ELSEVIER

Available online at www.sciencedirect.com
SCIENCE DIRECT

J. Math. Anal. Appl. 298 (2004) 538–558

Journal of MATHEMATICAL ANALYSIS AND APPLICATIONS

www.elsevier.com/locate/jmaa

Existence of traveling wave fronts of delayed lattice differential equations ☆

Jianhua Huang [a,1], Gang Lu [b], Xingfu Zou [c,*]

[a] College of Mathematics and Econometrics, Hunan University, Changsha, Hunan 410082, PR China
[b] Department of Mathematics, Central China Normal University, Wuhan, HuBei 430079, PR China
[c] Department of Applied Mathematics, University of Western Ontario, London, Ontario, Canada N6A 5B7

Received 7 August 2003

Available online 20 July 2004

Submitted by C.V. Pao

Abstract

Existence of traveling wave fronts for delayed lattice differential equations is established by Schauder fixed point theorem. The main result is applied to a delayed and discretely diffusive model for the population of Daphnia magna.

Keywords: Schauder fixed point theorem; Quasimonotone; Upper solution; Lower solution; Supersolution; Subsolution

1. Introduction

Traveling wave solutions for lattice differential equations (LDEs) without time delay have been extensively and intensively studied in the last decade; see [1–9,12,14–16,18–20]. For delayed lattice differential equations, Wu and Zou [18] recently developed an iterative scheme and used an upper–lower solution method to prove the existence of traveling wave fronts of lattice differential equation. Both quasimonotone and weakened quasimonotone

☆ This work was supported by NNSF of China (No. 19971032) and NSERC of Canada.
* Corresponding author. On leave from the Department of Mathematics and Statistics, Memorial University of Newfoundland.

E-mail addresses: jhhuang32@yahoo.com (J. Huang), xzou@math.mun.ca, xzou@uwo.ca (X. Zou).

[1] Current address: Department of Mathematics, National University of Defense Technology, Changsha, Hunan 410073, PR China.

doi:10.1016/j.jmaa.2004.05.027

nonlinearities were explored in [18]. This technique was used in Zou [23] to a system of delayed differential equations on higher dimensional lattices. By the same approach, Hsu et al. [13] generalized the results of [18] for quasimonotone case to more general equations which include scalar functional differential equations of retarded, advanced and mixed type, but which possess the quasimonotonicity with respect to the delayed terms in the nonlinearity. The technique was also employed successfully by Weng et al. [17] to a system of delayed lattice differential equations with *global* interactions, which is derived from the population's age structure of the species. Hsu et al. [13] employed the shooting method to obtain the existence of traveling wave solution, which include scalar functional differential equations of retarded, advanced and mixed type, but which deals with the quasimonotonicity with respect to the delayed terms in the nonlinearity.

The approach developed in [18] has computational convenience, since the iteration only involves solving first order linear ordinary differential equations and generates a monotone sequence that converges to a profile function for the wave front. But the iteration scheme requires the existence of a pair of upper–lower solutions to the wave equation with the upper solution being monotonically nondecreasing and converging to the two distinct equilibria as $t \to -\infty$ and $t \to +\infty$, respectively. Such requirements on the upper–lower solutions have restricted the applicability of the above approach. Therefore, as far as existence of traveling wave fronts go, it is desirable to relax some restrictions on the upper–lower solutions.

In this paper, we will consider the existence of traveling wave fronts of the following system of delayed differential equations:

$$\frac{d}{dt}u_n(t) = f\big((u_n)_t\big) + \sum_{j=1}^{m} a_j\big[g\big(u_{n-j}(t)\big) + g\big(u_{n+j}(t)\big) - 2g\big(u_n(t)\big)\big],$$
$$n \in Z, \tag{1.1}$$

where Z is the integer lattice, $m \geqslant 1$ is an integer, a_j, $1 \leqslant j \leqslant m$, are positive real numbers, $g : R \to R$ and $f : X \to R$ are given mappings to be specified later, where $X = C([-\tau, 0]; R)$ is the Banach space of continuous functions defined on $[-\tau, 0]$ equipped with the super-norm, $\tau \geqslant 0$ is a given constant. Also, for any $\phi \in C(R, R)$, we use the notation ϕ_t to denote the element in X define by $\phi_t(s) = \phi(t + s)$ for $s \in [-\tau, 0]$. When $m = 1$, $a_1 = d$, $\tau = 0$ and $g(x) = x$, system (1.1) becomes

$$\frac{d}{dt}u_n(t) = f\big(u_n(t)\big) + d\big[u_{n-1}(t) + u_{n+1}(t) - 2u_n(t)\big], \quad n \in Z, \tag{1.2}$$

which represents the spatial discretization of the scalar reaction–diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial t^2} + f\big(u(x,t)\big).$$

System (1.2) is a model for population genetics where spatially discrete populations of diploid individuals are considered. System (1.2) is also used to model propagation of nerve pulses in myelinated axons, where traveling wave fronts are a crucial aspect (see Bell and Cosner [2]). For results on traveling wave fronts of (1.2), see Britton [3], Chi et al. [6], Hankerson and Zinner [12], Keener [14], Zinner [20,21], Zinner et al. [22].

As in [18] and [13], we will tackle the existence via the corresponding wave equation, and using the upper–lower solution technique. Discussed will be both quasimonotone and weakened quasimonotone nonlinearities in the sense of [18]. But, instead of establishing a monotone sequence convergent to a wave front profile function, we will employ the Schauder fixed point theorem to the operator used by Wu and Zou [18], in a properly chosen subset in a Banach space in $C(R, R)$ equipped with the exponential decay norm. The subset is obtained from a pair of upper–lower solutions, which are less restrictive than what are required in [18] and [13]. This makes searching for the upper–lower solutions easier than in [18] and [13]. For example, when the reaction term satisfies the quasimonotonicity, we will prove that existence of a *supersolution* and a *subsolution* (which may even not be continuous) satisfying certain conditions will guarantee the existence of a required pair of upper–lower solutions stated above; when the reaction term only satisfies the weakened quasimonotonicity, our existence result could also be less demanding for the upper solution, as demonstrated in our example in Section 5 (see Remark 5.1), where a delayed and discretely diffusive population model for Daphnia magna is considered.

The rest of this paper is organized as follows. In Section 2, we do some preparation necessary for the later sections. Section 3 is devoted to establishing the existence of traveling wave front solutions in the case of quasimonotone nonlinearities. Section 4 is parallel to Section 3, but deals with the case of weakened quasimonotone nonlinearities. Finally, application of the main results to a delayed and discretely diffusive model for the population of Daphnia magna is given in Section 5.

2. Preliminaries

A traveling wave solution of (1.1) is a solution of the form $u_n(t) = \phi(t + nc)$, where c is a given positive constant and $\phi : R \to R$ is a differentiable function satisfying the following mixed functional differential equation:

$$\frac{d}{dt}\phi(t) = f(\phi_t) + \sum_{j=1}^{m} a_j\big[g\big(\phi(t + r_j)\big) + g\big(\phi(t - r_j)\big) - 2g\big(\phi(t)\big)\big], \tag{2.1}$$

where $r_j = jc,\ j = 1, \ldots, m$. If ϕ is monotone and satisfies the following asymptotic boundary condition:

$$\lim_{t\to-\infty} \phi(t) = \phi_- \quad \text{and} \quad \lim_{t\to+\infty} \phi(t) = \phi_+, \tag{2.2}$$

then the corresponding traveling wave solution is called a traveling wave front. Therefore, (1.1) has a traveling wave front if and only if (2.1) has a monotone solution on R satisfying the asymptotic boundary condition (2.2).

Without loss of generality, we assume $\phi_- = 0$ and $\phi_+ = K$, therefore condition (2.2) can be replaced by

$$\lim_{t\to-\infty} \phi(t) = 0 \quad \text{and} \quad \lim_{t\to+\infty} \phi(t) = K. \tag{2.3}$$

For convenience of statements, we make the following hypothesis:

(A1) $f(\tilde{0}) = f(\tilde{K}) = 0$ with $0 < K$.
(A2) There exist a positive constants $L > 0$ such that

$$\left|f(\phi) - f(\psi)\right| \leqslant L\|\phi - \psi\|$$

for $\phi, \psi \in C([-\tau, 0], R)$ with $0 \leqslant \phi(s), \psi(s) \leqslant K$, $s \in [-\tau, 0]$.
(A3) $g : [0, K] \to R$ is continuously differentiable, with $0 \leqslant g'(x) \leqslant g'(0)$, $g(0) = 0$.

Here and in the sequel, for any $u \in R$, $\tilde{u}$ will denote the constant function on $[-\tau, 0]$ taking the value u for all $s \in [-\tau, 0]$.

Let $\rho > 0$ and equip $C(R, R)$ with the norm $\|\cdot\|$ defined by $|\phi|_\rho = \sup_{t\in R} |\phi(t)|e^{-\rho|t|}$. Denote

$$B_\rho(R, R) = \left\{\phi \in C(R, R)\colon \sup_{t\in R}\left|\phi(t)\right|e^{-\rho|t|} < \infty\right\}.$$

Then, it is easily seen that $B_\rho(R, R)$ is a Banach space.

