Jordan all-derivable points in the algebra of all upper triangular matrices

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ABSTRACT

Let $\mathcal{M}_n$ be the algebra of all $n \times n$ upper triangular matrices. We say that $\varphi \in L(\mathcal{M}_n)$ is a Jordan derivable mapping at $G$ if $\varphi(ST + TS) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)$ for any $S, T \in \mathcal{M}_n$ with $ST = G$. An element $G \in \mathcal{M}_n$ is called a Jordan all-derivable point of $\mathcal{M}_n$ if every Jordan derivable linear mapping $\varphi$ at $G$ is a derivation. In this paper, we show that every element in $\mathcal{M}_n$ is a Jordan all-derivable point.

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1. Introduction and preliminaries

Let $\mathcal{A}$ and $\mathcal{B}$ be two rings with unit $I_1$ and $I_2$, respectively, and let $\mathcal{M}$ be a faithful $(\mathcal{A}, \mathcal{B})$-bimodule. We write

$$\mathcal{T} = \left\{ \begin{bmatrix} X & W \\ 0 & Y \end{bmatrix} : X \in \mathcal{A}, W \in \mathcal{M}, Y \in \mathcal{B} \right\}. $$

Under the usual matrix addition and multiplication, $\mathcal{T}$ will be called a triangular algebra. Let $H$ and $K$ be two complex Hilbert spaces. $B(H, K)$ stands for the set of all bounded linear operators from $H$ into $K$, and abbreviate $B(H, H)$ to $B(H)$. We may regard every element of $B(M, N)$ as of $B(H, K)$ for any closed
If two members of point for the strong operator topology. An and Hou [1] showed that every Jordan derivable mapping CSL algebras is a generalized derivation; (2) every invertible operator in nest algebras is an all-derivable point of

We describe some of the results related to ours. It was proved by Zhu et al. [19] that

Zhu and Xiong in [12,16] showed that (1) every norm-continuous generalized derivable mapping at $0$ on finite CSL algebras is a generalized derivation; (2) every invertible operator in nest algebras is an all-derivable point for the strong operator topology. An and Hou [1] showed that every Jordan derivable mapping $\varphi$ at $0$ in some idempotents on triangular algebras is a derivation. For other results, see [2–8,10–15,17,18].

It is the aim of this paper to show the following two statements: (1) In the triangular algebra $\mathcal{T}$, we obtain that $0, I$ and $\begin{bmatrix} I_1 & X_0 \\ 0 & I_2 \end{bmatrix}$ are Jordan all-derivable points in Section 2. (2) Let $\mathcal{N}$ be a complete nest on $H$ with $\dim(H_-)^{\perp} = 1$. We obtain some sufficient conditions that $G \in \text{alg} \mathcal{N}$ is a Jordan all-derivable point of $\text{alg} \mathcal{N}$ in Section 3. (3) In Section 4, we prove that every element in $\mathcal{T}_n \mathcal{M}_n$ is a Jordan all-derivable point.

2. Jordan all-derivable points in ring algebras

In this section, we always assume that the two rings $\mathcal{A}$ and $\mathcal{B}$ are of characteristic not 2 and 3, and there exists a positive integer $n$ such that $nl_1 - X$ and $nl_2 - Y$ are invertible if $X \in \mathcal{A}$ and $Y \in \mathcal{B}$.

**Theorem 2.1.** If $\varphi : \mathcal{T} \rightarrow \mathcal{T}$ is a Jordan derivable mapping at $0$, then $\varphi$ is a derivation.

**Proof.** If $T = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \mathcal{T}$, we write

$$
\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) = \begin{bmatrix} \varphi_{11}(X) + \delta_{11}(Y) + \tau_{11}(Z) \\ 0 \\ \varphi_{12}(X) + \delta_{12}(Y) + \tau_{12}(Z) \\ \varphi_{22}(X) + \delta_{22}(Y) + \tau_{22}(Z) \end{bmatrix}.
$$

For any $X \in \mathcal{A}$, $W \in \mathcal{M}_n$, let $S = \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}$ and $T = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}$. Then $ST = 0$ and we have

$$
\begin{bmatrix} \varphi_{11}(XW) & \varphi_{12}(XW) \\ 0 & \varphi_{22}(XW) \end{bmatrix} = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)
$$

$$
= \begin{bmatrix} \varphi_{11}(W) & \varphi_{12}(W) \\ 0 & \varphi_{22}(W) \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) \\ 0 & \delta_{22}(X) \end{bmatrix} \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix}
$$

$$
+ \begin{bmatrix} 0 & W \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) \\ 0 & \delta_{22}(X) \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \varphi_{11}(W) \\ 0 & \varphi_{12}(W) \end{bmatrix}.
$$

The above matrix equation implies the following three equations

$$
\varphi_{11}(XW) = \varphi_{11}(W)X + X\varphi_{11}(W),
$$

(1)
\[ \varphi_{12}(XW) = \delta_{11}(X)W + X\varphi_{12}(W) + W\delta_{22}(X), \]
\[ \varphi_{22}(XW) = 0 \]

for any \( X \in \mathcal{A}, W \in \mathcal{M} \). Taking \( X = I_1 \), we get that \( \varphi_{11}(W) = 0 \) and \( \varphi_{22}(W) = 0 \) for any \( W \in \mathcal{M} \).

For any \( X \in \mathcal{A}, W \in \mathcal{M} \) and \( Y \in \mathcal{B} \), if we take \( S = \begin{bmatrix} 0 & W \\ 0 & Y \end{bmatrix} \) and \( T = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \), then \( ST = 0 \). Thus we have

\[
\begin{bmatrix} 0 & \varphi_{12}(XW) \\ 0 & 0 \end{bmatrix} = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)
\]
\[
= \begin{bmatrix} \tau_{11}(Y) & \varphi_{12}(W) + \tau_{12}(Y) \\ 0 & \tau_{22}(Y) \end{bmatrix} \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) \\ 0 & \delta_{22}(X) \end{bmatrix} \begin{bmatrix} 0 & W \\ 0 & Y \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & W \\ 0 & Y \end{bmatrix} \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) \\ 0 & \delta_{22}(X) \end{bmatrix} + \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \tau_{11}(Y) & \varphi_{12}(W) + \tau_{12}(Y) \\ 0 & 0 \end{bmatrix}.
\]

The above matrix equation implies the following three equations

\[ X\tau_{11}(Y) + \tau_{11}(Y)X = 0, \] (4)
\[ \delta_{22}(X)Y + Y\delta_{22}(X) = 0, \] (5)
\[ \varphi_{12}(XW) = \delta_{11}(X)W + \delta_{12}(X)Y + W\delta_{22}(X) + X\tau_{12}(Y) + X\varphi_{12}(W) \] (6)

for any \( X \in \mathcal{A}, W \in \mathcal{M} \) and \( Y \in \mathcal{B} \).

Taking \( X = I_1 \) in Eq. (4), we get that \( \tau_{11}(Y) = 0 \) for any \( Y \in \mathcal{B} \). Similarly, taking \( Y = I_2 \) in Eq. (5), we obtain \( \delta_{22}(X) = 0 \) for any \( X \in \mathcal{A} \).

By Eqs. (2) and (6), we get \( X\tau_{12}(Y) + \delta_{12}(X)Y = 0 \). Thus \( \delta_{12}(X) = -X\tau_{12}(I_2) \) and \( \tau_{12}(Y) = -\delta_{12}(I_1)Y \).

It follows from \( \delta_{22}(X) = 0 \) and Eq. (2) that \( \varphi_{12}(XW) = \delta_{11}(X)W + X\varphi_{12}(W) \) for any \( X \in \mathcal{A} \) and \( W \in \mathcal{M} \). By the above equation, we have

\[ \delta_{11}(X_1X_2)W + X_1X_2\varphi_{12}(W) = \varphi_{12}(X_1X_2W) = \delta_{11}(X_1)X_2W + X_1\varphi_{12}(X_2W) \]
\[ = \delta_{11}(X_1)X_2W + X_1\delta_{12}(X_2)W + X_1X_2\varphi_{12}(W) \] (7)

for any \( X_1, X_2 \in \mathcal{A} \) and \( W \in \mathcal{M} \). Since \( \mathcal{M} \) is a faithful \((\mathcal{A}, \mathcal{B})\)-bimodule, we see that \( \delta_{11}(X_1X_2) = \delta_{11}(X_1)X_2 + X_1\delta_{11}(X_2). \)

For any \( Y \in \mathcal{B}, W \in \mathcal{M} \), and for any invertible \( X \in \mathcal{A} \), let \( S = \begin{bmatrix} 0 & -X^{-1}WY \\ 0 & Y \end{bmatrix} \) and \( T = \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix} \).

