|\phi\|_{Y'_{d,k}} = 2^{dk/2}\sup_m 2^{-m/2}\left\|\varphi\big(2^{-m}\cdot\big)\phi_k\right\|_{L^2_x}, \tag{14}$$

are duals to the spaces $Y_{d,k}$.

From the fact that $\|F\|_{L^2(|x|\sim 2^m)} \sim \|\varphi(2^{-m}\cdot)F\|_{L^2}$ we can replace at will, $\|F\|_{L^2(|x|\sim 2^m)}$, with the comparable expression, $\|\varphi(2^{-m}\cdot)F\|_{L^2}$. We will often do this in the sequel, in order to make use of localizations in both the space and frequency variables.

We shall show:

Theorem 1.2. *There is a constant $C = C(n)$, so that for every function $\phi\in\mathcal{S}(\mathbf{R}^n)$ and $\gamma, d \geqslant 0$, we have*

$$\|\phi\|_{Y^{\gamma,d}} \leqslant C\sum_m 2^{m/2}\left\|\varphi\big(2^{-m}\cdot\big)D_x^{\gamma-d/2}\phi(t,x)\right\|_{L^2_x}. \tag{15}$$

We conclude this section proving the following corollary.

Corollary 1.2. *There is a constant $C = C(n)$, so that for every function $\phi\in\mathcal{S}(\mathbf{R}^n)$ and $\gamma, d \geqslant 0$, we have*

$$\sup_m 2^{-m/2}\left\|\varphi\big(2^{-m}\cdot\big)D_x^{\gamma+d/2}\phi(t,x)\right\|_{L^2_x} \leqslant C_n\|\phi\|_{\overline{Y}^{\gamma,d}}. \tag{16}$$

Proof. This corollary is an easy consequence of the previous one. We can give its proof in one line. Consider the mapping $\phi\to D_x^{-\gamma}\phi$, and take the estimate dual to (15). That proves (16). □

Remark 1. We see that we can also assume that the functions ϕ are dependent on the time variable, that is $\phi = \phi(t,x)$. We just replace in (10), (12) and (14) the L^2_x norm by the mixed one $L^2_tL^2_x$, this does not cause confusion, because the

[2] As noted in [10], the expressions $\phi\to\|\phi\|_{Y_{d,k}}$ are not faithfull norms, in the sense that may be zero, even for some $\phi\neq 0$. On the other hand, they satisfy all the other norm requirements and $\phi\to(\sum_k 2^{\gamma k}\|\phi_k\|^2_{Y_{d,k}})^{1/2}$ is a norm!

whole proof only depends on what happens in the space variable (see, fore more details, the proofs of Lemma 4.1 and Theorem 1.2. The estimates (15) and (16) remain valid and defined in this way, the spaces $Y^{\gamma,d}$ (now based on $L_t^2 L_x^2$) are related to the others listed above and connected with the evolution problem (5). If we choose $\gamma = 0$, we are in the case of (8). Furthermore, the smoothing spaces in (7) are a particular case of the previous one; that is, $\gamma = 0$ and $d = 1$ while the others in (8) are derived if we pick $\gamma = 0$ and $d = \tilde{d}$. Finally, for the wave equation, we may consider $\gamma = \tilde{\gamma} - 1/2$ and $d = 1$. The authors say that it is possible to use these results in many other situations.

Remark 2. Theorem 1.2 and Corollary 1.2 generalize completely the estimates proved in Lemma 1 in the paper [10], which correspond to (15) and (16) in the case $\gamma = 0$ and $d = 1$.

1.1. *Outline of paper*

Section 2 is devoted to presenting the ancillary theory used in the whole paper. In Section 3 we prove Theorem 1.1, using two important introductory lemmas. In Section 4, we give the proof of the embedding Theorem 1.2. Finally, in Appendix A we present an alternative proof of Lemma 3.2.

2. Notations and preliminaires

In this section we introduce some important tools and useful notation

1. Given any two positive real numbers a, b, we write $a \lesssim b$ to indicate $a \leqslant Cb$, with $C > 0$. In the same way we say that $a \sim b$, if there exist positive constants c_1, c_2, so that $c_1 a \leqslant b \leqslant c_2 a$.

2. The Fourier transform and its inverse are defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i x\cdot\xi}\, dx,$$

$$f(x) = \int_{\mathbf{R}^n} \hat{f}(\xi) e^{2\pi i x\cdot\xi}\, d\xi.$$

3. Consider a positive, decreasing, function $\psi : \mathbf{R}_+ \to \mathbf{R}_+$, which is smooth away from zero, supported in $\{\xi\colon\ 0 \leqslant \xi \leqslant 2\}$ and equals 1 for all $0 \leqslant \xi \leqslant 1$. Set $\varphi(\xi) \equiv \psi(\xi) - \psi(2\xi)$. The function φ is smooth and satisfies:

$$\operatorname{supp} \varphi(\xi) \subseteq \{1/2 \leqslant |\xi| \leqslant 2\}, \qquad \sum_{k\in\mathbf{Z}} \varphi\big(2^{-k}\xi\big) = 1, \quad \forall \xi \neq 0. \tag{17}$$