Denote

$$C_{[0,K]}(R, R) = \left\{\phi \in C(R, R)\colon\ 0 \leqslant \phi(s) \leqslant K,\ s \in R\right\}.$$

Let $\mu > 0$, which will be specified in Sections 3 and 4. Define $H : B_{[0,K]}(R, R) \to B_\rho(R, R)$ by

$$H(\phi)(t) = \mu\phi(t) + f(\phi_t) + \sum_{j=1}^{m} a_j\left[g\left(\phi(t + r_j)\right) + g\left(\phi(t - r_j)\right) - 2g\left(\phi(t)\right)\right].$$

Via H, we can define $F : C_{[0,K]}(R, R) \to C_{[0,K]}(R, R)$ by

$$F(\phi)(t) = e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\phi)(s)\, ds.$$

It is easy to show that under (A1)–(A3), $F : C_{[0,K]}(R, R) \to C_{[0,K]}(R, R)$ is well defined. For any $\phi \in C_{[0,K]}(R, R)$, $F(\phi)$ satisfies

$$\left(F(\phi)\right)' + \mu F(\phi) - H(\phi)(t) = 0. \tag{2.4}$$

Thus, if $F(\phi) = \phi$, i.e., ϕ is a fixed point of F, then (2.1) has a solution. If this solution satisfies the boundary condition (2.3) and is monotone, then we obtain the existence of traveling wave front solution of (1.1).

As mentioned in the Introduction, we will use a pair of upper–lower solutions of (2.1) to construct a subset of $C(R, R)$ in which the Schauder fixed point theorem can be applied to the related operator. To this end, we need to make it clear what upper and lower solutions mean.

Definition 2.1. A continuous function $\phi : R \to [0, K]$ is called an upper solution of (2.1) if it is differentiable almost everywhere, and satisfies

$$\frac{d}{dt}\phi(t) \geqslant f(\phi_t) + \sum_{j=1}^{m} a_j\left[g\left(\phi(t + r_j)\right) + g\left(\phi(t - r_j)\right) - 2g\left(\phi(t)\right)\right], \quad \text{a.e. on } R.$$

Lower solution $\phi(t)$ of (2.1) can be similarly defined by reversing the inequality in the above inequality.

Definition 2.2. A function $\phi: R \to R$ is called a supersolution of (2.1) if there exist finitely many constants T_i, $i = 1, \dots, p$, such that $\phi(t)$ is differentiable in $R/\{T_i,\ i = 1, \dots, p\}$ and satisfies

$$\frac{d}{dt}\phi(t) \geqslant f(\phi_t) + \sum_{j=1}^{m} a_j\big[g\big(\phi(t+r)\big) + g\big(\phi(t-r_j)\big) - 2g\big(\phi(t)\big)\big]$$
$$\text{for } t \in R/\{T_i,\ i = 1, \dots, p\}.$$

A subsolution of (2.1) is defined by reversing the inequality in the above inequality.

In what follows, we assume that an upper solution $\bar{\phi}(t)$ and a lower solution $\underline{\phi}(t)$ of (2.1) are given so that

(H1) $0 \leqslant \underline{\phi} \leqslant \bar{\phi} \leqslant K$, with $\lim_{t\to-\infty} \underline{\phi} = 0$, $\lim_{t\to\infty} \bar{\phi} = K$.
(H2) $f(\tilde{u}) \neq 0$ for $u \in (0, \inf_{t\in R} \bar{\phi}(t)] \cup [\sup_{t\in R} \underline{\phi}(t), K)$.

3. Quasimonotone nonlinearities

In this section, we will consider (2.1) with the following quasimonotonicity:

(QM) There exists a constant $\mu > 0$ such that for any $\phi, \psi \in X$ with $0 \leqslant \phi(s) \leqslant \psi(s) \leqslant K$ for $s \in [-\tau, 0]$, one has

$$f(\psi) - f(\phi) + \mu\big[\psi(0) - \phi(0)\big] \geqslant 2A\big[g\big(\psi(0)\big) - g\big(\phi(0)\big)\big],$$

where $A = \sum_{j=1}^{m} a_j$.

Without loss of generality, for (QM) we will always choose $\mu > 1$ and $\mu > \rho$ in the rest of the paper. Assuming (QM), the operator H defined in Section 2 enjoys the following nice properties.

Lemma 3.1 (Wu and Zou [18]). *Assume that* (A1)–(A3) *and* (QM) *are satisfied. Then*

(i) $0 \leqslant H(\phi)(t) \leqslant f(\tilde{K}) + \mu K$ *for* $\phi \in C_{[0,K]}(R, R)$.
(ii) $H(\phi)(t)$ *is nondecreasing in* $t \in R$, *if* $\phi \in C_{[0,K]}(R, R)$ *is nondecreasing in* $t \in R$.
(iii) $H(\phi)(t) \leqslant H(\psi)(t)$ *for* $t \in R$, *if* $\phi, \psi \in C_{[0,K]}(R, R)$ *are given so that* $\phi(t) \leqslant \psi(t)$ *for* $t \in R$.

As a direct consequence of Lemma 3.1, the operator F defined in Section 2 also shares with H the above nice properties. In other words, we have the following lemma.

Lemma 3.2. *Assume that* (A1)–(A3) *and* (QM) *hold. Then*

(i) $0 \leqslant F(\phi)(t) \leqslant f(\tilde{K}) + \mu K$ *for* $\phi \in C_{[0,K]}(R, R)$.
(ii) $F(\phi)(t)$ *is nondecreasing in* $t \in R$, *if* $\phi \in C_{[0,K]}(R, R)$ *is nondecreasing in* $t \in R$.
(iii) $F(\phi)(t) \leqslant F(\psi)(t)$ *for* $t \in R$, *if* $\phi, \psi \in C_{[0,K]}(R, R)$ *are given so that* $\phi(t) \leqslant \psi(t)$ *for* $t \in R$.

Next, we further explore the operator F in Lemmas 3.3–3.5.

Lemma 3.3. *Assume that* (A1)–(A3) *hold. Then* $F : C_{[0,K]}(R, R) \to B_\rho(R, R)$ *is continuous with respect to the norm* $|\cdot|_\rho$ *in* $B_\rho(R, R)$.

Proof. We first prove that $H : C_{[0,K]}(R, R) \to B_\rho(R, R)$ is continuous. For any $\varepsilon > 0$, choose $\delta > 0$ such that $\delta < \varepsilon/N$, where $N = Le^{\rho c\tau} + \mu + 2Ag'(0) + 2g'(0)\sum_{j=1}^{m}(a_j e^{\rho r_j})$. If $\phi, \psi \in C_{[0,K]}(R, R)$ satisfy

$$|\phi - \psi|_\rho = \sup_{t\in R}\bigl|\phi(t) - \psi(t)\bigr|e^{-\rho|t|} < \delta,$$

then

$$\begin{aligned}
&\bigl|H\bigl(\phi(t)\bigr) - H\bigl(\psi(t)\bigr)\bigr| \\
&\quad= \Biggl|f(\phi_t) - f(\psi_t) + \mu\bigl(\phi(t) - \psi(t)\bigr) + \sum_{j=1}^{m} a_j\bigl\{\bigl[g\bigl(\phi(t + r_j)\bigr) - g\bigl(\psi(t + r_j)\bigr)\bigr] \\
&\qquad+ \bigl[g\bigl(\phi(t - r_j)\bigr) - g\bigl(\psi(t - r_j)\bigr)\bigr] + 2\bigl[g\bigl(\psi(t)\bigr) - g\bigl(\phi(t)\bigr)\bigr]\bigr\}\Biggr| \\
&\quad\leqslant \bigl|f(\phi_t) - f(\psi_t)\bigr| + \mu\bigl|\phi(t) - \psi(t)\bigr| + \sum_{j=1}^{m} a_j\bigl|g\bigl(\phi(t + r_j)\bigr) - g\bigl(\psi(t + r_j)\bigr)\bigr| \\
&\qquad+ \sum_{j=1}^{m} a_j\bigl|g\bigl(\phi(t - r_j)\bigr) - g\bigl(\psi(t - r_j)\bigr)\bigr| + 2\sum_{j=1}^{m} a_j\bigl|g\bigl(\psi(t)\bigr) - g\bigl(\phi(t)\bigr)\bigr| \\
&\quad\leqslant \sum_{j=1}^{m} a_j g'(0)\bigl[\bigl|\phi(t + r_j) - \psi(t + r_j)\bigr|\bigl|\phi(t - r_j) - \psi(t - r_j)\bigr|\bigr] \\
&\qquad+ \bigl|f(\phi_t) - f(\psi_t)\bigr| + \bigl(\mu + 2Ag'(0)\bigr)\bigl|\phi(t) - \psi(t)\bigr|.
\end{aligned}$$

It is easily seen that

$$\begin{aligned}
&\bigl|\phi(t - r_j) - \psi(t - r_j)\bigr|e^{-\rho|t|} + \bigl|\phi(t + r_j) - \psi(t + r_j)\bigr|e^{-\rho|t|} \\
&\quad\leqslant 2e^{\rho r_j}\bigl|\phi(s) - \psi(s)\bigr|_\rho.
\end{aligned} \tag{3.1}$$

Thus, it follows from (3.1) that

$$
\begin{aligned}
&\bigl|H\bigl(\phi(t)\bigr) - H\bigl(\psi(t)\bigr)\bigr|e^{-\rho|t|} \\
&\quad \leqslant \bigl|f(\phi_t) - f(\psi_t)\bigr|e^{-\rho|t|} + \bigl(\mu + 2Ag'(0)\bigr)\bigl|\phi(t) - \psi(t)\bigr|e^{-\rho|t|} \\
&\qquad + \sum_{j=1}^{m} a_j g'(0)\bigl[\bigl|\phi(t+r_j) - \psi(t+r_j)\bigr| + \bigl|\phi(t-r_j) - \psi(t-r_j)\bigr|\bigr]e^{-\rho|t|} \\
&\quad \leqslant \bigl|f(\phi_t) - f(\psi_t)\bigr|e^{-\rho|t|} + \left(\mu + 2Ag'(0) + 2\sum_{j=1}^{m}(a_j e^{\rho r_j})g'(0)\right)\bigl|\phi(t) - \psi(t)\bigr|_\rho \\
&\quad \leqslant L\|\phi_t - \psi_t\|_{X_c} e^{-\rho|t|} + \left(\mu + 2Ag'(0) + 2\sum_{j=1}^{m}(a_j e^{\rho r_j})g'(0)\right)\bigl|\phi(t) - \psi(t)\bigr|_\rho \\
&\quad \leqslant L \sup_{s\in[-c\tau,0]}\bigl|\phi(t+s) - \psi(t+s)\bigr|e^{-\rho|t|} \\
&\qquad + \left(\mu + 2Ag'(0) + 2\sum_{j=1}^{m}(a_j e^{\rho r_j})g'(0)\right)\bigl|\phi(t) - \psi(t)\bigr|_\rho \\
&\quad \leqslant L \sup_{\theta\in R}\bigl|\phi(\theta) - \psi(\theta)\bigr|e^{-\rho|\theta|}e^{\rho c\tau} \\
&\qquad + \left(\mu + 2Ag'(0) + 2\sum_{j=1}^{m}(a_j e^{\rho r_j})g'(0)\right)\bigl|\phi(t) - \psi(t)\bigr|_\rho \\
&\quad \leqslant \left(Le^{\rho c\tau} + \mu + 2Ag'(0) + 2\sum_{j=1}^{m}(a_j e^{\rho r_j})g'(0)\right)\bigl|\phi(t) - \psi(t)\bigr|_\rho \\
&\quad \leqslant N\delta < \varepsilon.
\end{aligned}
$$

Therefore, $H : C_{[0,K]}(R, R) \to B_\rho(R, R)$ is continuous.