Then \( ST = 0 \). So we have

\[
\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)
\]
\[
= \begin{bmatrix} 0 & -\varphi_{12}(X^{-1}WY) + \tau_{12}(Y) \\ 0 & \tau_{22}(Y) \end{bmatrix} \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) + \varphi_{12}(W) \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -X^{-1}WY \\ 0 & Y \end{bmatrix}
\]
\[
+ \begin{bmatrix} 0 & -X^{-1}WY \\ 0 & Y \end{bmatrix} \begin{bmatrix} \delta_{11}(X) & \delta_{12}(X) + \varphi_{12}(W) \\ 0 & 0 \end{bmatrix}
\]
\[
+ \begin{bmatrix} X & W \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & -\varphi_{12}(X^{-1}WY) + \tau_{12}(Y) \\ 0 & \tau_{22}(Y) \end{bmatrix}.
\]

From the above matrix equation and \( X\tau_{12}(Y) + \delta_{12}(X)Y = 0 \), we have the equation

\[ \delta_{11}(X)X^{-1}WY + X\varphi_{12}(X^{-1}WY) = \varphi_{12}(W)Y + W\tau_{22}(Y). \] (8)
If we take $X = I_1$ in the above equation, then $\delta_{11}(l_1)WY + \psi_{12}(W) = \psi_{12}(W)Y + W\tau_{22}(Y)$. Letting $X = I_1$ in Eq. (2), we have $\delta_{11}(l_1)W = 0$ for any $W \in \mathcal{M}$. Thus $\delta_{11}(l_1) = 0$. So we have $\psi_{12}(W) = \psi_{12}(W)Y + W\tau_{22}(Y)$. For any $Y_1, Y_2 \in \mathcal{B}$, from the above equation, we have

$$
\psi_{12}(W)Y_1 Y_2 + W\tau_{22}(Y_1 Y_2) = \psi_{12}(WY_1 Y_2) = \psi_{12}(W)Y_1 Y_2 + W\tau_{22}(Y_1) Y_2 + WY_1 \tau_{22}(Y_2). \tag{9}
$$

So $\tau_{22}(Y_1 Y_2) = \tau_{22}(Y_1) Y_2 + Y_1 \tau_{22}(Y_2)$. Now by the similar computations as that in Section 2 of [1], one can show that $\psi$ is a derivation. \□

**Theorem 2.2.** If $\psi : \mathcal{T} \to \mathcal{T}$ be a Jordan derivable mapping at $S = \begin{bmatrix} I_1 & X_0 \\ 0 & I_2 \end{bmatrix}$. Then $\psi$ is a derivation.

**Proof.** For any invertible element $X \in \mathcal{A}$ and $Y \in \mathcal{B}$, let $S = \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ and $T = \begin{bmatrix} X^{-1} & X^{-1}X_0 \\ 0 & Y^{-1} \end{bmatrix}$. Then $ST = G$ and we have

\[
\begin{bmatrix}
2\delta_{11}(l_1) + 2\tau_{11}(l_2) + \psi_{11}(X_0 + X^{-1}X_0Y) & 2\delta_{12}(l_1) + 2\tau_{12}(l_2) + \psi_{12}(X_0 + X^{-1}X_0Y) \\
0 & 2\delta_{22}(l_1) + \psi_{22}(X_0 + X^{-1}X_0Y) + 2\tau_{22}(l_2)
\end{bmatrix}
= \psi(ST + TS) = \psi(S)T + S\psi(T) + \psi(T)S + T\psi(S)
\]

\[
= \begin{bmatrix}
\delta_{11}(X) + \tau_{11}(Y) & \delta_{12}(X) + \tau_{12}(Y) \\
0 & \delta_{22}(X) + \tau_{22}(Y)
\end{bmatrix} \begin{bmatrix}
X^{-1} & X^{-1}X_0 \\
0 & Y^{-1}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\delta_{11}(X^{-1}) + \tau_{11}(Y^{-1}) + \psi_{11}(X^{-1}X_0) \\
0 \\
X^{-1}X_0 & Y^{-1}
\end{bmatrix}
\begin{bmatrix}
\delta_{11}(X) + \tau_{11}(Y) & \delta_{12}(X) + \tau_{12}(Y) \\
0 & \delta_{22}(X) + \tau_{22}(Y)
\end{bmatrix}
\]

The above matrix equation implies the following three equations

\[
2\delta_{11}(l_1) + \psi_{11}(X_0 + X^{-1}X_0Y) + 2\tau_{11}(l_2)
= (\delta_{11}(X) + \tau_{11}(Y))X^{-1} + (\delta_{11}(X^{-1}) + \psi_{11}(X^{-1}X_0) + \tau_{11}(Y^{-1}) \delta_{11}(X) + \tau_{11}(Y)). \tag{10}
\]

\[
2\delta_{12}(l_1) + \psi_{12}(X_0 + X^{-1}X_0Y) + 2\tau_{12}(l_2)
= (\delta_{12}(X) + \tau_{12}(Y))Y^{-1} + (\delta_{12}(X^{-1}) + \psi_{12}(X^{-1}X_0) + \tau_{12}(Y^{-1}) \delta_{12}(X) + \tau_{12}(Y)). \tag{11}
\]

and

\[
2\delta_{22}(l_1) + \psi_{22}(X_0 + X^{-1}X_0Y) + 2\tau_{22}(l_2)
= (\delta_{22}(X^{-1}) + \psi_{22}(X^{-1}X_0) + \tau_{22}(Y^{-1})) + Y + (\delta_{22}(X^{-1}) + \psi_{22}(X^{-1}X_0) + \tau_{22}(Y^{-1}) \delta_{22}(X) + \tau_{22}(Y)). \tag{12}
\]
Letting $X = I_1$ and $Y = I_2$ in Eq. (10), we have $\delta_{11}(I_1) + \tau_{11}(I_2) = 0$. Putting $X = \frac{1}{2}I_1$ and $Y = I_2$ in Eq. (10), we get that $\varphi_{11}(X_0) = \tau_{11}(I_2)$. Taking $X = I_1$ and $Y = \frac{1}{2}I_2$ in Eq. (10) and using the above two results, we obtain $\varphi_{11}(X_0) = 0$. So $\delta_{11}(I_1) = \tau_{11}(I_2) = \varphi_{11}(X_0) = 0$. It follows from Eq. (10) that

$$\varphi_{11}(X^{-1}X_0Y) = (\delta_{11}(X) + \tau_{11}(Y))X^{-1} + (\delta_{11}(X^{-1}) + \varphi_{11}(X^{-1}X_0)$$

$$+ \tau_{11}(Y^{-1}))X + X(\delta_{11}(X^{-1}) + \varphi_{11}(X^{-1}X_0)$$

$$+ \tau_{11}(Y^{-1})) + X^{-1}(\delta_{11}(X) + \tau_{11}(Y)).$$

(13)

Letting $X = I_1$ and $Y = I_2$ in Eq. (12), we have $\delta_{22}(I_1) + \tau_{22}(I_2) = 0$. Putting $X = I_1$ and $Y = \frac{1}{2}I_2$ in Eq. (12), we get that $\varphi_{22}(X_0) = 2\delta_{22}(I_1)$. Taking $X = \frac{1}{2}I_1$ and $Y = I_2$ in Eq. (12), we obtain $\delta_{22}(I_1) + \varphi_{22}(X_0) = 0$. So we have $\delta_{22}(I_1) = \tau_{22}(I_2) = \varphi_{22}(X_0) = 0$. It follows that

$$\varphi_{22}(X^{-1}X_0Y) = (\delta_{22}(X) + \tau_{22}(Y))Y^{-1} + (\delta_{22}(X^{-1}) + \varphi_{22}(X^{-1}X_0)$$

$$+ \tau_{22}(Y^{-1}))Y + Y(\delta_{22}(X^{-1}) + \varphi_{22}(X^{-1}X_0)$$

$$+ \tau_{22}(Y^{-1})) + Y^{-1}(\delta_{22}(X) + \tau_{22}(Y)).$$

(14)

Letting $X = I_1$, $Y = I_2$ in Eq. (11), we have $\delta_{12}(I_1) + \tau_{12}(I_2) = 0$. Putting $Y = I_2$ and $Y = \frac{1}{2}I_2$ respectively in Eq. (11), we get that

$$\varphi_{12}(X_0) = 2\delta_{12}(X) + \tau_{12}(I_2) + X^{-1}\delta_{12}(X) + \frac{1}{2} \delta_{12}(X^{-1}) + \tau_{12}(I_2) + X\delta_{12}(X^{-1})$$

$$+ X\varphi_{12}(X^{-1}X_0) + 2X\tau_{12}(I_2) + \frac{1}{2}X^{-1}\tau_{12}(I_2) + X^{-1}X_0\varphi_{22}(X) + \delta_{11}(X)X^{-1}X_0$$