We use $\varphi(\xi)$ to create a *frequency space localization*. Similarly, we can introduce the notion of *physical space localization* by means of the function $\varphi(x)$ which satisfies (17). In higher dimensions, we slightly abuse notation and denote a function with similar properties by the same symbol; namely, $\varphi(\xi) = \varphi(|\xi|)$, $\psi(x) = \psi(|x|)$, etc. Note that for $n > 1$, $\psi : \mathbf{R}^n \to \mathbf{R}$ is a smooth function even at zero. The kth Littlewood–Paley projection operators P_k, $P_{\leqslant k}$ are defined by

$$\widehat{P_k f}(\xi) = \varphi\big(2^{-k}\xi\big)\hat{f}(\xi),$$

and

$$\widehat{P_{\leqslant k} f}(\xi) = \psi\big(2^{-k}\xi\big)\hat{f}(\xi),$$

we also write $P_{<k}$ for $P_{\leqslant k-1}$. Let us observe that the kernel of P_k is integrable, smooth and real-valued for every k and that every P_k is bounded on every L^p space, $1 \leqslant p \leqslant \infty$, and that it commutes with constant coefficient differential operators. We may also consider $P_{>k} := \sum_{l>k} P_l$, which essentially restricts the Fourier transform to frequencies $\gtrsim 2^k$. In the paper we shall use, without any discussion, the notation f_k, $f_{\leqslant k}$, $f_{<k}$, to indicate respectively $P_k f$, $P_{\leqslant k} f$, $P_{<k} f$, and, for any fixed $N \geqslant 1$, we usually write $f_{k-N\leqslant\cdot\leqslant K+N}$ for $P_{\leqslant k+N} f - P_{<k-N} f$. An obvious modification give also the operators $P_{<k+N} f - P_{\leqslant k-N} f$. and consequently we have $f_{k-N<\cdot<K+N}$. We notice also that one can obtain the representations

$$P_k f(x) = 2^{nk} \int \hat{\varphi}\big(2^k y\big) f(x-y)\, dy \tag{18}$$

and

$$P_{\leqslant k} f(x) = 2^{nk} \int \hat{\psi}\big(2^k y\big) f(x-y)\, dy, \tag{19}$$

for any $f \in \mathcal{S}(\mathbf{R}^n)$. Another helpful observation is that, for the differential operator D_x^s defined via the multiplier $|\xi|^s$, we have the identity

$$D_x^s P_k u = 2^{ks} \tilde{P}_k u, \tag{20}$$

where $\tilde{\varphi}(\xi) = \varphi(\xi)|\xi|^s$. Notice that the operator $\tilde{P}_k$ has kernel satisfying the same estimate of P_k. In view of this we say that the two operators are equivalent and we make no difference in the use of them.

5. The Littlewood–Paley theory just presented enables us to introduce the Bernstein inequality for $\mathbf{R}^n$. Given a function $f \in \mathcal{S}(\mathbf{R}^n)$, for any number $k \in \mathbf{Z}$ the estimate

$$\|P_{\leqslant k} f\|_{L^q} \lesssim 2^{nk(\frac{1}{p}-\frac{1}{q})} \|P_{\leqslant k} f\|_{L^p} \tag{21}$$

is valid provided that $1 \leqslant p \leqslant q \leqslant \infty$. This inequality remains true if we replace the operator $P_{\leqslant k}$ by the other one P_k introduced above.

6. We say a function f belongs to $L^r(X)$ if the norm

$$\|f\|_{L^r_x(X)} = \left(\int_X |f(x)|^r \, dx \right)^{1/r}$$

is finite; we will usually omit the domain of integration if $X = \mathbf{R}^n$. In a similar way we say a function u belongs to $L^p_t L^q_x$ if the mixed space–time norm,

$$\|u\|_{L^q_t L^r_x} = \left(\int_{\mathbf{R}} \left(\int_{\mathbf{R}^n} |u(t,x)|^r \, dx \right)^{q/r} dt \right)^{1/q},$$

is finite.

7. The Hardy–Littlewood maximal function is defined by

$$M(f)(x) = \sup_{r>0} \frac{1}{\mu(B(x,r))} \int_{B(x,r)} |f(y)| \, dy, \tag{22}$$

where $B(x,r)$ is the ball of radius r centered at x and μ is the classical Lebesgue measure on $\mathbf{R}^n$. We remember also that the operator M is bounded on L^p for $1 < p \leqslant \infty$ (we refer to [9] or [20] for more details).

3. Proof of the Theorem 1.1

First of all we need the following lemma:

Lemma 3.1. *Given any function $f \in \mathcal{S}(\mathbf{R}^n)$ and real number $0 < s < 1$, there exists a positive constant $C_{s,n}$ so that, for all x and y in $\mathbf{R}^n$,*

$$|f(x) - f(y)| \leqslant C_{s,n} |x-y|^s \big(M(|D|^s f)(x) + M(|D|^s f)(y) \big), \tag{23}$$

where $M(\cdot)$ is the Hardy–Littlewood maximal operator.