In order to prove Lemma 3.3, we need to estimate $|F(\phi)(t) - F(\psi)(t)|$. By the definition of $F(\phi)(t)$, we have

$$
\begin{aligned}
\bigl|F\bigl(\phi(t)\bigr) - F\bigl(\psi(t)\bigr)\bigr| &= \left|e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}H\bigl(\phi(s)\bigr)\,ds - e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}H\bigl(\psi(s)\bigr)\,ds\right| \\
&\leqslant e^{-\mu t}\int_{\infty}^{t} e^{\mu s}\bigl|H\bigl(\phi(s)\bigr) - H\bigl(\psi(t)\bigr)\bigr|\,ds \\
&\leqslant \bigl|H\bigl(\phi(s)\bigr) - H\bigl(\psi(s)\bigr)\bigr|_\rho e^{-\mu t}\int_{-\infty}^{t} e^{\mu s + \rho|s|}\,ds.
\end{aligned}
$$

(a) If $t < 0$, we obtain

$$
\bigl|F\bigl(\phi(t)\bigr) - F\bigl(\psi(t)\bigr)\bigr| \leqslant \frac{1}{\mu - \rho}e^{-\rho t}\bigl|H\bigl(\phi(s)\bigr) - H\bigl(\psi(s)\bigr)\bigr|_\rho,
$$

and hence

$$\left|F(\phi(t)) - F(\psi(t))\right|e^{-\rho|t|} \leqslant \frac{1}{\mu-\rho}\left|H(\phi(t)) - H(\psi(t))\right|_\rho.$$

(b) If $t > 0$, it follows that

$$\begin{aligned}&\left|F(\phi(t)) - F(\psi(t))\right| \\ &\quad\leqslant e^{-\mu t}\left[\frac{1}{\mu-\rho} - \frac{1}{\mu+\rho} + \frac{1}{\mu+\rho}e^{(\mu+\rho)t}\right]\left|H(\phi(s)) - H(\psi(s))\right|_\rho.\end{aligned}$$

Therefore, we have

$$\begin{aligned}&\left|F(\phi(t)) - F(\psi(t))\right|e^{-\rho|t|} \\ &\quad\leqslant \left[\left(\frac{1}{\mu-\rho} - \frac{1}{\mu+\rho}\right)e^{-(\mu+\rho)t} + \frac{1}{\mu+\rho}\right]\left|H(\phi(t)) - H(\psi(t))\right|_\rho \\ &\quad\leqslant \frac{1}{\mu-\rho}\left|H(\phi(t)) - H(\psi(t))\right|_\rho.\end{aligned}$$

Thus, by using the fact that H is continuous in $B_\rho(R, R)$, it follows that F is also continuous with respect to the norm $|\cdot|_\rho$. The proof is completed. □

We further assume that the upper–lower solutions $\bar{\phi}(t)$ and $\underline{\phi}(t)$ satisfy

(H3) $\sup_{s\leqslant t}\underline{\phi}(s) \leqslant \bar{\phi}(t)$ for all $t \in R$.

Then the set

$$\Gamma[\underline{\phi}, \bar{\phi}] = \left\{\phi \in C_{[0,K]}(R, R);\ \begin{array}{l}\text{(i) } \phi \text{ is nondecreasing in } R, \\ \text{(ii) } \underline{\phi}(t) \leqslant \phi(t) \leqslant \bar{\phi}(t) \text{ for all } t \in R,\end{array}\right\}$$

is nonempty, since (H3) implies that $\phi_0 = \sup_{s\leqslant t}\underline{\phi}(s) \in \Gamma[\underline{\phi}, \bar{\phi}]$.

Remark 3.1. If $\phi \in C_{[0,K]}(R, R)$ is nondecreasing, then $\sup_{s\leqslant t}\phi(s) = \phi(t)$. Therefore, (H3) is implied by (H1) if either $\bar{\phi}(t)$ or $\underline{\phi}$ is nondecreasing (assuming $\underline{\phi} \leqslant \bar{\phi}(t)$ for all $t \in R$).

Lemma 3.4. *If* (QM) *holds, then* $F(\Gamma[\underline{\phi}, \bar{\phi}]) \subset \Gamma[\underline{\phi}, \bar{\phi}]$.

Proof. Since $\bar{\phi}(t)$ is a upper solution, we have

$$\bar{\phi}'(t) + \mu\bar{\phi}(t) - H\bar{\phi}(t) \geqslant 0. \tag{3.2}$$

By (2.4), we know that

$$F'(\bar{\phi}) + \mu F(\bar{\phi}) - H\bar{\phi}(t) = 0. \tag{3.3}$$

Combining (3.2) and (3.3) gives

$$\left(F(\bar{\phi}) - \bar{\phi}\right)' + \mu\left(F(\bar{\phi}) - \bar{\phi}\right) \leqslant 0. \tag{3.4}$$

Let $w(t) = F(\bar{\phi}) - \bar{\phi}$, and denote $r(t) = w'(t) + \mu w(t)$. Then, it follows from (3.4) that $r(t) \leqslant 0$. Since $w(t)$ is bounded on $(-\infty, \infty)$,

$$w(t) = \int_{-\infty}^{t} e^{-\mu(t-s)} r\, ds \leqslant 0,$$

which implies that $F(\bar{\phi}) \leqslant \bar{\phi}$. By a similar argument, we can prove that $F(\underline{\phi}) \geqslant \underline{\phi}$. Combining this with Lemma 3.2(ii), we see that $F(\Gamma[\underline{\phi}, \bar{\phi}]) \subset \Gamma[\underline{\phi}, \bar{\phi}]$. The proof is completed. □

Lemma 3.5. *If* (QM) *holds, then the operator* $F : \Gamma[\underline{\phi}, \bar{\phi}] \to \Gamma[\underline{\phi}, \bar{\phi}]$ *is compact.*

Proof. By Lemma 3.2(i), $F(\phi)$ is uniformly bounded for $\phi \in \Gamma[\underline{\phi}, \bar{\phi}]$. Since

$$\begin{aligned} 0 \leqslant F'(\phi)(t) &= -\mu e^{-\mu t} \int_{-\infty}^{t} e^{\mu s} H(\phi)(s)\, ds + H(\phi)(t) = -\mu F(\phi)(t) + H(\phi)(t) \\ &\leqslant -\mu F(0)(t) + H(K)(t) \leqslant H(K)(t) - H(0)(t) = \mu K, \end{aligned}$$

we know that $F(\phi)$ is equicontinuous for $\phi \in \Gamma[\underline{\phi}, \bar{\phi}]$. But since $\Gamma[\underline{\phi}, \bar{\phi}]$ consists of functions defined on $R = (-\infty, \infty)$ which is not compact, the Ascoli–Arzela lemma cannot be applied directly. For each integer $n > 0$, consider the "truncation" F_n of F define by

$$F_n(\phi)(t) = \begin{cases} F(\phi)(t), & t \in [-n, n], \\ F(\phi)(-n), & t \in (-\infty, -n), \\ F(\phi)(n), & t \in (n, +\infty). \end{cases}$$

Obviously, $F_n(\phi)$ is also uniformly bounded and equicontinuous for $\phi \in \Gamma[\underline{\phi}, \bar{\phi}]$. For each $\phi \in \Gamma[\underline{\phi}, \bar{\phi}]$, $F_n(\phi)$ has the compact "support" $[-n, n]$. By the Ascoli–Arzela lemma, we know $F_n : \Gamma[\underline{\phi}, \bar{\phi}] \to \Gamma[\underline{\phi}, \bar{\phi}]$ is compact. Using the estimate

$$\sup_{t \in R} \left|(F_n\phi)(t) - (F\phi)(t)\right| e^{-\rho|t|} = \sup_{|t|>n} \left|(F_n\phi)(t) - (F\phi)(t)\right| e^{-\rho|t|} \leqslant 2K e^{-\rho n},$$

we know $F_n \to F$ uniformly in $\Gamma[\underline{\phi}, \bar{\phi}]$. Therefore, by Proposition 2.12 in [19], the limit operator $F : \Gamma[\underline{\phi}, \bar{\phi}] \to \Gamma[\underline{\phi}, \bar{\phi}]$ is also compact. The proof is completed. □

Theorem 3.1. *Assume that* (A1)–(A3) *and* (QM) *hold. If* (2.1) *has an upper solution* $\bar{\phi}$ *and a lower solution* $\underline{\phi}$ *satisfying* (H1)–(H3)*, then* (2.1)–(2.3) *has a solution, i.e.,* (1.2) *has a traveling wave front solution.*