(15)

and

$$\varphi_{12}(X_0) = \delta_{12}(X) + \tau_{12}(I_2) + X^{-1}\delta_{12}(X) + \varphi_{12}(X_{12}) + \tau_{12}(I_2) + X^{-1}\tau_{12}(I_2)$$

$$+ X\delta_{12}(X^{-1}) + X\varphi_{12}(X^{-1}X_0) + X\tau_{12}(I_2) + X^{-1}X_0\varphi_{22}(X) + \delta_{11}(X)X^{-1}X_0.$$ 

(16)

Thus $\delta_{12}(X) - \frac{1}{2}X^{-1}\tau_{12}(I_2) + X\tau_{12}(I_2) - \frac{1}{2}\delta_{12}(X^{-1}) = 0$. So

$$\delta_{12}(X) + X\tau_{12}(I_2) = \frac{1}{2}(X^{-1}\tau_{12}(I_2) + \delta_{12}(X^{-1}))$$

for any invertible $X \in A$. Replacing $X$ by $X^{-1}$ in the above equation, we get $\delta_{12}(X^{-1}) + X^{-1}\tau_{12}(I_2) = \frac{1}{2}(X\tau_{12}(I_2) + \delta_{12}(X))$. So we have $\delta_{12}(X) + X\tau_{12}(I_2) = \frac{1}{2}(X\tau_{12}(I_2) + \delta_{12}(X))$, thus $\delta_{12}(X) + X\tau_{12}(I_2) = 0$, that is $\delta_{12}(X) = -X\tau_{12}(I_2)$ for any invertible $X \in A$. For any $X \in A$, there is a positive integer $n$ such that $nI_1 - X$ is invertible. Thus $\delta_{12}(nI_1 - X) = -(nI_1 - X)\tau_{12}(I_2)$. Since $\delta_{12}(I_1) + \tau_{12}(I_2) = 0$, we have $\delta_{12}(X) = -X\tau_{12}(I_2)$ for any $X \in A$.

Similarly, Letting $X = I_1$ and $X = 2I_1$ respectively in Eq. (11), we get that

$$\varphi_{12}(X_0Y) = \delta_{12}(I_1)Y + \tau_{12}(Y) - 2\delta_{12}(I_1)Y^{-1} - 2\tau_{12}(Y^{-1})$$

$$+ X_0\tau_{22}(Y) + \varphi_{12}(X_0)Y + \tau_{11}(Y)X_0.$$ 

(17)

Letting $X = I_1$ and $X = 2I_1$, respectively, in Eq. (13), we have $\varphi_{11}(X_0Y) = 2\tau_{11}(Y) + 2\tau_{11}(Y^{-1})$ and

$$\varphi_{11}(X_0Y) = 2\tau_{11}(Y) + 2\tau_{11}(Y^{-1}) + 8\tau_{11}(Y^{-1}) = 0.$$ 

So $\tau_{11}(Y^{-1}) = 0$. For any $Y \in B$, there is a positive integer $n$ such that $nI_2 - Y$ is invertible. Thus $\tau_{11}(nI_2 - Y) = 0$, that is $\tau_{11}(Y) = 0$ for any $Y \in B$. Letting $Y = I_2$ and $Y = 2I_2$ respectively in Eq. (14), we have $2\varphi_{22}(X^{-1}X_0) + 4\delta_{22}(X) + 4\delta_{22}(X^{-1}) = 0$ and

$$2\varphi_{22}(X^{-1}X_0) + \delta_{22}(X) + 4\delta_{22}(X^{-1}) = 0.$$ 

Thus $\delta_{22}(X) = 0$ for any invertible $X \in A$. By imitating the above proof, we obtain $\delta_{22}(X) = 0$ for any $X \in A$. 


For any \( Y \in \mathcal{B}, \ W \in \mathcal{M} \), and for any invertible \( X \in \mathcal{A} \), letting \( S = \begin{bmatrix} X & XW \\ 0 & Y \end{bmatrix} \) and \( T = \begin{bmatrix} X^{-1} & X^{-1}X_0 - WY^{-1} \\ 0 & Y^{-1} \end{bmatrix} \), then \( ST = G \). Thus we have

\[
\begin{bmatrix}
\varphi_{11}(X^{-1}X_0Y) & \varphi_{12}(X_0 + X^{-1}X_0Y) \\
0 & \varphi_{22}(X^{-1}X_0Y)
\end{bmatrix}
= \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)
\]

\[
= \begin{bmatrix}
\delta_{11}(X) + \varphi_{11}(XW) & \delta_{12}(X) + \varphi_{12}(XW) + \tau_{12}(Y) \\
0 & \varphi_{22}(XW) + \tau_{22}(Y)
\end{bmatrix}
\begin{bmatrix}
X^{-1} & X^{-1}X_0 - WY^{-1} \\
0 & Y^{-1}
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\delta_{11}(X^{-1}) & \varphi_{12}(X^{-1}X_0 - WY^{-1}) \\
0 & \varphi_{22}(X^{-1}X_0 - WY^{-1})
\end{bmatrix}
\begin{bmatrix}
X & XW \\
0 & Y
\end{bmatrix}
\]

\[
+ \begin{bmatrix}
\delta_{11}(X^{-1}) & \varphi_{12}(X^{-1}X_0 - WY^{-1}) \\
0 & \varphi_{22}(X^{-1}X_0 - WY^{-1})
\end{bmatrix}
\begin{bmatrix}
X^{-1}X_0 - WY^{-1} \\
0
\end{bmatrix}
\]

The above matrix equation implies the following three equations

\[
\varphi_{11}(X^{-1}X_0Y) = (\delta_{11}(X^{-1}) + \varphi_{11}(X^{-1}X_0 - WY^{-1}))X + X(\delta_{11}(X^{-1})
+ \varphi_{11}(XW))X^{-1},
\]

\[
(18)
\]

\[
\varphi_{12}(X_0 + X^{-1}X_0Y) = (\delta_{11}(X) + \varphi_{11}(XW))(X^{-1}X_0 - WY^{-1}) + (\delta_{12}(X) + \varphi_{12}(XW)
+ \tau_{12}(Y))Y^{-1} + (\delta_{11}(X^{-1}) + \varphi_{11}(X^{-1}X_0 - WY^{-1}))XW + (\delta_{12}(X^{-1})
+ \varphi_{12}(X^{-1}X_0 - WY^{-1}) + \tau_{12}(Y^{-1}))Y + X(\delta_{12}(X^{-1})
+ \varphi_{12}(X^{-1}X_0 - WY^{-1}) + \tau_{12}(Y^{-1})) + XW(\varphi_{22}(X^{-1}X_0 - WY^{-1})
+ \tau_{22}(Y^{-1})) + X^{-1}(\delta_{12}(X) + \varphi_{12}(XW) + \tau_{12}(Y))
+ (X^{-1}X_0 - WY^{-1})(\varphi_{22}(XW) + \tau_{22}(Y))
\]

\[
(19)
\]

\[
\varphi_{22}(X^{-1}X_0Y) = (\varphi_{22}(XW) + \tau_{22}(Y))Y^{-1} + (\varphi_{22}(X^{-1}X_0 - WY^{-1}) + \tau_{22}(Y^{-1}))Y
+ Y(\varphi_{22}(X^{-1}X_0 - WY^{-1}) + \tau_{22}(Y^{-1})) + Y^{-1}(\varphi_{22}(XW) + \tau_{22}(Y)).
\]

\[
(20)
\]

Letting \( X = 2I_1, \ Y = I_2 \) in Eqs. (18) and (20), we have \( \varphi_{11}(W) = 0 \) and \( \varphi_{22}(W) = 0 \) for any \( W \in \mathcal{M} \). Taking \( Y = \frac{1}{2}I_2 \) and \( Y = I_2 \), respectively, in Eq. (19), we have

\[
\varphi_{12}(X_0) = \delta_{11}(X)X^{-1}X_0 - 2\delta_{11}(X)W + 2\varphi_{12}(XW) + X^{-1}(\varphi_{12}(XW)
+ X\varphi_{12}(X^{-1}X_0) - 2X\varphi_{12}(W) + \delta_{11}(X^{-1})XW)
\]

\[
(21)
\]
and
\[ \varphi_{12}(X_0) = \delta_{11}(X)X^{-1}X_0 - \delta_{11}(X)W + \varphi_{12}(XW) + X^{-1}\varphi_{12}(XW) \]
\[ + X\varphi_{12}(X^{-1}X_0) - X\varphi_{12}(W) + \delta_{11}(X^{-1})XW. \] (22)