Proof. If $f \in \mathcal{S}(\mathbf{R}^n)$, we can easily calculate its inverse Fourier transform, obtaining

$$f(x) = \int \hat{f}(\xi) e^{2\pi i \langle x,\xi \rangle} \, d\xi = \int |\xi|^{-s} |\xi|^s \hat{f}(\xi) e^{2\pi i \langle x,\xi \rangle} \, d\xi. \tag{24}$$

If $0 < \alpha < n$,

$$|\xi|^{-\alpha} = c_{\alpha,n} \widehat{|x|^{\alpha-n}}(\xi).$$

Therefore, we can rewrite the right-hand side of (24) as

$$\int |\xi|^{-s} \big(|\xi|^s \hat{f}(\xi) \big) e^{2\pi i \langle x,\xi \rangle} \, d\xi = c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} \, dt. \tag{25}$$

Combining (24) and (26), we have the representation

$$f(x) = c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} \, dy, \tag{26}$$

where the kernel in (26) is generally referred to as a Riesz potential, so, we can write:

$$\begin{aligned} f(x) - f(y) &= c_{s,n} \int \frac{|D|^s f(t)}{|x-t|^{n-s}} \, dt - c_{s,n} \int \frac{|D|^s f(t)}{|y-t|^{n-s}} \, dt \\ &= c_{s,n} \int |D|^s f(y) \left(\frac{1}{|x-t|^{n-s}} - \frac{1}{|t-y|^{n-s}} \right) dt \\ &= I + II, \end{aligned} \tag{27}$$

where

$$I = c_{s,n} \int\limits_{|x-t|\leqslant 3|x-y|} |D|^s f(t)\left(\frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}}\right) dt, \tag{28}$$

and

$$II = c_{s,n} \int\limits_{|x-t|\geqslant 3|x-y|} |D|^s f(t)\left(\frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}}\right) dt. \tag{29}$$

We treat I first. We notice that it is easy to obtain the following inclusion of sets:

$$\{y\colon\ |x-t| \leqslant 3|x-y|\} \subseteq \{y\colon\ |t-y| \leqslant 4|x-y|\}.$$

We can write:

$$|I| \leqslant c_{s,n}\left(\int\limits_{|x-t|\leqslant 3|x-y|} \frac{||D|^s f(t)|}{|x-t|^{n-s}} + \int\limits_{|t-y|\leqslant 4|x-y|} \frac{||D|^s f(t)|}{|y-t|^{n-s}}\, dt\right).$$

Given any $R \in \mathbf{R}_+$ and $0 < s < 1$, we introduce the following function:

$$\Phi_1^R(z) = \frac{1}{|z|^{n-s}} \chi_{\{|z|\leqslant 4R\}}; \tag{30}$$

where $\chi_{\{|z|\leqslant 4R\}}$ is the characteristic function of the set $\{|z| \leqslant 4R\}$. We need to use the pointwise estimate given in the following proposition (see [20, Chapter III, Section 2.2], or [9, Chapter II, Section 4]).

Proposition 3.1. *if ϕ is a positive, radial decreasing function (on $(0,\infty)$) and integrable, then we have*

$$\sup_{R>0}\left|\Phi^R * g(z)\right| \leqslant \left\|\Phi^R\right\|_{L^1} M(g)(z).$$

In our case, from this lemma we get $|\Phi_1^R * g(z)| \leqslant \|\Phi_1^R\|_{L^1} M(g)(z)$; i.e.,

$$\int\limits_{|z-t|\leqslant R} \frac{|g(t)|}{|z-t|^{n-s}}\, dt \leqslant \left\|\Phi_1^R\right\|_{L^1} M(g)(z), \tag{31}$$

for all $g \in \mathcal{S}(\mathbf{R}^n)$. Then, coupling the estimates (30), (31) with $g = |D|^s f$ and taking $|x-y| \sim R$, we obtain the inequalities

$$|I| \leqslant C_{s,n}\left(\left\|\Phi_1^R\right\|_{L^1} M\left(|D|^s f\right)(x) + \left\|\Phi_1^R\right\|_{L^1} M\left(|D|^s f\right)(y)\right) \leqslant C_{s,n} R^s\left(M\left(|D|^s f\right)(x) + M\left(|D|^s f\right)(y)\right), \tag{32}$$

where the third inequality in the above estimate (32) is achieved using the fact that $R^s = c_n \|\Phi_1^R\|_{L^1}$.

Now we look at II. We have from (29) the following estimate

$$|II| \leqslant c_{s,n} \int\limits_{|x-t|\geqslant 3|x-y|} \left||D|^s f(t)\right|\left|\frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}}\right| dt. \tag{33}$$

We notice now that if $|x-y| \geqslant 3|x-t|$, then the triangle inequality easily yields

$$2|x-y| \leqslant |x-t|, \qquad |y-t| \leqslant 2|x-t|, \qquad 2|x-t| \leqslant 3|y-t|,$$

which amount to saying that $|x-t| \sim |y-t|$. This enables us to obtain the following pointwise estimates:

$$\left|\frac{1}{|x-t|^{n-s}} - \frac{1}{|y-t|^{n-s}}\right| \lesssim \frac{|x-t|^{n-s} - |y-t|^{2n-2s}}{|x-t|^{n-s}} \lesssim |x-y| \frac{|x-t|^{n-s-1}}{|x-t|^{2n-2s}}. \tag{34}$$

Replacing the estimate (34) in the inequality (33), we arrive at

$$|II| \leqslant c_{s,n}|x-y| \int\limits_{|x-t|\geqslant 3|x-y|} \frac{|D|^s f(t)}{|x-t|^{n-s+1}}\, dt. \tag{35}$$