Proof. Obviously, $\Gamma[\underline{\phi}, \bar{\phi}]$ is a bounded subset of $B_\rho(R, R)$. It is easy to verify that $\Gamma[\underline{\phi}, \bar{\phi}]$ is closed and convex. By Lemma 3.3–3.5 and Schauder fixed point theorem, we know that F has a fixed point ϕ in $\Gamma[\underline{\phi}, \bar{\phi}]$. In order to prove that this fixed point ϕ gives a traveling wave front, we need to verify the asymptotic boundary condition (2.3). Obviously

$$\begin{aligned} &0 \leqslant \phi_1 := \lim_{t \to -\infty} \phi(t) = \inf_{t \in R} \phi(t) \leqslant \inf_{t \in R} \bar{\phi}(t), \\ &\sup_{t \in R} \underline{\phi}(t) \leqslant \sup_{t \in R} \phi_2 = \lim_{t \to +\infty} \phi(t) =: \phi(t) \leqslant K. \end{aligned}$$

Since $\bar{\phi}_1$ and $\bar{\phi}_2$ are zeros of $f(\cdot)$, by (H2), we have

$$\phi_1 = \lim_{t\to-\infty}\phi(t) = 0, \qquad \phi_2 = \lim_{t\to\infty}\phi(t) = K.$$

Finally, the fact that $\phi(t) \in \Gamma[\underline{\phi}, \bar{\phi}]$ implies that $\phi(t)$ is monotone, and therefore it presents a traveling wave front. The proof is completed. □

Remark 3.2. Unlike in Wu and Zou [18], we do not require that $\bar{\phi}(t)$ be monotone and satisfy $\lim_{t\to-\infty}\bar{\phi}(t) = 0$. This brings some convenience in searching for the upper–lower solutions.

Lemma 3.6. *Assume that* (A1)–(A3) *and* (QM) *hold. Assume* $(\underline{\phi}, \bar{\phi})$ *is a pair of supersolution and subsolution of* (2.1) *satisfying* (H1)–(H3) *and*

(H4) $\bar{\phi}(t^-) \leqslant \bar{\phi}(t^+)$ *and* $\underline{\phi}(t^+) \leqslant \underline{\phi}(t^-)$ *for all* $t \in R$.

Then, $(F(\underline{\phi}(t)), F(\bar{\phi}(t)) \in C_{[0,K]}(R, R)$ *is a pair of upper solution and lower solution of* (2.1) *satisfying* (H1)–(H3).

Proof. Let $t > -\infty$ be such that $\bar{\phi}$ is continuous at t, and let $T_p < T_{p-1} < \cdots < T_1$ be all the discontinuous points of $\bar{\phi}$ in $(-\infty, t)$. Denote $T_{p+1} = -\infty$. Then, by the definition of supersolution, $\bar{\phi}'(s) + \mu\bar{\phi}(s) - H(\bar{\phi}(s)) \geqslant 0$ for $s \in (T_{k+1}, T_k)$, $0 \leqslant k \leqslant p$. Hence

$$\begin{aligned}
F(\bar{\phi})(t) &\leqslant e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}\bar{\phi}'(s)\,ds + \mu e^{-\mu t}\int_{-\infty}^{t} e^{\mu s}\bar{\phi}(s)\,ds \\
&= e^{-\mu t}\sum_{j=1}^{p}\left(\int_{T_{j+1}}^{T_j} e^{\mu s}\bar{\phi}'(s)\,ds\right) + \mu e^{-\mu t}\sum_{j=1}^{p}\left(\int_{T_{j+1}}^{T_j} e^{\mu s}\bar{\phi}(s)\,ds\right) \\
&\quad + e^{-\mu t}\int_{T_1}^{t} e^{\mu s}\bar{\phi}'(s)\,ds + \mu e^{-\mu t}\int_{T_1}^{t} e^{\mu s}\bar{\phi}(s)\,ds \\
&= \bar{\phi}(t) - e^{-\mu t}e^{\mu T_1}\bar{\phi}\big(T_1^+\big) + e^{-\mu t}\sum_{j=1}^{p}\big[e^{\mu T_j}\bar{\phi}\big(T_j^-\big) - e^{\mu T_{j+1}}\bar{\phi}\big(T_{j+1}^+\big)\big] \\
&= \bar{\phi}(t) + e^{-\mu t}\sum_{j=1}^{p}\big[e^{\mu T_j}\bar{\phi}\big(T_j^-\big) - e^{\mu T_j}\bar{\phi}\big(T_j^+\big)\big] - e^{\mu(T_{p+1}-t)}\bar{\phi}\big(T_{p+1}^+\big) \\
&= \bar{\phi}(t) + e^{-\mu t}\left\{\sum_{j=1}^{p}e^{\mu T_j}\big[\bar{\phi}\big(T_j^-\big) - \bar{\phi}\big(T_j^+\big)\big] - e^{\mu T_{p+1}}\bar{\phi}\big(T_{p+1}^+\big)\right\} \\
&= \bar{\phi}(t) + e^{-\mu t}\left\{\sum_{j=1}^{p}e^{\mu T_j}\big[\bar{\phi}\big(T_j^-\big) - \bar{\phi}\big(T_j^+\big)\big]\right\} \leqslant \bar{\phi}(t).
\end{aligned}$$

By Lemma 3.1(iii),

$$F'(\bar{\phi})(t) + \mu F(\bar{\phi})(t) - H\big(F(\bar{\phi})\big)(t) \geqslant F'(\bar{\phi})(t) + \mu F(\bar{\phi})(t) - H(\bar{\phi})(t) = 0,$$

which implies that $F(\bar{\phi})(t)$ is a upper solution of (2.1). Similarly, we can also prove that $F(\underline{\phi})(t)$ is a lower solution of (2.1). The above also confirms that $0 \leqslant \underline{\phi} \leqslant F(\underline{\phi}) \leqslant F(\bar{\phi}) \leqslant K$. By L'Hospital's rule, one can easily verify that $\lim_{t\to-\infty} F(\underline{\phi})(t) = 0$ and $\lim_{t\to\infty} F(\bar{\phi})(t) = K$. Noting that $\inf_{t\in R} F(\bar{\phi})(t) \leqslant \inf_{t\in R} \bar{\phi}(t)$ and $\sup_{t\in R} F(\underline{\phi})(t) \geqslant \sup_{t\in R} \underline{\phi}(t)$, one sees that the pair $(F(\underline{\phi})(t), F(\bar{\phi})(t))$ satisfies (H2). For (H3), let $\phi_0(t) = \sup_{s\leqslant t} \underline{\phi}(t)$. Then $\phi_0(t)$ is nondecreasing and $\underline{\phi}(t) \leqslant \phi_0(t) \leqslant \bar{\phi}(t)$. By Lemma 3.2 and Remark 3.1, $\sup_{s\leqslant t} F(\underline{\phi})(s) \leqslant \sup_{s\leqslant t} F(\phi_0)(s) = F(\phi_0)(t) \leqslant F(\bar{\phi})(t)$, meaning that the pair $(F(\underline{\phi})(t), F(\bar{\phi})(t))$ also satisfies (H3). The proof is completed. □

Combining Lemma 3.6 with Theorem 3.1, we immediately obtain the following result.

Theorem 3.2. *Assume that* (A1)–(A3) *and* (QM) *hold. If* (2.1) *has a supersolution* $\bar{\phi}(t)$ *and subsolution* $\underline{\phi}(t)$ *satisfying* (H1)–(H4), *then* (2.1)–(2.3) *has a solution, i.e.,* (1.1) *has a traveling wave front solution.*

Remark 3.3. From the proof of Theorem 3.1, we know that supersolution and subsolution may have finite many discontinuous points, and thus, one can expect that searching for such a pair of supersolution and subsolution would be easier than searching for a pair of upper and lower solutions required in Theorem 3.1.

4. Nonquasimonotone nonlinearities

The quasimonotonicity condition (QM) plays an important role in Section 3. But in many models arising from practical problems, (QM) may not be satisfied. In this section, we will relax (QM) to a weaker condition (QM*) which is given below:

(QM*) there exists a constant $\mu > 0$ such that for any $\phi, \psi \in X$ with $0 \leqslant \phi(s) \leqslant \psi(s) \leqslant K$ and $[\phi(s) - \psi(s)]e^{\mu s}$ nondecreasing in $s \in [-\tau, 0]$, one has

$$f(\phi) - f(\psi) + \mu\big[\phi(0) - \psi(0)\big] \geqslant 2A\big[g\big(\phi(0)\big) - g\big(\psi(0)\big)\big],$$

where $A = \sum_{j=1}^{m} a_j$.

Parallel to Lemmas 3.1–3.3, we can establish the following Lemmas 4.1–4.3.