Thus \( \varphi_{12}(XW) = X\varphi_{12}(W) + \delta_{11}(X)W \) for every \( W \in \mathcal{M} \) and \( X \in \mathcal{A} \). Similarly, letting \( X = I_1 \) and \( X = 2I_1 \), respectively, in Eq. (19), we have
\[ \varphi_{12}(WY^{-1}) = \varphi_{12}(W)Y^{-1} + W\tau_{22}(Y^{-1}). \] So we have \( \varphi_{12}(WY) = \varphi_{12}(W) + W\tau_{22}(Y) \) for every \( W \in \mathcal{M} \) and \( Y \in \mathcal{B} \). By Eq. (17) and \( \varphi_{12}(XW) = X\varphi_{12}(W) + \delta_{11}(X)W \), we have \( \delta_{12}(I_1)Y + \tau_{12}(Y) = 2(\delta_{12}(I_1)Y^{-1} + \tau_{12}(Y^{-1})) \). Hence \( \tau_{12}(Y) = -\delta_{12}(I_1)Y \). By \( \varphi_{12}(XW) = X\varphi_{12}(W) + \delta_{11}(X)W \) and \( \varphi_{12}(WY) = \varphi_{12}(W) + W\tau_{22}(Y) \), we get that \( \delta_{11} \) and \( \tau_{22} \) are derivations. That shows that \( \varphi \) is a derivation. \( \square \)

**Corollary 2.3.** Let \( \varphi : \mathcal{T} \to \mathcal{T} \) be a Jordan derivable mapping at \( I = \begin{bmatrix} I_1 & 0 \\ 0 & I_2 \end{bmatrix} \). Assume that the two rings \( \mathcal{A} \) and \( \mathcal{B} \) are of characteristic not 2 and 3, and for every \( X \in \mathcal{A}, Y \in \mathcal{B} \), there exists an integer \( n \) such that \( nI_1 - X \) and \( nI_2 - Y \) are invertible. Then \( \varphi \) is a derivation.

**Proof.** By imitating the Proof of Theorem 2.2, we can prove that \( \varphi \) is a derivation. Hence \( I \) is a Jordan all-derivation point. \( \square \)

### 3. Jordan all-derivable points in nest algebras

**Theorem 3.1.** Let \( \mathcal{N} \) be a complete nest on \( H \) with \( \dim(H_{-})^{\perp} = 1 \), and let \( P = P(H_{-}), Q = P((H_{-})^{\perp}) \). \( \mathcal{A} = P\mathcal{alg N}P \). The following statements hold:

1. If \( F \) is a Jordan all-derivable point of \( \mathcal{A} \) and \( 0 \neq F_0 \in B((H_{-})^{\perp}, (H_{-})^{\perp}) \), then \( G = \begin{bmatrix} F & D \\ 0 & F_0 \end{bmatrix} \) is a Jordan all-derivable point of \( \mathcal{alg N} \).
2. If \( F \) is a Jordan all-derivable point of \( \mathcal{A} \) and \( D \in B(H^\perp, H_{-}) \), then \( G = \begin{bmatrix} F & D \\ 0 & 0 \end{bmatrix} \) is a Jordan all-derivable point of \( \mathcal{alg N} \).
3. If \( 0 \neq F_0 \in B((H_{-})^{\perp}, (H_{-})^{\perp}) \), then \( G = \begin{bmatrix} 0 & 0 \\ 0 & F_0 \end{bmatrix} \) is a Jordan all-derivable point of \( \mathcal{alg N} \).
4. If \( 0 \neq D \in B(H^\perp, H_{-}) \), then \( G = \begin{bmatrix} 0 & D \\ 0 & 0 \end{bmatrix} \) is a Jordan all-derivable point of \( \mathcal{alg N} \).

**Proof.** We may write \( H_0 = (H_{-})^{\perp} = \text{span}(g) \), where \( g \in (H_{-})^{\perp} \) with \( \|g\| = 1 \). Obviously, \( \mathcal{A} \) is a nest algebra in \( B(H_{-}) \). Simultaneously, we may regard \( \mathcal{A} \) as a subalgebra of \( \mathcal{alg N} \). Naturally, for any \( S \in \mathcal{alg N} \), then \( S \) can be represented as \( 2 \times 2 \) operator matrix relative to the orthogonal decomposition \( H = H_{-} \oplus H_0 \) as follows
\[ S = \begin{bmatrix} X & Y \\ Z & 0 \end{bmatrix} = \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix}, \]
where \( X = PSP, Y = PSQ = y \otimes g \) for some \( y \in H_{-}, Z = QSQ = zg \otimes g \) for some \( z \in C \). Let \( \varphi : \mathcal{alg N} \to \mathcal{alg N} \) be a linear mapping. Then there exist \( A(X) \in \mathcal{A}, a(X) \in H_{-} \) and \( \alpha(X) \in C \) such that
\[
\varphi(\begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix}) = P\varphi(PSP)P + P\varphi(PSP)Q + Q\varphi(PSP)Q = A(X) + a(X) \otimes g + \alpha(X)g \otimes g = \begin{bmatrix} A(X) & a(X) \otimes g \\ 0 & \alpha(X)g \otimes g \end{bmatrix};
\]

and there exist \( B(y) \in A, b(y) \in H_- \) and \( \beta(y) \in C \) such that

\[
\varphi \begin{bmatrix} 0 & \gamma \\ 0 & 0 \end{bmatrix} = P \varphi(PSQ)P + P \varphi(PSQ)Q + Q \varphi(PSQ)Q \\
= B(y) + b(y) \otimes g + \beta(y)g \otimes g \\
= \begin{bmatrix} B(y) & b(y) \otimes g \\ 0 & \beta(y)g \otimes g \end{bmatrix}.
\]

Simultaneously, there exist \( C_0 \in A, c_0 \in H_- \) and \( \gamma_0 \in C \) such that

\[
\varphi \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} = \varphi(zg \otimes g) \\
= z [P \varphi(g \otimes g)P + P \varphi(g \otimes g)Q + \varphi(g \otimes g)Q] \\
= z [C_0 + c_0 \otimes g + \alpha_0 g \otimes g] \\
= z \begin{bmatrix} C_0 & c_0 \otimes g \\ 0 & \gamma_0 g \otimes g \end{bmatrix}.
\]

Since \( \varphi \) is a linear mapping on \( \text{alg } N, A(\cdot), a(\cdot) \) and \( \alpha(\cdot) \) are linear mappings from \( B(H_-) \) into \( B(H_-), H_- \) and \( C \), respectively, and \( B(\cdot), b(\cdot) \) and \( \beta(\cdot) \) are linear mappings from \( H_- \) into \( B(H_-), H_- \) and \( C \), respectively.

For any \( S = PSP + PSQ + QSQ = \begin{bmatrix} P & PSQ \\ 0 & QSQ \end{bmatrix} \in \text{alg } N \) and \( T = PTP + PTQ + QTQ = \begin{bmatrix} PTP & PTQ \\ 0 & QTQ \end{bmatrix} \in \text{alg } N \), we always write \( X = PSP, Y = PSQ = y \otimes g, Z = QSQ = zg \otimes g, U = PTP, V = PTQ = v \otimes g, W = QTQ = wg \otimes g \) in the rest of the section.

The proof of the statement (1). Let \( F \) be a Jordan all-derivable point of \( A \), and let \( \varphi \) be a Jordan derivable mapping at \( G = [F \quad D] \in \text{alg } N \). It is easy to verify that \( A(\cdot) \) is a Jordan derivable mapping at \( F = PGP \) on \( A \). Without loss of generality, we may assume that \( F_0 = g \otimes g \) and \( D = d \otimes g \).

If \( ST = G = [F \quad d \otimes g] \otimes g, \) then we have

\[
\begin{bmatrix} F & d \otimes g \\ 0 & g \otimes g \end{bmatrix} = G = ST \\
= \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix} \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix} \\
= \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix},
\]

i.e., \( XU = F, Xv + wy = d, zw = 1 \). Especially, if we take \( S = \begin{bmatrix} \mu^{-1} X & \mu d \otimes g \\ 0 & \mu g \otimes g \end{bmatrix} \) and \( T = \begin{bmatrix} \mu U & 0 \\ 0 & \mu^{-1} g \otimes g \end{bmatrix} \), then \( ST = G \). Since \( \varphi \) is a Jordan derivable mapping at \( G \), we have \( \varphi(G + TS) = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + T\varphi(S) + \varphi(T)S \).