Let us introduce, for $R \in \mathbf{R}_+$ and $0 < s < 1$, the following function

$$\Phi_2^R(z) = \frac{1}{|z|^{n-s+1}} \chi_{\{|z|\geqslant 4R\}}, \tag{36}$$

where $\chi_{\{|z|\geqslant 4R\}}$ is the characteristic function of the set $\{|z|\leqslant 4R\}$, by the same argument used to prove the inequality (31), we see that the right-hand side of the (35) satisfies the following

$$\int\limits_{|z-t|\geqslant R} \frac{||D|^s f(t)|}{|z-t|^{n-s}}\,dt \leqslant \left\|\Phi_2^R\right\|_{L^1} M\big(|D|^s f\big)(z). \tag{37}$$

We thus have the following chain of inequalities

$$|II| \leqslant C_{s,n} R\big(\left\|\Phi_2^R\right\|_{L^1} M\big(|D|^s f\big)(x) + \left\|\Phi_2^R\right\|_{L^1} M\big(|D|^s f\big)(y)\big) \leqslant C_{s,n} R^s\big(M\big(|D|^s f\big)(x) + M\big(|D|^s f\big)(y)\big), \tag{38}$$

with $R^{s-1} = c_n\|\Phi_2^R\|_{L^1}$ and $|x-y|\sim R$. Finally, coupling the estimates (32) and (38), we obtain the bound (23) as desired. □

Remark 3. We notice that this lemma can be read as "local Hölder continuity." Up to this point, we have considered functions that are small at infinity. It is quite natural to ask what happens for more general ones. Given a function f such that $|D|^s f \in L^\infty$, we obtain from (23) the following:

$$\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^s} < \infty. \tag{39}$$

This means that the function is Hölder continuous of order $0<s<1$. Furthermore, the Hölder continuous norm is related to another one, involving the frequency decomposition (17). Set

$$\|f\|_{\mathrm{Lip}(s)} = \sup_k 2^{sk}\|P_k f\|_{L^\infty}.$$

We know that this norm and the norm (39) are equivalent, provided that $0<s<1$. Thus, from these considerations, we get that $\|f\|_{\mathrm{Lip}(s)}$ is bounded whenever $|D|^s f$ is in the space L^∞. On the other hand if we choose a again the assumption $|D|^s f\in L^\infty$, for any $p<\infty$ we get another kind of continuity, more precisely

$$\left\|\sup_{x\neq y} \frac{|f(x)-f(y)|}{|x-y|^s}\right\|_{L^\infty} < \infty. \tag{40}$$

This underlines the fact that the pointwise estimate (23) is more general and could be applied to other cases.

Lemma 3.1 allows us to prove:

Lemma 3.2. *Let be two functions $f, g \in \mathcal{S}(\mathbf{R}^n)$ and a function $h \in L^1(\mathbf{R}^n)$ with $\operatorname{supp} h \subseteq B(0,R)$. Define*

$$\mathcal{H}(x) = \int h(y)\big(f(x)-f(x-y)\big)g(x-y)\,dy, \tag{41}$$

then there exists a positive constant $C_{s,n}$, so that the following estimate

$$\|\mathcal{H}\|_{L^r} = C_{s,n} R^s \|h\|_{L^1}\left\||D|^s f\right\|_{L^p}\|g\|_{L^q} \tag{42}$$

is satisfied, provided that $1\leqslant p,q,r\leqslant\infty$, $\frac{1}{p}+\frac{1}{q}=\frac{1}{r}$ and $0<s<1$.

Proof. We split the proof in two cases, $1<p\leqslant\infty$, and $p=1$.

Case $1<p\leqslant\infty$. From the definition of the function h, we can rewrite the function $\mathcal{H}(x)$ as

$$\mathcal{H}(x) = \int\limits_{|y|\leqslant R} h(y)\big(f(x)-f(x-y)\big)g(x-y)\,dy, \tag{43}$$

and thus, using the Hölder inequality, one has the bound

$$\|\mathcal{H}\|_{L^r} \leqslant C\|h\|_{L^1} \sup_{|y|\leqslant R}\left(\int\left|\big(f(x)-f(x-y)\big)g(x-y)\right|^r\right)^{1/r} \leqslant C\|h\|_{L^1}\sup_{|y|\leqslant R}\left(\int\left|f(x)-f(x-y)\right|^p\right)^{1/q}\|g\|_{L^q}. \tag{44}$$

Now, using the pointwise estimate (23) in (44), we obtain

$$\|\mathcal{H}\|_{L^r} \leqslant C\|h\|_{L^1} R^s \sup_{|y|\leqslant R}\left\|M\big(|D|^s f\big)(\cdot) + M\big(|D|^s f\big)(\cdot - y)\right\|_{L^p}\|g\|_{L^q}. \tag{45}$$

Finally, by the well-known estimate for the Hardy–Littlewood maximal function $\|M(f)\|_{L^p} \lesssim \|f\|_{L^p}$, with $1<p\leqslant\infty$, we obtain (42) as desired.