Lemma 4.1 (Wu and Zou [18]). *Assume that* (A1)–(A3) *and* (QM*) *hold. Then for any* $\phi(t)$ *satisfying*

(I) $\phi(t)$ *is nondecreasing in* R; *and* $0 \leqslant \phi(t) \leqslant K$,
(II) $e^{\mu t}[\phi(t+s) - \phi(s)]$ *is nondecreasing in* $t \in R$ *for every* $s > 0$,

the following hold:

(i) *$H(\phi)(t) \geqslant 0$,*
(ii) *$H(\phi)(t)$ is nondecreasing in $t \in R$,*
(iii) *$H(\psi)(t) \leqslant H(\phi)(t)$ for $t \in R$ if $\psi \in C(R, R^n)$ satisfies that $0 \leqslant \psi(t) \leqslant \phi(t) \leqslant K$ and that $e^{\mu t}[\phi(t) - \psi(t)]$ is nondecreasing in $t \in R$.*

Lemma 4.2. *Assume that* (A1)–(A3) *and* (QM*) *hold. Then for any $\phi(t)$ satisfying*

(I) *$\phi(t)$ is nondecreasing in R; and $0 \leqslant \phi(t) \leqslant K$,*
(II) *$e^{\mu t}[\phi(t+s) - \phi(s)]$ is nondecreasing in $t \in R$ for every $s > 0$,*

we have

(i) *$F(\phi)(t) \geqslant 0$,*
(ii) *$F(\phi)(t)$ is nondecreasing in $t \in R$,*
(iii) *$F(\psi)(t) \leqslant F(\phi)(t)$ for $t \in R$ if $\psi \in C(R, R^n)$ satisfies that $0 \leqslant \psi(t) \leqslant \phi(t) \leqslant K$ and that $e^{\mu t}[\phi(t) - \psi(t)]$ is nondecreasing in $t \in R$.*

Note that the continuity of the map $F: C_{[0,K]}(R, R) \to B_\rho(R, R)$ does not depend on (QM) and thus remains true. Similar to Section 3, we now construct a subset of $C_{[0,K]}(R, R)$. For this purpose, in the rest of this section, we assume that there are an upper solution $\bar{\phi}(t)$ and a lower solution $\underline{\phi}(t)$ satisfying (H1)–(H2) and the following additional assumption:

(H5) The set $\Gamma^*[\underline{\phi}(t), \bar{\phi}(t)]$ is nonempty, where

$$\Gamma^*\big[\underline{\phi}(t), \bar{\phi}(t)\big] = \left\{ \phi \in C(R, R);\ \begin{array}{rl} \text{(i)} & \phi(t) \text{ is nondecreasing in } R, \\ \text{(ii)} & \underline{\phi}(t) \leqslant \phi(t) \leqslant \bar{\phi}(t), \\ \text{(iii)} & e^{\mu t}[\bar{\phi}(t) - \phi(t)] \text{ and } e^{\mu t}[\phi(t) - \underline{\phi}(t)] \\ & \text{are nondecreasing in } t \in R. \\ \text{(iv)} & e^{\mu t}[\phi(t+s) - \phi(t)] \text{ is nondecreasing} \\ & \text{in } t \in R \text{ for every } s > 0 \end{array} \right\}.$$

Lemma 4.3. *Assume that* (A1)–(A3) *and* (QM*) *hold. Then $\Gamma^*[\underline{\phi}, \bar{\phi}]$ is closed, bounded, convex subset of $B_\rho(R, R)$.*

Proof. Boundedness and convexity can be easily shown by their definitions. We next prove that $\Gamma^*[\underline{\phi}, \bar{\phi}]$ is closed. Assume $\phi_n \in \Gamma^*[\underline{\phi}, \bar{\phi}]$ and $\phi_n \to \phi$ in norm $|\cdot|_\rho$, i.e.,

$$\lim_{n\to+\infty} \sup_{t\in R} \big|\phi_n(t) - \phi(t)\big| e^{-\rho|t|} = 0.$$

By $\phi_n \in \Gamma^*[\underline{\phi}, \bar{\phi}]$, we know that for any $t_1, t_2 \in R$ with $t_1 \leqslant t_2$,

$$e^{\mu t_2}\big[\phi_n(t_2+s) - \phi_n(t_2)\big] \geqslant e^{\mu t_1}\big[\phi_n(t_1+s) - \phi_n(t_1)\big],$$
$$e^{\mu t_2}\big[\bar{\phi}(t_2) - \phi_n(t_2)\big] \geqslant e^{\mu t_1}\big[\bar{\phi}(t_1) - \phi_n(t_1)\big],$$
$$e^{\mu t_2}\big[\phi_n(t_2) - \underline{\phi}(t_2)\big] \geqslant e^{\mu t_1}\big[\phi_n(t_1) - \underline{\phi}(t_1)\big].$$

Taking limit $n \to \infty$, we have

$$e^{\mu t_2}\big[\phi(t_2+s)-\phi_n(t_2)\big] \geqslant e^{\mu t_1}\big[\phi(t_1+s)-\phi_n(t_1)\big],$$
$$e^{\mu t_2}\big[\bar{\phi}(t_2)-\phi(t_2)\big] \geqslant e^{\mu t_1}\big[\bar{\phi}(t_1)-\phi(t_1)\big],$$
$$e^{\mu t_2}\big[\phi(t_2)-\underline{\phi}(t_2)\big] \geqslant e^{\mu t_1}\big[\phi(t_1)-\underline{\phi}(t_1)\big],$$

which implies that $\phi(t)$ satisfies (iii) and (iv) of $\Gamma^*[\underline{\phi},\bar{\phi}]$. It is easy to show that $\phi(t)$ satisfies (i) of $\Gamma^*[\underline{\phi},\bar{\phi}]$. Hence $\phi \in \Gamma^*[\underline{\phi},\bar{\phi}]$ and therefore, $\Gamma^*[\underline{\phi},\bar{\phi}]$ is closed. This completes the proof. □

Lemma 4.4. *If* (QM*) *holds, then* $F(\Gamma^*[\underline{\phi},\bar{\phi}]) \subset \Gamma^*[\underline{\phi},\bar{\phi}]$.

Proof. Let $\phi(t) \in \Gamma^*[\underline{\phi},\bar{\phi}]$. By Lemma 4.2(ii), $F(\phi)(t)$ is nondecreasing in $t \in R$. Lemma 4.2(iii) implies

$$F(\underline{\phi})(t) \leqslant F(\phi)(t) \leqslant F(\bar{\phi})(t).$$

Repeating the proof of Lemma 3.4 gives

$$F(\bar{\phi})(t) \leqslant \bar{\phi}(t), \qquad F(\underline{\phi})(t) \geqslant \underline{\phi}(t).$$

Hence

$$\underline{\phi}(t) \leqslant F(\phi)(t) \leqslant \bar{\phi}(t),$$

which implies that $F(\phi)(t)$ satisfying (ii) of $\Gamma^*[\underline{\phi},\bar{\phi}]$.

Next, we will verify the fourth condition of $\Gamma^*[\underline{\phi},\bar{\phi}]$ for $F(\phi)$. For any $s > 0$, we have

$$e^{\mu t}\big[F(\phi)(t+s)-F(\phi)(t)\big] = e^{-\mu s}\int_{-\infty}^{t+s} e^{\mu\theta}H(\phi)(\theta)\,d\theta - \int_{-\infty}^{t} e^{\mu\theta}H(\phi)(\theta)\,d\theta$$
$$= e^{-\mu s}\int_{-\infty}^{t} e^{\mu(\xi+s}H(\phi)(\xi+s)\,d\xi - \int_{-\infty}^{t} e^{\mu\theta}H(\phi)(\theta)\,d\theta$$
$$= \int_{-\infty}^{t} e^{\mu\theta}\big[H(\phi)(\theta+s)-(H(\phi)(\theta)\big]\,d\theta.$$

By Lemma 4.1, we obtain

$$\frac{d}{dt}\big[e^{\mu t}\big(F(\phi)(t+s)-F(\phi)(t)\big)\big] = e^{\mu t}\big[H(\phi)(t+s)-H\big(\phi(t)\big)\big] \geqslant 0,$$

which implies (iv) of $\Gamma[\underline{\phi},\bar{\phi}]^*$ holds for $F(\phi)$.

Repeating the proof of Proposition 4.1 in Wu and Zou [18], we know $e^{\mu t}[\bar{\phi}(t) - F(\phi)(t)]$ and $e^{\mu t}[F(\phi) - \underline{\phi}(t)]$ are nondecreasing in $t \in R$. Thus, $F(\phi)(t) \in \Gamma^*[\underline{\phi},\bar{\phi}]$, and therefore $F(\Gamma^*[\underline{\phi},\bar{\phi})) \subset \Gamma^*(\underline{\phi},\bar{\phi}]$. □

Lemma 4.5. *If* (QM*) *holds, then* $F:\Gamma[\underline{\phi},\bar{\phi}]^*\to\Gamma[\underline{\phi},\bar{\phi}]^*$ *is compact.*

The proof is similar to that of Lemma 3.5 and hence is omitted here.

Now, by Lemmas 3.3 and 4.3–4.5, the Schauder's fixed point theorem, and the same argument as in the proof of Theorem 3.1, we obtain the following result.

Theorem 4.1. *Assume that* (A1)–(A3) *and* (QM*) *hold. If* (2.1) *has an upper solution* $\bar{\phi}(t)$ *and a lower solution* $\underline{\phi}(t)$ *satisfying* (H1), (H2) *and* (H5), *then* (2.1)–(2.3) *has a solution, i.e.,* (1.1) *has a traveling wave front solution.*

Remark 4.1. Comparing with the results in nonquasimonotone case of Wu and Zou [18], we do not require that the upper solution $\bar{\phi}(t)$ belongs to the profile Γ^*, in which the upper solution is required to be satisfied $e^{\mu t}[\bar{\phi}(t+s)-\bar{\phi}(t)]$ is nondecreasing in $t\in R$ for all $s>0$, and $\lim_{t\to-\infty}\bar{\phi}(t)=0$. This brings some convenience in searching for the upper–lower solutions.

5. An example

Consider the following lattice differential equation with time delay:

$$\frac{d}{dt}u_n(t)=d\big[u_{n+1}(t)-2u_n(t)+u_{n-1}(t)\big]+u_n(t)\left(\frac{1-u_n(t-\tau)}{1+\gamma u_n(t-\tau)}\right),\quad n\in Z. \tag{5.1}$$

System (5.1) can be considered as the spatial discretization of the reaction diffusion equation

$$\frac{\partial u(x,t)}{\partial t}=d\frac{\partial^2 u(x,t)}{\partial x^2}+u(x,t)\left(\frac{1-(x,t-\tau)}{1+\gamma u(x,t-\tau)}\right), \tag{5.2}$$

which has been used to model the population of Daphnia magna (see, e.g., Feng and Lu [10], Gourley [11] and references cited therein).