So we have

\[
\begin{bmatrix} A(F + UX) + \mu^2 B(Ud) + B(d) + 2C_0 & * \\ 0 & * \end{bmatrix} \\
= \begin{bmatrix} \mu^{-1} A(X) + \mu B(d) + \mu C_0 & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \mu U & 0 \\ 0 & \mu^{-1} g \otimes g \end{bmatrix} \\
+ \begin{bmatrix} \mu A(U) + \mu^{-1} C_0 & * \\ 0 & * \end{bmatrix} \begin{bmatrix} \mu^{-1} X & \mu d \otimes g \\ 0 & \mu g \otimes g \end{bmatrix}
\]

\[.\]
follows from the main theorem in [10] that 

\[ A \in \text{ for any } \]

If we multiply the above equation by 1930 and let \( \xi \in \), then it follows that \( A(\cdot) \) is a Jordan derivation on \( \mathcal{A} \). Note that \( \mathcal{A} \) is a nest algebra. It follows from the main theorem in [10] that \( A(\cdot) \) is an inner derivation. Hence there exists an operator \( A \in B(H_{-}) \) such that \( A(A) = XA - AX \), for \( A \in \mathcal{A} \). Furthermore, the following matrix equation holds

\[
\varphi\left(\begin{bmatrix} F & d \otimes g \\ 0 & g \otimes g \end{bmatrix} + [UX, (Uy + zw) \otimes g] \right) = \varphi(G + TS) = \varphi(S)T + S \varphi(T) + T \varphi(S)
\]

\[
= \left(\begin{array}{cc}
XA - AX & a(X) \otimes g \\
0 & \alpha(X)g \otimes g
\end{array}\right) + \left(\begin{array}{cc}
0 & b(y) \otimes g \\
0 & \beta(y)g \otimes g
\end{array}\right) + z\left(\begin{array}{cc}
0 & c_0 \otimes g \\
0 & \gamma_0 g \otimes g
\end{array}\right) - \left(\begin{array}{cc}
0 & c_0 \otimes g \\
0 & \gamma_0 g \otimes g
\end{array}\right)
\]

\[
+ [U, v \otimes g] \left(\begin{array}{cc}
XA - AX & a(X) \otimes g \\
0 & \alpha(X)g \otimes g
\end{array}\right) + \left(\begin{array}{cc}
0 & b(y) \otimes g \\
0 & \beta(y)g \otimes g
\end{array}\right) + w\left(\begin{array}{cc}
0 & c_0 \otimes g \\
0 & \gamma_0 g \otimes g
\end{array}\right)
\]

for any \( XU = F \), \( Xv + wy = d \), \( zw = 1 \). The above matrix equation implies that

\[
a(F + UX) + b(d + Uy) + zb(v) = (XA - AX)v + (UA - AU)y
\]

\[
+ w(a(X) + b(y)) + U(a(X) + b(v) + zc_0)
\]

\[
+ X(a(U) + b(v) + wc_0) + z(a(U) + b(v))
\]

\[
+ (\alpha(U) + \beta(v) + w\gamma_0)y + w\gamma_0 c_0
\]

\[
+ (\alpha(X) + \beta(y) + \gamma_0)\varphi(U) \quad \text{(23)}
\]

\[
\alpha(F + UX) + \beta(d + Uy + zv) = 2w(\alpha(X) + \beta(y)) + 2z(\alpha(U) + \beta(v)) + 2\gamma_0
\]

\[
\quad \text{(24)}
\]

for any \( X, U \in B(H_{-}), v, y \in H_{-} \) and \( w, z \in C \) with \( XU = F \), \( Xv + wy = d \), \( zw = 1 \).

If we take \( X = \xi I \) (0 \( \neq \xi \in C \), \( U = \xi^{-1}F \), \( v = y + \xi^{-1}d \), \( z = -\xi^{-1} \) and \( w = -\xi \) in Eq. (24), then

\[
2\alpha(F) + \beta(d) + \xi^{-1}\beta(Fy + y) + \xi^{-2}\beta(d) = -2\xi^2\alpha(I) - 2\xi\beta(y) - 2\xi^{-2}\alpha(F) + 2\gamma_0.
\]

If we multiply the above equation by \( \xi^2 \) and let \( \xi \to 0 \), we obtain \( 2\alpha(F) + \beta(d) = 0 \). Thus the above is the unique solution of the above equation. Furthermore, we have

\[
\xi^{-1}\beta(Fy + y) + 2\xi^2\alpha(I) + 2\xi\beta(y) = 2\gamma_0.
\]

If we multiply the above equation by \( \xi \) and let \( \xi \to 0 \), we get that \( \beta(Fy + y) = 0 \). Thus we have

\[
\alpha(F) + \beta(d) = 0.
\]
\[2\xi^2 \alpha(I) + 2\xi \beta(y) = 2\gamma_0.\]

Letting \(\xi \to 0\), we have \(\gamma_0 = 0\). Then we have
\[\beta(y) = -\xi \alpha(I),\]
\[\alpha(UX + F) = 2w \alpha(X) + 2\alpha(U)\]
for any \(X, U \in B(H_-), v, y \in H_-\) and \(w, z \in C\) with \(XU = F, Xv + wy = d, zw = 1\). For any invertible operator \(X \in A\), if we take \(U = X^{-1}F, y = -\xi Xv + \xi d, z = \xi\) and \(w = \xi^{-1}\) in the above equation, then we get that
\[\alpha(F + X^{-1}FX) = 2\xi \alpha(x^{-1}F) + 2\xi^{-1} \alpha(x).\]

If we multiply the above equation by \(1/w\) and let \(\xi \to 0\), we get that \(\alpha(X) = 0\). It follows from Lemma 2.1 in [19] that \(\alpha(X) = 0\) for any \(X \in A\).

For any invertible operator \(X \in A\) and complex number \(0 \neq w \in C\), if we take \(U = X^{-1}F, y = w^{-1}d, v = 0\) and \(z = w^{-1}\) in Eq. (23), then
\[a(F + X^{-1}FX) + w^{-1}b(X^{-1}Fd) = wXc + w^{-1}a(X^{-1}F) + wXc + w^{-1}(X^{-1}FA - AX^{-1}F)d + X^{-1}Fa(X) + w^{-1}X^{-1}Fb(d) + w^{-1}X^{-1}Fc_0.\]  

(25)

If we multiply the above equation by \(1/w\) and let \(w \to \infty\), then \(a(X) = -Xc_0\) for any invertible operator \(X \in A\). By Lemma 2.1 in [19], \(a(X) = -Xc_0\) for any \(X \in A\). In particular, \(a(I) + c_0 = 0\). For arbitrary invertible operator \(X \in A\), if we take \(U = X^{-1}F, y = d, v = 0\) and \(z = w = 1\) in Eq. (23), then we get that
\[b(X^{-1}Fd) = (X^{-1}FA - AX^{-1}F)d + X^{-1}Fb(d).\]

For any invertible operator \(X \in A\) and for any \(v \in H_-\), taking \(z = w = 1\) and \(Xv + y = d\) in Eq. (23), we have
\[b(Xv) - Xb(v) - (XA - AX)v = b(X^{-1}Fxv) - X^{-1}Fb(Xv) - (X^{-1}FA - AX^{-1}F)Xv.\]

Replacing \(X\) by \(2X\) in the above equation, then
\[2(b(Xv) - Xb(v) - (XA - AX)v) = b(X^{-1}Fxv) - X^{-1}Fb(Xv) - (X^{-1}FA - AX^{-1}F)Xv.\]

Thus \(b(Xv) = Xb(v) + (XA - AX)v\). It follows from Lemma 2.1 in [19] that
\[b(Xv) = Xb(v) + (XA - AX)v\]  

(26)

for any \(X \in A\) and \(v \in H_-\).

In other words, we have
\[
\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} X & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right) \\
= \begin{bmatrix} XA - AX & -Xc_0 \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b(y) \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & zc_0 \otimes g \\ 0 & 0 \end{bmatrix} \\
= \begin{bmatrix} XA - AX & -(Xc_0 + b(y) + zc_0) \otimes g \\ 0 & 0 \end{bmatrix}. 
\]

(27)

for any \(S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \in \text{alg } \mathcal{N}\).

We now claim that \(\varphi\) is a Jordan derivable mapping at \(\tilde{G} = \begin{bmatrix} I & d \otimes g \\ 0 & g \otimes g \end{bmatrix}\). In fact, for any \(S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}\) and \(T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix} \in \text{alg } \mathcal{N}\) with \(ST = \tilde{G}\), then
\[
\begin{bmatrix}
I & d \otimes g \\
0 & g \otimes g
\end{bmatrix}
= \tilde{G} = ST =
\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\begin{bmatrix}
U & V \\
0 & W
\end{bmatrix}
= \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix}
\begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}
= \begin{bmatrix}
XU & (Xv + wy) \otimes g \\
0 & zwg \otimes g
\end{bmatrix}.
\]

i.e., \(XU = I\), \(Xv + wy = d\), \(zw = 1\). Thus we have
\[
\varphi(\tilde{G} + TS) = \varphi\left(\begin{bmatrix}
I + UX \\
0
\end{bmatrix}
\begin{bmatrix}
d + Uy + zv \\
2g \otimes g
\end{bmatrix}
\right) = \begin{bmatrix}
UXA - AUX & (-UXc_0 + b(d + Uy + zv) + c_0) \otimes g \\
0 & 0
\end{bmatrix}.
\]