Case $p = 1$. Consider, for fixed $R \in \mathbf{R}_+$, the function $\Phi \in L^1(\mathbf{R}^n)$ given by

$$\Phi = \Phi_1^R + \Phi_2^R, \tag{46}$$

where Φ_1^R and Φ_2^R are as in (30) and (36). From (27), (32) and (38) we have

$$\left|f(x) - f(x-y)\right| \leqslant |I + II| \lesssim \left(\Phi * \left||D|^s f(x)\right| + \Phi * \left||D|^s f(x-y)\right|\right), \tag{47}$$

with I and II as in the (28) and (29). From the pointwise estimate (47) and then following the same line of the proof of the previous case we arrive at

$$\|\mathcal{H}\|_{L^r} \leqslant C\|h\|_{L^1} \sup_{|y|\leqslant R} \left\|\Phi * \left||D|^s f(\cdot)\right| + \Phi * \left||D|^s f(\cdot - y)\right|\right\|_{L^p} \|g\|_{L^q}. \tag{48}$$

Finally an application of the Young inequality gives the desired estimate (42). □

3.1. Proof of the Theorem 1.1

Let us introduce the smooth, positive functions

$$\chi_j(x) = \varphi\left(2^{-j}x\right), \quad j \geqslant 1, \qquad \chi_0(x) = \sum_{j\leqslant 0} \varphi\left(2^{-j}x\right),$$

with φ defined as in (17), and set

$$P = \sum_{j\geqslant 0} P\chi_j = \sum_{j\geqslant 0} Q_j, \tag{49}$$

where $Q_j = P\chi_j$. Consider now, for any $k \in \mathbf{Z}$, the rescaled functions $(Q_j)_{2^{-k}} = 2^{nk}Q_j(2^k x)$ so that $\operatorname{supp}(Q_j)_{2^{-k}} \subseteq \{|x| \leqslant 2^{-k}2^j\}$, and they also satisfy:

$$\int \left|(Q_j)_{2^{-k}}\right| dx \lesssim 2^{-j(\varepsilon+s)}. \tag{50}$$

To prove (2) start with the fact that

$$\left\| \int P_k(y)\left(f(x) - f(y)\right)g(x-y)\,dy \right\|_{L^r}$$

is less than or equal to

$$\sum_{j\geqslant 0} \left\| \int (Q_j)_{2^{-k}}\left(f(x) - f(y)\right)g(x-y)\,dy \right\|_{L^r}. \tag{51}$$

Because of Lemma 3.2, each term of the series (51) is less than or equal to a constant times

$$2^{(j-k)s}\left\|(Q_j)_{2^{-k}}\right\|_{L^1} \left\||D|^s f\right\|_{L^p} \|g\|_{L^q} \lesssim 2^{(j-k)s}2^{-j(\epsilon+s)} \left\||D|^s f\right\|_{L^p} \|g\|_{L^q} \sim 2^{-ks}2^{-j\epsilon} \left\||D|^s f\right\|_{L^p} \|g\|_{L^q},$$

and therefore (51) is no bigger than a constant times $2^{-ks}\||D|^s f\|_{L^p}\|g\|_{L^q}$. This complete the proof of the theorem.

4. Proof of the embedding Theorem 1.2

First we introduce the following lemma, in which we improve a result proved in [10]. Namely, we have:

Lemma 4.1. *If f is a Schwartz function satisfying the property* $\operatorname{supp}_\xi \hat{f}(\xi) \subseteq \{|\xi| \lesssim 2^k\}$, *then*

$$\sum_m 2^{am} \left\|\left[P_k, \varphi\left(2^{-m}\cdot\right)\right] f\right\|_{L^2} \leqslant C \left\||D|^{-a} f\right\|_{L^2} \tag{52}$$

holds for all integers k and all $a \in (-n/2, 1)$.

Proof. We shall distinguish two cases. For $m \leqslant -k$ we use the estimate (3)—with $r = 2$, $p = \infty$, $q = 2$, $0 < s < 1$—to obtain

$$\left\|\left[P_k, \varphi\left(2^{-m}\cdot\right)\right] f\right\|_{L^2} \lesssim 2^{-ks} \left\||D|^s\varphi\left(2^{-m}\cdot\right)\right\|_{L^2} \|f\|_{L^\infty}. \tag{53}$$

From Young's inequality we get

$$\left\||D|^s\varphi\left(2^{-m}\cdot\right)\right\|_{L^2} \leqslant C2^{m(n/2-s)};$$

and, from Bernstein's inequality, we derive

$$\|f\|_{L^\infty} \lesssim 2^{nk/2}\|f\|_{L^2}.$$

Indeed, from these two estimates we obtain

$$2^{am}\big\|\big[P_k, \varphi\big(2^{-m}\cdot\big)\big]f\big\|_{L^2} \lesssim 2^{m(a+n/2-s)}2^{k(n/2-s)}\|f\|_{L^2}. \tag{54}$$

Now, given $a \in (-n/2, 1)$, we can find an $s \in (0,1)$ so that $a + n/2 - s > 0$. With this s chosen, we get the estimates:

$$\sum_{m\leqslant -k} 2^{am}\big\|\big[P_k, \varphi\big(2^{-m}\cdot\big)\big]f\big\|_{L^2} \lesssim 2^{k(n/2-s)} \sum_{m\leqslant -k} 2^{m(a+n/2-s)}\|f\|_{L^2} \lesssim 2^{-ak}\|f\|_{L^2}. \tag{55}$$