If $\gamma=0$, system (5.1) becomes

$$\frac{d}{dt}u_n(t)=d\big[u_{n+1}(t)-2u_n(t)+u_{n-1}(t)\big]+u_n(t)\big[1-u_n(t-\tau)\big],\quad n\in Z, \tag{5.3}$$

which was also proposed as a model for propagation of nerve pulses in myelinated axons where the membrane is excitable only at spatially discrete sites (see, e.g., Bell [1], Bell and Cosner [2], Chi et al. [6], Keener [14], Zinner [20–22] Wu and Zou [18] and references cited therein).

Substituting $u_n(t)=\phi(t+cn)$ into (5.1) leads to

$$\phi'(t)=d\big[\phi(t+c)+\phi(t-c)-2\phi(t)\big]+\phi(t)\left(\frac{1-\phi(t-\tau)}{1+\gamma\phi(t-\tau)}\right). \tag{5.4}$$

We are interested in solutions of (5.4) satisfying

$$\lim_{t\to-\infty}\phi(t)=0,\qquad \lim_{t\to\infty}\phi(t)=1. \tag{5.5}$$

For $\phi \in C([-\tau, 0], R)$, denote

$$f\big(\phi(x)\big)(x) = \phi(x)(0)\left(\frac{1-\phi(x)(-\tau)}{1+\gamma\phi(x)(-\tau)}\right).$$

Obviously, (A1)–(A3) are satisfied for this functional f with $k = 1$. We verify that f satisfies (QM*).

Lemma 5.1. *When $\tau \geqslant 0$ is sufficiently small, $f(\phi)$ satisfies* (QM*).

Proof. For any $\phi, \psi \in C([-\tau, 0], R)$ with $0 \leqslant \psi(s) \leqslant \phi(s) \leqslant 1$ and $e^{\mu s}[\phi(s) - \psi(s)]$ nondecreasing in $s \in [-\tau, 0]$, we have

$$\begin{aligned}
f(\phi) - f(\psi) &= \phi(0)\left(\frac{1-\phi(-\tau)}{1+\gamma\phi(-\tau)}\right) - \psi(0)\left(\frac{1-\psi(-\tau)}{1+\gamma\psi(-\tau)}\right) \\
&= \frac{(\phi(0)-\psi(0))[1-\gamma\phi(-\tau)\psi(-\tau)-\phi(-\tau)]}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))} \\
&\quad + \frac{\gamma(\phi(0)\psi(-\tau)-\psi(0)\phi(-\tau)) - \psi(0)[\phi(-\tau)-\psi(-\tau)]}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))} \\
&\geqslant \frac{(\phi(0)-\psi(0))[1-\gamma\phi(-\tau)\psi(-\tau)-\phi(-\tau)]}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))} \\
&\quad + \frac{-\gamma e^{\mu\tau}(\phi(0)-\psi(0)) - \psi(0)e^{\mu\tau}(\phi(0)-\psi(0))}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))} \\
&= \frac{(\phi(0)-\psi(0))[1-\gamma\phi(-\tau)\psi(-\tau)-\phi(-\tau)-\psi(0)e^{\mu\tau}-\gamma\phi(0)e^{\mu\tau}]}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))]} \\
&\geqslant \frac{(\phi(0)-\psi(0))[-\gamma-(1+\gamma)e^{\mu\tau}]}{(1+\gamma\phi(-\tau))(1+\gamma\psi(-\tau))} \\
&\geqslant \big[-\gamma-(1+\gamma)e^{\mu\tau}\big]\big(\phi(0)-\psi(0)\big).
\end{aligned}$$

If we choose

$$\mu > 1 + 2\gamma, \tag{5.6}$$

then, for sufficiently small τ, (5.6) implies

$$\mu \geqslant \gamma + (1+\gamma)e^{\mu\tau}. \tag{5.7}$$

Thus

$$f(\phi) - f(\psi) + \mu\big(\phi(0)-\psi(0)\big) \geqslant \big[\mu - \gamma - (1+\gamma)e^{\mu\tau}\big]\big(\phi(0)-\psi(0)\big) \geqslant 0,$$

and this completes the proof. □

In order to apply Theorem 4.1, we need to find a pair of upper and lower solutions of (5.4) required by the theorem. Note that by letting

$$\phi(ct) \Rightarrow \phi(t) \quad \text{and} \quad c^{-1} \Rightarrow c, \tag{5.8}$$

Eq. (5.4) is transformed to the following equivalent equation:

$$c\phi'(t) = d\big[\phi(t+1) + \phi(t-1) - 2\phi(t)\big] + \phi(t)\left(\frac{1-\phi(t-c\tau)}{1+\gamma\phi(t-c\tau)}\right). \tag{5.9}$$

Indeed, (5.9) corresponds to the traveling wave solutions of (5.1) with the form $u_n(t) = \phi(n+ct)$. Thus, in what follows, we will work on (5.9) for a required pair of upper and lower solutions. To this end, the following result about the characteristic equation of (5.9) at 0 will be employed.

Lemma 5.2 [18, Proposition 4.3]. *Let $\Delta(\lambda) = c\lambda - 1 - d[e^{\lambda} + e^{-\lambda} - 2]$, $\lambda \in R$, where $d > 0$. Then there exists $c^* = c^*(d) > 0$ such that*

(i) *if $c < c^*$, $\Delta(\lambda)$ has no real zeros*;
(ii) *if $c = c^*$, $\Delta(\lambda)$ has precisely one double zeros*;
(iii) *if $c > c^*$, $\Delta(\lambda)$ have exactly two real zeros $0 < \lambda_1 < \lambda_2$, and $\Delta(\lambda) > 0$ for all $\lambda \in (\lambda_1, \lambda_2)$.*

From now on, we will assume $c > c^*$ and use λ_1 and λ_2 to construct the upper and lower solutions.

Let $\alpha \in (0,1)$ (to be specified later). For $M > 1$ and $\varepsilon > 0$, denote $t^* = \frac{1}{\varepsilon}\ln\frac{1}{M}$ and define

$$\bar{\phi}(t) = \begin{cases} e^{\lambda_1 t}, & t \leqslant 0, \\ 1, & t > 0, \end{cases} \qquad \underline{\phi}(t) = \begin{cases} \alpha(1-Me^{\varepsilon t})e^{\lambda_1 t}, & t \leqslant t^*, \\ 0, & t > t^*. \end{cases}$$

Let $\gamma > 0$ be given and ε be such that

$$0 < \varepsilon < \min\{\ln 2, \lambda_1, \lambda_2 - \lambda_1\}. \tag{5.10}$$

For sufficiently small $\tau \geqslant 0$, choose $M > 1$ such that

$$M > \frac{\alpha(\gamma+1)e^{-\lambda_1 c\tau}}{\Delta(\lambda_1+\varepsilon)}. \tag{5.11}$$

Noticing that $\lambda_1 > 0$ and $d > \frac{1}{1-e^{-\lambda_1}}$, direct calculation shows that $\frac{1}{\lambda_1}\ln\frac{d-1}{d} + 1 > 0$. It follows that

Lemma 5.3. *Assume $d > \frac{1}{1-e^{-\lambda_1}}$ and let $\tau \geqslant 0$ be sufficiently small. Then $\bar{\phi}(t)$ is an upper solution of* (5.9).

Proof. Assume τ is sufficiently small such that $c\tau \leqslant 1$, and $0 < \tau < \tau^* = \frac{1}{c}\big[\frac{1}{\lambda_1}\ln\frac{d-1}{d} + 1\big]$. Since $0 \leqslant \bar{\phi}(t-c\tau) \leqslant 1$, $\frac{1}{1+\gamma\bar{\phi}(t-c\tau)} < 1$ and thus,

$$\begin{aligned} &c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \frac{\bar{\phi}(t)[1-\bar{\phi}(t-c\tau)]}{1+\gamma\bar{\phi}(t-c\tau)} \\ &\quad\geqslant c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1-\bar{\phi}(t-c\tau)\big]. \end{aligned}$$

We have five cases to verify.

(i) For $t > 1$, $\bar{\phi}(t) = \bar{\phi}(t+1) = \bar{\phi}(t-1) = \bar{\phi}(t-c\tau) = 1$. Obviously,

$$c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1 - \bar{\phi}(t-c\tau)\big] = 0.$$

(ii) For $c\tau < t \leqslant 1$, $\bar{\phi}(t) = \bar{\phi}(t+1) = \bar{\phi}(t-c\tau) = 1$, $\bar{\phi}(t-1) = e^{\lambda_1(t-1)}$, and hence,

$$\begin{aligned}&c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1 - \bar{\phi}(t-c\tau)\big]\\&\quad = -d\big[\bar{\phi}(t-1) - 1\big] = d\big[1 - e^{\lambda_1(t-1)}\big] > 0.\end{aligned}$$

(iii) For $0 < t < c\tau$, $\bar{\phi}(t) = \bar{\phi}(t+1) = 1$, $\bar{\phi}(t-1) = e^{\lambda_1(t-1)}$, and $\bar{\phi}(t-c\tau) = e^{\lambda_1(t-c\tau)} < 1$. Since $d > \frac{1}{1-e^{-\lambda_1}} > 1$, $0 < \tau < \tau^*$ and $0 < t < c\tau$, we have $c\tau < c\tau^* = \frac{1}{\lambda_1}\ln\frac{d-1}{d} + 1$. Direct calculation shows that $1 - \frac{d}{d-1}e^{\lambda_1(t-1)} \geqslant 0$. Therefore,

$$\begin{aligned}&c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1 - \bar{\phi}(t-c\tau)\big]\\&\quad \geqslant c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\\&\quad = -d\big[-1 + e^{\lambda_1(t-1)}\big] - 1 = (d-1)\left[1 - \frac{d}{d-1}e^{\lambda_1(t-1)}\right] \geqslant 0.\end{aligned}$$