Notice that
\[
\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\begin{bmatrix}
UF & V \\
0 & W
\end{bmatrix}
= G
\]
and \(\varphi\) is a Jordan derivable mapping at \(G\). Hence we have
\[
\varphi\left(\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\begin{bmatrix}
UF & V \\
0 & W
\end{bmatrix}
+ \begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\varphi\left(\begin{bmatrix}
UF & V \\
0 & W
\end{bmatrix}
\right)
\right) + \begin{bmatrix}
UF & V \\
0 & W
\end{bmatrix}
\varphi\left(\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}\right) = \varphi(\tilde{G} + TS)
\]
\[
= \begin{bmatrix}
(F + UFx)A - A(F + UFx) & (b(UFy + zv) + 2c_0 - (F + UFx)c_0 + b(d)) \otimes g \\
0 & 0
\end{bmatrix}.
\]

Let us calculate
\[
\varphi\left(\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\begin{bmatrix}
U(I - F) & 0 \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}
\varphi\left(\begin{bmatrix}
U(I - F) & 0 \\
0 & 0
\end{bmatrix}\right)
\right) + \begin{bmatrix}
U(I - F) & 0 \\
0 & 0
\end{bmatrix}
\varphi\left(\begin{bmatrix}
X & Y \\
0 & Z
\end{bmatrix}\right) = \begin{bmatrix}
A(F + UFx) - (F + UFx)A + UXA - AUX & E_1 \otimes g \\
0 & 0
\end{bmatrix}.
\]

where \(E_1 = -(UX - UFx - F + I)c_0 + (U - UF)b(y) + (UA - AU)y - (UFA - AUF)y\). We then have by Eqs. (29) + (30) that
\[
\varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S) = \begin{bmatrix}
UXA - AUX & E_2 \otimes g \\
0 & 0
\end{bmatrix}.
\]

where \(E_2 = -(UX - I)c_0 + (U - UF)b(y) + b(d) + (UA - AU)y - (UFA - AUF)y + zb(v) + b(UFy)\). By replacing \(X\) and \(y\) in Eq. (26) by \(U\) and \(y\), respectively, we have \(b(UFy) = Ub(y) + (UA - AU)y\). Similarly, we have \(b(UFy) = Ub(y) + (UFA - AUF)y\). Thus \(E_2 = -UXc_0 + c_0 + b(d) + zb(v) + b(Uy)\). So \(\varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S) = \varphi(\tilde{G} + TS)\). It follows from Theorem 2.2 that \(\varphi\) is a derivation. Hence \(G\) is an Jordan all-derivable point of \(alg \mathcal{N}\).

The proof of the statement (2). For any \(S = \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix}\) and \(T = \begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}\) with \(ST = G\), then we have
\[
\begin{bmatrix}
F & d \otimes g \\
0 & 0
\end{bmatrix}
= G = ST = \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix}
\begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}
= \begin{bmatrix}
XU & (Xv + wy) \otimes g \\
0 & zwg \otimes g
\end{bmatrix}.
\]
i.e., \( XU = F \), \( Xv + wy = d \) and \( zw = 0 \). Especially, if we take \( S = \begin{bmatrix} \mu^{-1}X & \mu d \otimes g \\ 0 & 0 \end{bmatrix} \) and \( T = \begin{bmatrix} \mu U & 0 \\ 0 & \mu^{-1}g \otimes g \end{bmatrix} \) with \( XU = F \), then \( ST = G \). Since \( \varphi \) is a Jordan derivable mapping at \( G \), we have

\[
\varphi(G + TS) = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + T\varphi(S) + \varphi(T)S, \quad \text{i.e.,}
\]

\[
\begin{bmatrix}
\alpha(F + UX) + \mu^2B(Ud) + B(d) \\
0
\end{bmatrix} = \begin{bmatrix}
\mu^{-1}A(X) + \mu B(d) \\
0
\end{bmatrix} \begin{bmatrix}
\mu U & 0 \\
0 & \mu^{-1}g \otimes g
\end{bmatrix}
+ \begin{bmatrix}
\mu U \\
0
\end{bmatrix} \begin{bmatrix}
\mu^{-1}A(U) + \mu^{-1}C_0 \\
0
\end{bmatrix} \begin{bmatrix}
\mu^{-1}X & \mu d \otimes g \\
0 & 0
\end{bmatrix}
+ \begin{bmatrix}
\mu U \\
0
\end{bmatrix} \begin{bmatrix}
\mu^{-1}A(X) + \mu B(d) \\
0
\end{bmatrix} \begin{bmatrix}
\mu U & 0 \\
0 & \mu^{-1}g \otimes g
\end{bmatrix}
+ \begin{bmatrix}
\mu^{-1}X & \mu d \otimes g \\
0 & 0
\end{bmatrix} \begin{bmatrix}
\mu U & 0 \\
0 & \mu^{-1}g \otimes g
\end{bmatrix}.
\]

So we have \( A(X)U + A(U)X + XA(U) + UA(X) + \mu^2(UB(d) + B(d)U - B(Ud)) + \mu^{-2}(XC_0 + C_0X) = A(F + UX) + B(d) \). It is easy to get that \( C_0X + XC_0 = 0 \) and \( UB(d) + B(d)U = B(Ud) \). If we take \( X = I \), \( U = F \) and \( X = F \), \( U = I \) in the above equation, respectively, we have \( C_0 = 0 \) and \( B(d) = 0 \). Furthermore, \( B(Ud) = 0 \). Hence \( B(y) = 0 \) for any \( y \in H_- \). It follows that \( A(X)U + A(U)X + XA(U) + UA(X) = A(F + UX) \). Since \( F \) is a Jordan all-derivable point of \( A, A(\cdot) \) is a Jordan derivation on \( A \). Note that \( A \) is a nest algebra. It follows from the main theorem in [10] that \( A(\cdot) \) is an inner derivation. Hence there exists an operator \( A \in B(H_-) \) such that

\[
A(X) = AX - AX, \quad \forall X \in A
\]

furthermore the following matrix equation holds

\[
\varphi \left( \begin{bmatrix}
F & d \otimes g \\
0 & 0
\end{bmatrix} + \begin{bmatrix}
UX & (Uy + zv) \otimes g \\
0 & 0
\end{bmatrix} \right) = \varphi(G + TS) = \varphi(S)T + S\varphi(T) + T\varphi(S) + \varphi(T)S
\]

\[
= \left( \begin{bmatrix}
X & a(X) \otimes g \\
0 & \alpha(X)g \otimes g
\end{bmatrix} + \begin{bmatrix}
0 & b(y) \otimes g \\
0 & \beta(y)g \otimes g
\end{bmatrix} + z \begin{bmatrix}
0 & c_0 \otimes g \\
0 & \gamma_0g \otimes g
\end{bmatrix} \right) \begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}
+ \left( \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix} \left( \begin{bmatrix}
UA - AU & a(U) \otimes g \\
0 & \alpha(U)g \otimes g
\end{bmatrix} + \begin{bmatrix}
0 & b(v) \otimes g \\
0 & \beta(v)g \otimes g
\end{bmatrix} + w \begin{bmatrix}
0 & c_0 \otimes g \\
0 & \gamma_0g \otimes g
\end{bmatrix} \right) \begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}
\right)
+ \left( \begin{bmatrix}
UA - AU & a(U) \otimes g \\
0 & \alpha(U)g \otimes g
\end{bmatrix} + \begin{bmatrix}
0 & b(v) \otimes g \\
0 & \beta(v)g \otimes g
\end{bmatrix} + w \begin{bmatrix}
0 & c_0 \otimes g \\
0 & \gamma_0g \otimes g
\end{bmatrix} \right) \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix}
+ \begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix} \left( \begin{bmatrix}
X & AX - AX \\
0 & \alpha(X)g \otimes g
\end{bmatrix} + \begin{bmatrix}
0 & b(y) \otimes g \\
0 & \beta(y)g \otimes g
\end{bmatrix} + z \begin{bmatrix}
0 & c_0 \otimes g \\
0 & \gamma_0g \otimes g
\end{bmatrix} \right)
\]

for any \( XU = F \), \( Xv + wy = d \) and \( zw = 0 \). The above matrix equation implies that

\[
a(F + UX) + b(d + Uy + zv) = (XA - AX)v + (a(X) + b(y) + zc_0)w + U(aX) + b(y) + zc_0 + (\alpha(X) + \beta(y) + z\gamma_0)w + (UA - AU)y + (a(U) + b(v) + wc_0)z
\]

\[
+ X(a(U) + b(v) + zc_0) + (\alpha(U) + \beta(v) + wc_0)z
\]

and

\[
\alpha(F + UX) + \beta(d + Uy + zv) = 2w(\alpha(X) + \beta(y)) + 2z(\alpha(U) + \beta(v))
\]