For $m > -k$ we follow a similar argument. We replace (54) by the following commutator estimate

$$\big\|\big[P_k, \varphi\big(2^{-m}\cdot\big)\big]f\big\|_{L^2} \lesssim 2^{-ks}\big\||D|^s\varphi\big(2^{-m}\cdot\big)\big\|_{L^\infty}\|f\|_{L^2}, \tag{56}$$

obtained from (3)—with $r = 2$, $p = 2$, $q = \infty$. In this way, since Young's inequality gives

$$\big\||D|^s\varphi\big(2^{-m}\cdot\big)\big\|_{L^\infty} \lesssim 2^{-ms},$$

we arrive at

$$\sum_{m>-k} 2^{am}\big\|\big[P_k, \varphi\big(2^{-m}\cdot\big)\big]f\big\|_{L^2} \lesssim 2^{-ks} \sum_{m>-k} 2^{m(a-s)}\|f\|_{L^2} \lesssim 2^{-ak}\|f\|_{L^2} \tag{57}$$

where we this time we choose $s \in (0,1)$ so that $a - s < 0$. Finally, this estimate and (55) lead to (52). □

4.1. Proof of Theorem 1.2

Taking into account the definition of $Y^{\gamma,d}$ and the mapping $\phi \to D_x^{-\gamma+d/2}\phi = g$, we need to show that, for every Schwartz function g,

$$\sum_k\bigg(\sum_m 2^{m/2}\big\|\varphi\big(2^{-m}\cdot\big)\tilde P_k g\big\|_{L^2_x}\bigg)^2 \lesssim \bigg(\sum_m 2^{m/2}\big\|\varphi\big(2^{-m}\cdot\big)g\big\|_{L^2_x}\bigg)^2, \tag{58}$$

where $\tilde P_k$ is the operator defined as in (20). We distinguish two cases. For $m \leqslant -k$ we get

$$\sum_k\bigg(\sum_{m\leqslant -k} 2^{m/2}\big\|\varphi\big(2^{-m}\cdot\big)\tilde P_k g\big\|_{L^2_x}\bigg)^2 \leqslant \sum_k 2^{-k}\|\tilde P_k g\|^2_{L^2_x} \lesssim \|D_x^{-1/2}g\|_{L^2_x}. \tag{59}$$

In the remaining case, $m > -k$, the decomposition

$$\varphi\big(2^{-m}x\big)\tilde P_k g = \tilde P_k\big(\varphi\big(2^{-m}\cdot\big)g\big) - \big[\tilde P_k, \varphi\big(2^{-m}\cdot\big)\big]g \tag{60}$$

is valid. For the first term on the right-hand side of (60), we get the estimate

$$\bigg(\sum_k\bigg(\sum_{m>-k} 2^{m/2}\big\|\tilde P_k\big(\varphi\big(2^{-m}\cdot\big)g\big)\big\|_{L^2_x}\bigg)^2\bigg)^{1/2} \lesssim \sum_m 2^{m/2}\bigg(\sum_k\big\|\tilde P_k\big(\varphi\big(2^{-m}\cdot\big)g\big)\big\|^2_{L^2_x}\bigg)^{1/2} \sim \sum_m 2^{m/2}\big\|\varphi\big(2^{-m}\cdot\big)g\big\|_{L^2_x}. \tag{61}$$

It remains to estimate the commutator in (60). In order to do that we use the following relation

$$\big[\tilde P_k, \varphi\big(2^{-m}\cdot\big)\big]g = \sum_{l\leqslant k-N}\tilde P_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big) + \big[\tilde P_k, \varphi\big(2^{-m}\cdot\big)\big]g_{k-N<\cdot<k+N} + \sum_{l\geqslant k+N}\tilde P_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big), \tag{62}$$

with $N \geqslant 3$, and in general we refer to the first term on the right-hand side of the above identity as "high-low interactions," while the second and the third are the "low-high interactions" and "high-high interactions." The second term arising above is easy to handle. By the commutator estimate (52) (with $a = 1/2$), we obtain

$$\sum_{m>-k} 2^{m/2}\big\|\big[\tilde P_k, \varphi\big(2^{-m}\cdot\big)\big]g_{k-N<\cdot<k+N}\big\|_{L^2} \lesssim 2^{-k/2}\|g_{k-N<\cdot<k+N}\|_{L^2_x}. \tag{63}$$

Squaring this estimate and taking the sum over k we find

$$\sum_k\bigg(\sum_{m>-k} 2^{m/2}\big\|\varphi\big(2^{-m}\cdot\big)g_{k-N<\cdot<k+N}\big\|_{L^2}\bigg)^2 \lesssim \big\|D_x^{-1/2}g\big\|^2_{L^2_x}, \tag{64}$$

Now we need to estimate the first and the third terms on the right-hand side of (62). In order to do that by the Plancherel identity and the properties of the operator P_l, we obtain the inequalities

$$\left\|\tilde{P}_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big)\right\|_{L_x^2} \lesssim \left\|\tilde{P}_k\big(\varphi\big(2^{-m}\cdot\big)\bar{P}_l * P_l g\big)\right\|_{L_x^2} \lesssim \left\|\int K_{k,m,l}(\xi,\eta)\widehat{P_l g}(\eta)\,d\eta\right\|_{L_\xi^2},$$

with $\bar{P}_l = P_{l-1} + P_l + P_{l+1}$ and kernel $K_{k,m,l}(\xi,\eta)$, which satisfies

$$\left|K_{k,m,l}(\xi,\eta)\right| \leqslant C2^{nm}\frac{\varphi_k(\xi)\varphi_l(\eta)}{2^{M(\max(k,l)+m)}}$$

for some $M > 0$ large enough. From these two estimates we get (for more detail on the argument used here we refer to the paper [11, Lemmas 5.1 and 8.1]):