(iv) For $-1 < t \leqslant 0$, $\bar{\phi}(t) = e^{\lambda_1 t}$, $\bar{\phi}(t+1) = 1$, and $e^{\lambda_1(t+1)} > 1$. From Lemma 5.2, it follows that

$$\begin{aligned}&c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1 - \bar{\phi}(t-c\tau)\big]\\&\quad \geqslant c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\\&\quad = c\lambda_1 e^{\lambda_1 t} - d\big[1 + e^{\lambda_1(t-1)} - 2e^{\lambda_1 t}\big] - e^{\lambda_1 t}\\&\quad \geqslant c\lambda_1 e^{\lambda_1 t} - d\big[e^{\lambda_1(t+1)} + e^{\lambda_1(t-1)} - 2e^{\lambda_1 t}\big] - e^{\lambda_1 t}\\&\quad = e^{\lambda_1 t}\big[c\lambda_1 - d(e^{\lambda_1} + e^{-\lambda_1} - 2) - 1\big] = 0.\end{aligned}$$

(v) For $t \leqslant -1$, we have

$$\begin{aligned}&c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\big[1 - \bar{\phi}(t-c\tau)\big]\\&\quad \geqslant c\bar{\phi}'(t) - d\big[\bar{\phi}(t+1) + \bar{\phi}(t-1) - 2\bar{\phi}(t)\big] - \bar{\phi}(t)\\&\quad = c\lambda_1 e^{\lambda_1 t} - d\big[e^{\lambda_1(t+1)} + e^{\lambda_1(t-1)} - 2e^{\lambda_1 t}\big] - e^{\lambda_1 t}\\&\quad = e^{\lambda_1 t}\big[c\lambda_1 - d(e^{\lambda_1} + e^{-\lambda_1} - 2) - 1\big] = 0.\end{aligned}$$

Combining with the above (i)–(v), we know that $\bar{\phi}(t)$ is an upper solution of (5.9), and this completes the proof. □

Lemma 5.4. *Let $\varepsilon > 0$ and $M > 0$ be such that* (5.10)–(5.11) *hold, and $\tau \geqslant 0$ be sufficiently small. Then, $\underline{\phi}(t)$ is a lower solution of* (5.3)*.*

Proof. Assume $c\tau \leqslant 1$. We verify the conclusion in the following five cases.

(i) For $t > 1 + t^*$, $\underline{\phi}(t-1) = \underline{\phi}(t) = \underline{\phi}(t+1) = \underline{\phi}(t-c\tau) = 0$. Obviously,

$$c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \frac{\underline{\phi}(t)[1 - \underline{\phi}(t-c\tau)]}{1 + \gamma\underline{\phi}(t-c\tau)} = 0.$$

(ii) For $t^* + c\tau < t < 1 + t^*$, $\underline{\phi}(t+1) = \underline{\phi}(t - c\tau) = \underline{\phi}(t) = 0$. It follows that

$$c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \frac{\underline{\phi}(t)[1-\underline{\phi}(t-c\tau)]}{1+\gamma\underline{\phi}(t-c\tau)}$$
$$= -\alpha d[1 - Me^{\varepsilon(t-1)}]e^{\lambda_1(t-1)} < 0.$$

(iii) For $t^* < t \leqslant t^* + c\tau$, $\underline{\phi}(t) = \underline{\phi}(t+1) = 0$. Since $Me^{\varepsilon(t-1)} < Me^{\varepsilon t^*} = 1$, we have

$$c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \frac{\underline{\phi}(t)[1-\underline{\phi}(t-c\tau)]}{1+\gamma\underline{\phi}(t-c\tau)}$$
$$= -d\alpha[1 - Me^{\varepsilon(t-1)}]e^{\lambda_1(t-1)} < 0.$$

(iv) For $t^* - 1 < t \leqslant t^*$, $\underline{\phi}(t) = \alpha(1 - Me^{\varepsilon t})e^{\lambda_1 t}$, $\underline{\phi}(t+1) = 0$. It follows that

$$\begin{aligned}
&c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \frac{\underline{\phi}(t)[1-\underline{\phi}(t-c\tau)]}{1+\gamma\underline{\phi}(t-c\tau)}\\
&\quad = c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \underline{\phi}(t) + \frac{(1+\gamma)\underline{\phi}(t-c\tau)\underline{\phi}(t)}{1+\gamma\underline{\phi}(t-c\tau)}\\
&\quad \leqslant c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \underline{\phi}(t) + (1+\gamma)\underline{\phi}(t-c\tau)\underline{\phi}(t)\\
&\quad = \alpha e^{\lambda_1 t}\big\{\big[c\lambda_1 - 1 - d(e^{-\lambda_1} - 2)\big] - Me^{\varepsilon t}\big[c(\varepsilon+\lambda_1) - 1 - d(e^{-(\lambda_1+\varepsilon)} - 2)\\
&\qquad + \alpha(1+\gamma)(1 - Me^{\varepsilon t})(1 - Me^{\varepsilon(t-c\tau)})e^{\lambda_1(t-c\tau)}\big]\big\}\\
&\quad \leqslant \alpha e^{\lambda_1 t}\big\{de^{\lambda_1} + \alpha(1+\gamma)e^{\lambda_1(t-c\tau)}[1 - Me^{\varepsilon t}]\\
&\qquad - Me^{\varepsilon t}\big[c(\lambda_1+\varepsilon) - 1 - d(e^{-(\lambda_1+\varepsilon)} - 2)\big]\big\}.
\end{aligned}$$

By (5.10) and $t^* - 1 < t < t^* < 0$, we have

$$e^{-\varepsilon} = e^{-\varepsilon}Me^{\varepsilon t^*} = Me^{\varepsilon(t^*-1)} < Me^{\varepsilon t} < Me^{\varepsilon t^*} = 1, \quad e^{\lambda_1 t} \leqslant e^{\varepsilon t} \leqslant e^{\varepsilon t^*} = \frac{1}{M}.$$

By (5.10) and (5.11) and Lemma 5.1, it follows that

$$c(\lambda_1+\varepsilon) - 1 - d(e^{-(\lambda_1+\varepsilon)} - 2) > de^{\lambda_1+\varepsilon} > 0.$$

Hence, we obtain

$$\begin{aligned}
&c\underline{\phi}'(t) - d\big[\underline{\phi}(t+1) + \underline{\phi}(t-1) - 2\underline{\phi}(t)\big] - \frac{\underline{\phi}(t)[1-\underline{\phi}(t-c\tau)]}{1+\gamma\underline{\phi}(t-c\tau)}\\
&\quad \leqslant \alpha e^{\lambda_1 t}\big\{de^{\lambda_1} + \alpha(1+\gamma)e^{\lambda_1(t-c\tau)}[1 - e^{-\varepsilon}]\\
&\qquad - e^{-\varepsilon}\big[c(\lambda_1+\varepsilon) - 1 - d(e^{-(\lambda_1+\varepsilon)} - 2)\big]\big\}\\
&\quad \leqslant \alpha e^{\lambda_1 t-\varepsilon}\big\{\alpha(1+\gamma)e^{\varepsilon+\lambda_1(t-c\tau)}[1 - e^{-\varepsilon}]\\
&\qquad - \big[c(\lambda_1+\varepsilon) - 1 - d(e^{-(\lambda_1+\varepsilon)} + e^{\lambda_1+\varepsilon} - 2)\big]\big\}\\
&\quad \leqslant \alpha e^{\lambda_1 t-\varepsilon}\left\{(1+\gamma)\frac{\alpha}{M}e^{\varepsilon-\lambda_1 c\tau}[1 - e^{-\varepsilon}] - \Delta(\varepsilon+\lambda_1)\right\}
\end{aligned}$$

$$\leqslant \frac{\alpha}{M} e^{\lambda_1 t-\varepsilon}\bigl\{\alpha(1+\gamma)e^{-\lambda_1 c\tau}[e^{\varepsilon}-1]-M\Delta(\varepsilon+\lambda_1)\bigr\}$$
$$\leqslant \frac{\alpha}{M} e^{\lambda_1 t-\varepsilon}\bigl\{\alpha(1+\gamma)e^{-\lambda_1 c\tau}-M\Delta(\varepsilon+\lambda_1)\bigr\}\leqslant 0.$$

(v) If $t \leqslant t^*-1<0$, by (5.10) and (5.11), we have

$$\begin{aligned}
&c\underline{\phi}'(t)-d\bigl[\underline{\phi}(t+1)+\underline{\phi}(t-1)-2\underline{\phi}(t)\bigr]-\frac{\underline{\phi}(t)[1-\underline{\phi}(t-c\tau)]}{1+\gamma\underline{\phi}(t-c\tau)}\\
&\quad\leqslant c\underline{\phi}'(t)-d\bigl[\underline{\phi}(t+1)+\underline{\phi}(t-1)-2\underline{\phi}(t)\bigr]-\underline{\phi}(t)+(1+\gamma)\underline{\phi}(t-c\tau)\underline{\phi}(t)\\
&\quad=\alpha e^{\lambda_1 t}\bigl[c\lambda_1-1-d(e^{-\lambda_1}+e^{\lambda_1}-2)\bigr]\\
&\qquad-\alpha M e^{(\varepsilon+\lambda_1)t}\bigl[c(\varepsilon+\lambda_1)-1-d(e^{\lambda_1+\varepsilon}+e^{-(\lambda_1+\varepsilon)}-2)\bigr]\\
&\qquad+\alpha^2(1+\gamma)(1-Me^{\varepsilon t})e^{\lambda_1 t}(1-Me^{\varepsilon(t-c\tau)})e^{\lambda_1(t-c\tau)}\\
&\quad\leqslant -\alpha M e^{(\varepsilon+\lambda_1)t}\Delta(\lambda_1+\varepsilon)+(\gamma+1)\alpha e^{\lambda_1(t-c\tau)}\\
&\quad=\alpha e^{(\varepsilon+\lambda_1)t}\bigl\{-M\Delta(\lambda_1+\varepsilon)+\alpha(\gamma+1)e^{-\lambda_1 c\tau}\bigr\}\leqslant 0.
\end{aligned}$$

Combining the above, we see that $\underline{\phi}$ is a lower solution of (5.9), and the proof is completed. □

For aforementioned λ_1 and $\varepsilon>0$, choose $\mu \geqslant \lambda_1$ (in addition to (5.6)) and $\alpha \in \bigl(\frac{\mu}{2(\lambda_1+\mu)},1\bigr)\subset(0,1)$. Let $M\geqslant 0$ be large such that (5.11) holds and $\sqrt{2}-1<\alpha M<M-1$.