for any \( X, U \in B(H_-), v, y \in H_- \) and \( w, z \in C \) with \( XU = F, Xv + wy = d, zw = 0 \).
If we take $X = \frac{1}{2}I$, $U = 2F$, $v = 2d$ and $z = w = 0$ in Eq. (33), then $2\alpha(F) + \beta(d) = 0$. For an arbitrary invertible operator $X \in A$, if we take $U = X^{-1}F$, $y = d$, $v = 0$ and $w = 1$ in Eq. (33), then we have $\alpha(X^{-1}FX) + \beta(X^{-1}Fd) = 2\alpha(X) + 2\beta(d)$. Replacing $X^{-1}$ by $2X^{-1}$ in the above equation, we have $\alpha(X^{-1}FX) + 2\beta(X^{-1}Fd) = \alpha(X) + 2\beta(d)$, so we get $\beta(X^{-1}Fd) = -\alpha(X)$. Let $2X$ be replaced, we have $\beta(X^{-1}Fd) = -4\alpha(X)$. So $\alpha(X) = 0$. It follows from Lemma 2.1 in [19] that $\alpha(X) = 0$ for any $X \in A$. For an arbitrary $y \in H_{\iota}$, taking $X = I$, $U = F$, $v = d$ and $z = w = 0$ in Eq. (33), we have $\beta(Fy) = 0$. For an arbitrary $y \in H_{\iota}$, taking $X = I$, $U = F$, $v = d - y$, $z = 0$ and $w = 1$ in Eq. (33), we have $\beta(Fy) = \beta(y)$. So $\beta(y) = 0$. Combining with Eq. (32), we see that

$$a(F + UX) + b(d + Uy + zv) = (Xa(X) - AX) = (a(X) + b(y) + zc_0) + U(a(x) + b(y) + zc_0) + (UA - AU)y + (a(U) + b(v) + wc_0)z + X(a(U) + b(v) + zc_0). \quad (34)$$

If we take $X = I$, $U = F$, $y = 0$, $v = d$ and $w = z = 0$ in Eq. (34), then $a(F) = Fa(I)$. If we take $X = I$, $U = F$, $y = 0$, $v = d$, $z = 0$ and $w = 1$ in Eq. (34), then $a(I) = -c_0$. If we take $X = F$, $U = I$, $y = d$, $v = 0$ and $z = 0$, $w = 1$ in Eq. (34), then $\gamma_y = 0$. For an arbitrary invertible operator $X \in A$ and complex number $\lambda \in C$, we may find a vector $v \in H_{\iota}$ such that $Xv = d$. If we take $U = X^{-1}F$, $y = 0$, $z = 0$ and $w = \lambda^{-1}$ in Eq. (34), let $\lambda \to 0$, then $a(X) = -Xc_0$ for any invertible operator $X \in A$. It follows from Lemma 2.1 in [19] that $a(X) = -Xc_0$ for any $X \in A$. For an arbitrary invertible operator $X \in A$, taking $U = X^{-1}F$, $y = d$, $v = 0$, $z = 0$ and $w = 1$ in Eq. (34), we have $b(X^{-1}Fd) = X^{-1}Fd (X^{-1}FA - AX^{-1}F)d$. For an arbitrary invertible operator $X \in A$ and vector $v \in H_{\iota}$, taking $U = X^{-1}F$, $Xv = d$, $z = 0$ and $w = 1$ in Eq. (34), we have

$$b(X^{-1}Fv) - X^{-1}Fd(Xv) - (X^{-1}FA - AX^{-1}F)Xv = b(Xv) - Xb(v) - (XA - AX)v.$$ Replacing $X$ by $2X$ in the above equation, we get $b(Xv) = Xb(v) + (XA - AX)v$. It follows from Lemma 2.1 in [19] that $b(Xv) = Xb(v) + (XA - AX)v$ for any $X \in A$ and $v \in H_{\iota}$. By imitating the proof of the statement (1), we can prove that $\varphi$ is a derivation. Hence $G$ is a Jordan all-derivable point of alg $A$.

The proof of the statement (3). Without loss of generality, we may assume that $F_0 = g \otimes g$. For any $S = \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix}$ and $T = \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix}$ with $ST = G$, then we have

$$\begin{bmatrix} 0 & 0 \\ 0 & zg \otimes g \end{bmatrix} = G = ST = \begin{bmatrix} X & y \otimes g \\ 0 & zg \otimes g \end{bmatrix} \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix} = \begin{bmatrix} XU & (Xv + wy) \otimes g \\ 0 & zwg \otimes g \end{bmatrix}.$$ i.e., $XU = 0$, $Xv + wy = 0$, $zw = 1$. Especially, if we take $S = \begin{bmatrix} \mu^{-1}X & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}$ and $T = \begin{bmatrix} \mu U & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}$, then $ST = G$. Since $\varphi$ is a Jordan derivable mapping at $G$, we have

$$\varphi(G + TS) = \varphi(ST + TS) = \varphi(S)T + S\varphi(T) + T\varphi(S) + \varphi(T)S.$$ So we have

$$\begin{bmatrix} A(UX) + 2C_0 \\ 0 \end{bmatrix} = \begin{bmatrix} \mu^{-1}A(X) + \mu C_0 & \mu U \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix} \begin{bmatrix} \mu^{-1}X & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}$$

$$+ \begin{bmatrix} \mu A(U) + \mu^{-1}C_0 & \mu^{-1}X \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix} \begin{bmatrix} \mu U & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}$$

$$+ \begin{bmatrix} \mu U & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix} \begin{bmatrix} \mu^{-1}A(X) + \mu C_0 & \mu U \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}$$

$$+ \begin{bmatrix} \mu^{-1}X & 0 \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix} \begin{bmatrix} \mu A(U) + \mu^{-1}C_0 & \mu U \\ 0 & \mu \otimes \mu \otimes g \end{bmatrix}.$$ Furthermore, we have $A(X)U + A(U)X + XA(U) + UA(X) + \mu^2 (UC_0 + C_0U) + \mu^{-2} (XC_0 + C_0X) = A(UX) + 2C_0$. It is easy to get that $C_0 = 0$. Thus $A(X)U + A(U)X + XA(U) + UA(X) = A(UX)$. Since
0 is a Jordan all-derivable point of $A$ in Theorem 2.1, it follows that $A(\cdot)$ is a Jordan derivation on $A$. Note that $A$ is a nest algebra. It follows from the theorem in [10] that $A(\cdot)$ is an inner derivation. Hence there exists an operator $A \in B(H_-)$ such that

$$A(X) = XA - AX, \quad \forall X \in A.$$ 

Furthermore, the following matrix equation holds

$$\varphi \left( \begin{bmatrix} 0 & 0 \\ 0 & g \otimes g \end{bmatrix} + \begin{bmatrix} UX & (Uy + zv) \otimes g \\ 0 & g \otimes g \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} XA - AX & a(X) \otimes g \\ 0 & \alpha(X)g \otimes g \end{bmatrix} + \begin{bmatrix} B(y) & b(y) \otimes g \\ 0 & \beta(y)g \otimes g \end{bmatrix} + z \begin{bmatrix} c_0 \otimes g \\ 0 \end{bmatrix} \right) \begin{bmatrix} U & v \otimes g \\ 0 & wg \otimes g \end{bmatrix}.$$ 

for any $XU = 0$, $Xv + wy = 0$ and $zw = 1$. The above matrix equation implies that

$$B(Uy + zv) = B(y)U + UB(y), \quad (35)$$

$$a(UX) + b(Uy + zv) = (XA - AX)v + (UA - AU)y + w(a(X) + b(y)) + wXc_0 + U(a(X) + b(y)) + X(a(U) + b(v)) + zUc_0 + z(a(U) + \beta(v)) \quad (36)$$

and

$$\alpha(UX) + \beta(Uy + zv) = 2w(\alpha(X) + \beta(y)) + 2z(\alpha(U) + \beta(v)) + 2\gamma_0$$

for any $X, U \in B(H_-), v, y \in H_-$ and $w, z \in \mathbb{C}$ with $XU = 0$, $Xv + wy = 0$ and $zw = 1$.

If we take $X = U = 0, y = v = 0$ and $z = w = 1$ in Eq. (35), then $C_0 = 0$. For an arbitrary $v \in H_-$, if we take $X = I, U = 0, v = y$ and $z = w = -1$ in Eq. (35), then $B(v) = 0$ for any $v \in H_-$. If we take $X = U = 0, y = v = 0$ and $z = w = 1$ in Eq. (37), then $\gamma_0 = 0$. For an arbitrary $X \in A$, if we take $U = 0, y = v = 0$ and $z = w = 1$ in Eq. (37), then $\alpha(X) = 0$ for any $X \in A$. For an invertible $v \in H_-$, if we take $X = U = 0, y = v = 0$ and $z = w = 1$, then $\beta(v) = 0$ for any $v \in H_-$. For an arbitrary $X \in A$, if we take $U = 0, y = v = 0$ and $z = w = 1$ in Eq. (37), then $\alpha(X) = -Xc_0$ for any $X \in A$. In particular, $a(I) + c_0 = 0$. Thus we have $b(Uy) = (XA - AX)v + wb(y) + Ub(y) + Xb(v) + (UA - AU)y$. For an arbitrary $v \in H_-$ and $X \in A$, if we take $U = 0, y = -Xv$ and $z = w = 1$ in Eq. (37), then $b(Xv) = Xb(v) + (XA - AX)v$.