$$\left\|\tilde{P}_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big)\right\|_{L_x^2} \lesssim 2^{-M(\max(k,l)+m)}\|P_l g\|_{L_x^2} \tag{65}$$

for somewhat smaller $M > 0$. We start with the term with indices k, l in the set $\{l \geqslant k+N\}$. Let us note that $l \geqslant k+N \geqslant -m+N$; that is, $l+m \geqslant N$. By the triangle inequality and the estimate (65), we get, for the corresponding terms in (62), the following:

$$\sum_k\left(\sum_{m>-k}2^{m/2}\left\|\sum_{l\geqslant k+N}\tilde{P}_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big)\right\|_{L_x^2}\right)^2 \lesssim \sum_k\left(\sum_{m>-k}2^{m/2}\sum_{l\geqslant k+N}2^{-(l+m)M}\|P_l g\|_{L_x^2}\right)^2. \tag{66}$$

Now, taking into account that the chains of inequalities

$$\begin{aligned}\frac{m}{2} - (l+m)M &\leqslant \frac{m}{2} - l - m - (l+m)(M-1) \leqslant -\frac{m}{2} - l - (l+m)(M-1) \leqslant -\frac{m}{2} - \frac{l}{2} - \frac{l}{2} - (l+m)(M-1)\\ &\leqslant -\left(\frac{m}{2}+\frac{l}{2}\right) - \frac{l}{2} - (l+m)(M-1) \leqslant -\frac{l}{2} - (l-k)(M-1)\end{aligned}$$

are valid, we see that (66) can be controlled by the following

$$\sum_k\left(\sum_{l\geqslant k+N}2^{-l/2}2^{-(l-k)(M-1)}\|P_l g\|_{L_x^2}\right)^2 = \sum_{l_1,l_2}2^{-l_1/2}\|P_{l_1}g\|_{L_x^2}2^{-l_2/2}\|P_{l_2}g\|_{L_x^2}\sum_{k\leqslant\min(l_1,l_2)-N}2^{-(M-1)(l_1+l_2-2k)}. \tag{67}$$

We claim that if $k \leqslant \min(l_1,l_2) - N$, then $l_1 + l_2 - 2k \geqslant |l_1 - l_2|$. By symmetry, it suffices assume $l_2 \geqslant l_1$, say $l_2 = l_1 + r$ with $r \geqslant 0$ and $k = l_1 - N - d$ with $d \geqslant 0$. From this we get

$$l_1 + l_2 - 2k = 2l_1 + r - 2l_1 + 2N + 2d = r + 2N + 2d \geqslant r = |l_1 - l_2|,$$

and by this it is easy to bound (67) by

$$\sum_{l_1,l_2}2^{-l_1/2}\|P_{l_1}g\|_{L^2}2^{-l_2/2}\|P_{l_2}g\|_{L_x^2}2^{-(M-1)|l_1-l_2|} \lesssim \sum_{l_1,l_2}2^{-l_1}\|P_{l_1}g\|_{L_x^2}^2 2^{-(M-1)|l_1-l_2|} \lesssim \|D_x^{-1/2}g\|_{L_x^2}^2. \tag{68}$$

So, from the estimates (66)–(68) one gets the desired

$$\sum_k\left(\sum_{m>-k}2^{m/2}\left\|\sum_{l\geqslant k+N}\tilde{P}_k\big(\varphi\big(2^{-m}\cdot\big)P_l g\big)\right\|_{L_x^2}\right)^2 \lesssim \|D_x^{-1/2}g\|_{L_x^2}^2. \tag{69}$$

It remains to evaluate the first term on the right-hand side of the (62). We follow the same line of the proof of the previous term, but we consider $l \leqslant k - N$. It is enough here to notice again that $k + m > 0$ and use the triangle inequality and the estimate (65). First we notice that in this regime we have

$$\frac{m}{2} - (k+m)M \leqslant \frac{m}{2} - \frac{k}{2} - \frac{m}{2} - (k+m)\left(M - \frac{1}{2}\right) \leqslant -\frac{k}{2} - (k+m)\left(M - \frac{1}{2}\right) \leqslant -\frac{k}{2},$$

and, choosing now $l_2 = l_1 + r$, $r \geqslant 0$ as in the previous case but with $k = l_1 + r + N + d$, $d \geqslant 0$,

$$l_1 + l_2 - 2k = 2l_1 + r - 2l_1 - 2r - 2N - 2d \leqslant r = |l_1 - l_2|.$$

This gives

$$\begin{aligned}
\sum_k \Big(\sum_{m>-k} 2^{m/2} \Big\| \sum_{l\leqslant k-N} \tilde{P}_k\big(\varphi(2^{-m}\cdot)P_l g\big)\Big\|_{L^2_x}\Big)^2 &\lesssim \sum_k \Big(\sum_{m>-k} 2^{m/2} \sum_{l\leqslant k-N} 2^{-(k+m)M} \|P_l g\|_{L^2_x}\Big)^2 \\
&\lesssim \sum_k \Big(\sum_{l\leqslant k-N} 2^{-k/2} \|P_l g\|_{L^2_x}\Big)^2 \\
&= \sum_{l_1,l_2} 2^{-l_1/2} \|P_{l_1} g\|_{L^2_x} 2^{-l_2/2} \|P_{l_2} g\|_{L^2_t L^2_x} \sum_{k\geqslant \max(l_1,l_2)+N} 2^{\frac{1}{2}(l_1+l_2-2k)} \\
&\lesssim \sum_{l_1,l_2} 2^{-l_1/2} \|P_{l_1} g\|_{L^2} 2^{-l_2/2} \|P_{l_2} g\|_{L^2_x} 2^{-\frac{1}{2}|l_1-l_2|} \\
&\lesssim \sum_{l_1,l_2} 2^{-l_1} \|P_{l_1} g\|^2_{L^2_x} 2^{-\frac{1}{2}|l_1-l_2|} \lesssim \big\|D_x^{-1/2} g\big\|^2_{L^2_x}.
\end{aligned} \tag{70}$$