Lemma 5.5. *For the parameters chosen as above, $\Gamma^*[\underline{\phi},\bar{\phi}]$ is nonempty.*

Proof. We claim that $\tilde{\phi}(t)=\frac{\alpha}{1+\alpha e^{-\lambda_1 t}}$ is in the set $\Gamma^*[\underline{\phi},\bar{\phi}]$. In fact,

$$\tilde{\phi}'(t)=\frac{\alpha^2\lambda_1 e^{-\lambda_1 t}}{[(1+\alpha e^{-\lambda_1 t}]^2}>0$$

implies that $\tilde{\phi}(t)$ is nondecreasing in R. Also

$$\bar{\phi}(t)-\tilde{\phi}(t)=e^{\lambda_1 t}-\frac{\alpha}{1+\alpha e^{-\lambda_1 t}}=\frac{e^{\lambda_1 t}}{1+\alpha e^{-\lambda_1 t}}>0,$$

which implies $\tilde{\phi}(t)\leqslant\bar{\phi}(t)$. By Proposition 4.6(ii) and (iii) in [18], we know that $\tilde{\phi}(t)\geqslant\underline{\phi}(t)$ and $e^{\mu t}[\tilde{\phi}(t)-\underline{\phi}(t)]$ is nondecreasing in $t\in R$. By Proposition 4.5(ii) in [18], we know that $e^{\mu t}[\tilde{\phi}(t+s)-\tilde{\phi}(t)]$ is nondecreasing for all $s>0$.

Next, we will verify that $e^{\mu t}[\bar{\phi}(t)-\tilde{\phi}(t)]$ is nondecreasing in $t\in R$. For $t>0$, $\bar{\phi}(t)=1$, and hence

$$\begin{aligned}
\frac{d}{dt}\bigl\{e^{\mu t}\bigl[\bar{\phi}(t)-\tilde{\phi}(t)\bigr]\bigr\}&=\frac{d}{dt}\left\{e^{\mu t}\left[1-\frac{\alpha}{1+\alpha e^{-\lambda_1 t}}\right]\right\}\\
&=\frac{e^{\mu t}\{(1-\alpha)\mu+\alpha(2\mu-\alpha\mu-\lambda_1\alpha)e^{-\lambda_1 t}+\mu\alpha^2e^{-2\lambda_1 t}\}}{[1+\alpha e^{-\lambda_1 t}]^2}\\
&\geqslant\frac{e^{\mu t}\{\alpha[\mu(1-\alpha)+\lambda_1(1-\alpha)]e^{-\lambda_1 t}\}}{[1+\alpha e^{-\lambda_1 t}]^2}>0.
\end{aligned}$$

For $t \leqslant 0$, $\bar{\phi}(t) = e^{\lambda_1 t}$, and hence

$$\frac{d}{dt}\left\{e^{\mu t}\left[\bar{\phi}(t) - \tilde{\phi}(t)\right]\right\} = \frac{d}{dt}\left\{e^{\mu t}\left[e^{\lambda_1 t} - \frac{\alpha}{1+\alpha e^{-\lambda_1 t}}\right]\right\}$$
$$= \frac{(\lambda_1+\mu)e^{(\lambda_1+\mu)t}\{(\mu+\lambda_1)+\alpha[2\alpha(\mu+\lambda_1)-\mu]e^{-\lambda_1 t}\}}{[1+\alpha e^{-\lambda_1 t}]^2} > 0.$$

Therefore, $e^{\mu t}[\bar{\phi}(t) - \tilde{\phi}(t)]$ is nondecreasing in $t \in R$.

From the above, see that $\tilde{\phi}(t) \in \Gamma^*[\underline{\phi}, \bar{\phi}]$, and this completes the proof. □

Lemma 5.5 verifies (H5). (H1) and (H2) can be easily verified for this pair of upper and lower solutions. Now, applying Theorem 4.1, we obtain the following result.

Theorem 5.1. *Assume $\tau > 0$ is sufficiently small. Then for every $c > c^*$ $(c < 1/c^*)$,* (5.2) *has a traveling wave front solution of the form $u_n(t) = \phi(n+ct)$ $(u_n(t) = \phi(t+cn))$ connecting* 0 *and* 1.

Remark 5.1. When the nonlinear reaction term only satisfies the weakened quasi-monotonicity (QM*), the corresponding main theorem in [18] (Theorem 4.1) requires that the upper solution $\bar{\phi}$ be such that $e^{\mu t}[\bar{\phi}(t+s) - \bar{\phi}(t)]$ is nondecreasing in $t \in R$ for all $s > 0$. This is a very demanding condition, and makes searching for the upper solution a hard job. But our Theorem 4.1 drops this condition, and thus allows us to choose simpler piecewise functions, as is shown in the above example.

Acknowledgment

The authors thank the referees for their valuable comments which have led to an improvement of the presentation of the paper.

References

[1] J. Bell, Some threshold results for models of myelinated nerves, Math. Biosci. 54 (1981) 181–190.
[2] J. Bell, C. Cosner, Threshold behavior and propagation for nonlinear differential–difference systems motivated by modeling myelinated axons, Quart. Appl. Math. 42 (1984) 1–14.
[3] N.F. Britton, Spatial structures and periodic traveling waves in an integro-differential reaction–diffusion population model, SIAM J. Appl. Math. 50 (1990) 1663–1688.
[4] J.W. Cahn, S.-N. Chow, E.S. Van Vleck, Spatially discrete nonlinear diffusions, Rocky Mountain J. Math. 25 (1995) 87–118.
[5] J.W. Cahn, J. Mallet-Paret, E.S. Van Vleck, Traveling wave solutions for systems of ODE's on a two-dimensional spatial lattice, SIAM J. Appl. Math. 59 (1998) 455–493.
[6] H. Chi, J. Bell, B. Hassard, Numerical solutions of a nonlinear advanced-delay differential equation from nerve condition theory, J. Math. Biol. 24 (1986) 583–601.
[7] S.-N. Chow, J. Mallet-Paret, W. Shen, Traveling waves in lattice dynamical systems, J. Differential Equations 149 (1998) 248–291.
[8] S.-N. Chow, W. Shen, Stability and bifurcation of traveling wave solutions in coupled map lattices, J. Dynam. Systems Appl. 4 (1995) 1–26.

[9] C.E. Elmer, E.S. Van Vleck, Computation of traveling waves for spatially discrete bistable reaction–diffusion equations, Appl. Numer. Math. 20 (1996) 157, 169.
[10] W. Feng, X. Lu, On diffusive population models with toxicants and time delays, J. Math. Anal. Appl. 233 (1999) 373–386.
[11] S.A. Gourley, Wave front solutions of diffusive delay model for populations of Daphnia magna, Comp. Math. Appl., in press.
[12] D. Hankerson, B. Zinner, Wave fronts for a cooperative triangonal system of differential equations, J. Dynam. Differential Equations 5 (1993) 359–373.
[13] C.-H. Hsu, S.S. Lin, W. Shen, Traveling waves in cellular neural networks, Internat. J. Bifur. Chaos 9 (1999) 1307–1319.
[14] J.P. Keener, Propagation and its failure to coupled systems of discrete excitable cells, SIAM J. Appl. Math. 47 (1987) 556–572.
[15] J. Mallet-Paret, Spatial patterns, spatial chaos, and traveling waves in lattice differential equations, in: S.J. van Strien, S.M. Verduyn Lunel (Eds.), Stochastic and Spatial Structures of Dynamical Systems, North-Holland, Amsterdam, 1996, pp. 105–129.
[16] J. Mallet-Paret, The global structure of traveling waves in spatially discrete dynamical systems, J. Dynam. Differential Equations 11 (1999) 49–127.
[17] P. Weng, H. Huang, J. Wu, Asymptotic speed of propagation of wave fronts in a lattice delay differential equation with global interaction, IMA J. Appl. Math. 68 (2003) 409–439.
[18] J. Wu, X. Zou, Asymptotic and periodic boundary value problems of mixed FDEs and wave solutions of lattice functional differential equations, J. Differential Equations 135 (1997) 315–357.
[19] E. Zeidler, Nonlinear Functional Analysis and Its Applications, I, Fixed-Point Theorems, Springer-Verlag, New York, 1986.
[20] B. Zinner, Stability of traveling wave fronts for the discrete Nagumo equation, SIAM J. Math. Anal. 22 (1991) 1016–1020.
[21] B. Zinner, Existence of traveling wave front solutions for the discrete Nagumo equation, J. Differential Equations 96 (1992) 1–27.
[22] B. Zinner, G. Harris, W. Hudson, Traveling wave fronts for the discrete Fisher's equation, J. Differential Equations 105 (1993) 46–62.
[23] X. Zou, Traveling wave fronts in spatially discrete reaction–diffusion equations on higher-dimensional lattices, in: Proceedings of the Third Mississippi State Conference on Differential Equations and Computational Simulations, Electron. J. Differential Equations 1 (1998) 211–221.