However, by Sobolev embedding and Hölder's inequality we obtain

$$\begin{aligned}
\big\|D_x^{-1/2} g\big\|_{L^2_x} \lesssim \|g\|_{L_x^{\frac{2n}{n+1}}} &= \Big(\sum_m \int_{2^{m-1}\leqslant|x|\leqslant 2^{m-1}} |g|^{\frac{2n}{n+1}}\, dx\Big)^{\frac{n+1}{2n}} \\
&\lesssim \Big(\sum_m \big(2^{mn}\big)^{\frac{1}{n+1}} \Big(\int_{2^{m-1}\leqslant|x|\leqslant 2^{m-1}} |g|^2\, dx\Big)^{\frac{2n}{2(n+1)}}\Big)^{\frac{n+1}{2n}} \\
&\lesssim \Big(\sum_m 2^{\frac{mn}{n+1}} \big\|\varphi(2^{-m}\cdot)g\big\|^{\frac{2n}{n+1}}_{L^2_x}\Big)^{\frac{n+1}{2n}} \lesssim \sum_m 2^{m/2} \big\|\varphi(2^{-m}\cdot)g\big\|_{L^2_x},
\end{aligned} \tag{71}$$

where the last estimate comes from the fact that $n \geqslant 2$ and the property $L^q L^p \subset L^p L^q$ if $q \leqslant p$.

Thus, the estimates (59), (64), (69), (66), (70) with (71) and (61) give the proof of Theorem 1.2.

Remark 4. Note that from the argument in the proof of the above theorem one can easily get the following estimate:

$$\big\|P(D) f_k\big\|_{Y^{\gamma,d}} \lesssim \|f_k\|_{Y^{\gamma,d}} = 2^{\gamma k}\|f\|_{Y_{d,k}}, \quad \forall k \in \mathbf{Z}, \tag{72}$$

valid for any pseudodifferential operator with symbol $P(\xi) \in C_0^\infty(\mathbf{R}^n)$.

Remark 5. Some generalizations of the weighted Sobolev norms used in the previous theorem can be seen in Theorems 1.6 and 1.7 in [11].

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Appendix A

In this section we shall show that the method used to prove the Lemma 3.2, case $p = 1$, can be applied in order to achieve the general result for $p \in (1, \infty]$. However the use of the Lemma 3.1 (we referred at this as "local Hölder continuity") is of interest in its own right, and such kind has not appeared in the literature. So, here we have the following

Lemma A.1 *(Version II of Lemma 3.2). Let be f, g two functions in $\in \mathcal{S}(\mathbf{R}^n)$ and h a function in $L^1(\mathbf{R}^n)$ with $\operatorname{supp} h \subseteq B(0, R)$. Define*

$$\mathcal{H}(x) = \int h(y)\big(f(x) - f(x-y)\big)g(x-y)\, dy, \tag{A.1}$$

then there exists a positive constant $C_{s,n}$, so that the following estimate

$$\|\mathcal{H}\|_{L^r} = C_{s,n} R^s \|h\|_{L^1} \big\||D|^s f\big\|_{L^p} \|g\|_{L^q} \tag{A.2}$$

is satisfied, provided that $1 \leqslant p, q, r \leqslant \infty$, $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$ and $0 < s < 1$.

Proof. We consider now the general case $1 \leqslant p \leqslant \infty$.

Following the line of the proof of the case $p = 1$, we use again the function $\Phi \in L^1(\mathbf{R}^n)$, with $R \in \mathbf{R}_+$ fixed, given by

$$\Phi = \Phi_1^R + \Phi_2^R,$$

where Φ_1^R and Φ_2^R are defined in (30) and (36). We repeat the proof of the Lemma 3.1, using (27), (32) (38) (notice that we did not use the Proposition 3.1 involving the Hardy–Littlewood maximal operator) and we arrive at the identity (47). We get easily, by this, the estimate

$$\|\mathcal{H}\|_{L^r} \leqslant C\|h\|_{L^1} \sup_{|y|\leqslant R} \left\| \Phi * \left| |D|^s f(\cdot) \right| + \Phi * \left| |D|^s f(\cdot - y) \right| \right\|_{L^p} \|g\|_{L^q}. \tag{A.3}$$

Now, by the Young inequality again, we bound the sum on the left-hand side of the above estimate (A.3) in the following way

$$\sup_{|y|\leqslant R} \left\| \Phi * \left| |D|^s f(\cdot) \right| + \Phi * \left| |D|^s f(\cdot - y) \right| \right\|_{L^p} \lesssim R^s \left\| |D|^s f \right\|_{L^p}. \tag{A.4}$$

This estimate and the (A.3) give the estimate (A.2). The alternative proof is now complete. □

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