In other words, we have

$$\varphi \left( \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix} \right) = \varphi \left( \begin{bmatrix} X & 0 \\ 0 & Z \end{bmatrix} + \begin{bmatrix} 0 & Y \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right) = \left[ \begin{bmatrix} Xc - CX & -Xc_0 \otimes g \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} b(y) \otimes g \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & zc_0 \otimes g \\ 0 & 0 \end{bmatrix} \right].$$

We now claim that $\varphi$ is a Jordan derivation on $\text{alg } \mathcal{N}$. We only need to show that $\varphi$ is a Jordan derivable mapping at $I_H$. In fact, for any $S = \begin{bmatrix} X & Y \\ 0 & Z \end{bmatrix}$ and $T = \begin{bmatrix} U & V \\ 0 & W \end{bmatrix}$ in $\text{alg } \mathcal{N}$ with $ST = I_H$, we have
\[
\begin{bmatrix}
1 & 0 \\
0 & g \otimes g
\end{bmatrix} = \begin{bmatrix}
XU & (Xv + wy) \otimes g \\
0 & zwg \otimes g
\end{bmatrix},
\]

i.e., \(XU = I, Xv + wy = 0\) and \(zw = 1\). Since
\[
\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}
\begin{bmatrix}
0 & V \\
W & 0
\end{bmatrix} = \begin{bmatrix}
0 & 0 \\
g \otimes g & 0
\end{bmatrix} = G \text{ and } \varphi \text{ is a Jordan}
\]
derivable mapping at \(G\), thus we have
\[
\varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)\begin{bmatrix}
0 & V \\
W & 0
\end{bmatrix} + \varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)\begin{bmatrix}
0 & V \\
W & 0
\end{bmatrix} \varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)
\]
\[+ \varphi\left(\begin{bmatrix}
0 & 0 \\
V & 0
\end{bmatrix}\right)\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix} = \varphi(G + TS) = \begin{bmatrix}
0 & (2c_0 + b(zv)) \otimes g \\
z & 0
\end{bmatrix}.
\]

On the other hand, we calculate
\[
\varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)\begin{bmatrix}
U & 0 \\
0 & 0
\end{bmatrix} + \varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)\begin{bmatrix}
U & 0 \\
0 & 0
\end{bmatrix} \varphi\left(\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix}\right)
\]
\[+ \varphi\left(\begin{bmatrix}
0 & 0 \\
U & 0
\end{bmatrix}\right)\begin{bmatrix}
X & 0 \\
0 & Z
\end{bmatrix} = \begin{bmatrix}
UXA - AUX & (-c_0 - UXc_0 + Ub(v) + UAy - AUy) \otimes g \\
0 & 0
\end{bmatrix}.
\]

By the equation \(b(Xv) = Xb(v) + (XA - AX)v\) for any \(v \in H_-\) and \(X \in \mathbb{A}\), thus \(b(Uy) = Ub(y) + (UA - AU)y\). On the other hand, we prove that
\[
\varphi(\lambda H + TS) = \varphi\left(\begin{bmatrix}
I & UX \\
0 & 2g \otimes g
\end{bmatrix} \otimes g \right) = \begin{bmatrix}
UXA - AUX & (-UXc_0 + b(Uy + zv) + c_0) \otimes g \\
0 & 0
\end{bmatrix}.
\]

It follows from the above two equations that \(\varphi(\lambda H) = \varphi(S)T + S\varphi(T) + \varphi(T)S + T\varphi(S)\). So \(\phi\) is Jordan derivable mapping at \(I_H\) and it follows from Theorem 2.3 that \(\phi\) is a derivation. Hence \(G\) is a Jordan all-derivable point of \(alg \mathbb{N}\).

(4) If \(\phi\) is a Jordan derivable mapping at \(G = \begin{bmatrix}
0 & D \\
0 & 0
\end{bmatrix}\) on \(alg \mathbb{N}\) \((D \neq 0)\). It is easy to verify that \(A(\cdot)\)

is a Jordan derivable mapping at 0 on \(\mathbb{A}\). By Theorem 2.1, \(\mathbb{A}\) is a Jordan derivation. Then there exists an operator \(A \in B(H_-)\) such that \(A(X) = AX - AX\) for any \(X \in \mathbb{A}\).

For any \(S = \begin{bmatrix}
X & y \otimes g \\
0 & zg \otimes g
\end{bmatrix}\) and \(T = \begin{bmatrix}
U & v \otimes g \\
0 & wg \otimes g
\end{bmatrix}\) with \(ST = G\), then the following three equations

hold by imitating the proof of the statement (1)
\[
B(d + Uy + zv) = B(y)U + UB(y) + z(C_0U + UC_0) + XB(v) + B(v)X + w(XC_0 + C_0X),
\]
\[
b(d + Uy + zv) + a(UX) = (XA - AX)v + zUC_0 + (UA - AU)y + wXC_0
\]
\[+ w(a(X) + b(y)) + X(a(U) + b(v)) + U(a(X) + b(y) + zy_0v) + w_0y,
\]
and
\[
\beta(d + Uy + zv) + \alpha(UX) = 2w(\alpha(X) + \beta(y)) + 2z(\alpha(U) + \beta(v))
\]
for any \(X, U \in B(H_-), v, y \in H_-\) and \(w, z \in \mathbb{C}\) with \(XU = 0, Xv + wy = d, zw = 0\).

If we take \(X = I, U = 0, v = 0, w = d, z = 0\) and \(w = 1\) in Eq. (40), then \(B(d) + 2C_0 = 0\). If we take \(X = I, U = 0, v = 0, w = d, z = 0\) and \(w = -1\) in Eq. (40), then \(B(d) - 2C_0 = 0\). Thus \(C_0 = B(d) = 0\). For an arbitrary \(v \in H_-\), if we take \(X = I, U = 0, v = d - v\) and \(z = 0\) and \(w = 1\) in Eq. (40), then \(B(v) = 0\) for any \(v \in H_-\).

For an arbitrary invertible operator \(X \in \mathbb{A}\), we may find a vector \(v \in H_-\) such that \(Xv = d\). If we take \(U = 0, y = 0, z = 0\) and \(w = 1\) in Eq. (42), then \(2\alpha(X) = \beta(d)\) for any invertible operator \(X \in \mathbb{A}\). Hence \(\alpha(X) = 0\). It follows from Lemma 2.1 in [19] that \(\alpha(X) = 0\) for any \(X \in \mathbb{A}\). For an arbitrary
y ∈ H−, if we take X = I, U = 0, v = d − y, z = 0 and w = 1 in Eq. (42), then β(y) = 0 for any y ∈ H−.

For an arbitrary X ∈ A, if we take U = 0, y = d, v = 0, z = 0 and w = 1 in Eq. (41), then we have

\[ a(X) + Xc_0 + \gamma_0 d = 0 \]

for any X ∈ B(H−). Furthermore, we have

\[ 2a(X) + 2Xc_0 + \gamma_0 d = a(2X) + 2Xc_0 + \gamma_0 d = 0. \]

It follows from the above two equations that \( a(X) = -Xc_0 \) and \( \gamma_0 = 0 \). In particular, \( a(l) + c_0 = 0 \).

For an arbitrary invertible operator X ∈ A and for any \( v \in H− \), taking \( U = 0, Xv + y = d, z = 0 \) and \( w = 1 \) in Eq. (41), we have \( b(Xv) = Xb(v) + (XA - AX)v \).

4. Jordan all-derivable points in the algebra of \( n \times n \) upper triangular matrices

In this section, we always write \( T_M_n \) for the algebra of all \( n \times n \) upper triangular matrices. We use the symbols \( H \) and \( \{e_i : i = 1, 2, \ldots, n\} \) to denote the Euclidean \( n \)-dimensional space and its normal orthogonal basis, respectively. We regard an \( n \times n \) matrix as an operator on Euclidean \( n \)-dimensional space \( H \), naturally. Thus \( T_M_n \) is a nest algebra associated with \( \mathcal{N} \), where \( \mathcal{N} = \{N_i : 0 ≤ i ≤ n\} \) and \( N_i = \text{span}\{e_j : 0 ≤ j ≤ i\} \) (\( e_0 = 0 \)).

**Lemma 4.1.** Let \( \mathcal{N} \) be a complete nest on \( H \), and let \( J \) be an invertible operator in \( B(H) \). Then \( G \in \text{alg} \mathcal{N} \) is a Jordan full-derivable point of \( \text{alg} \mathcal{N} \) if and only if \( JG^{-1} \) is a Jordan full-derivable point of the operator algebra \( J \text{alg} \mathcal{N}^{-1} \).

**Proof.** \( \Rightarrow \) Suppose that \( G \) is a Jordan full-derivable point of \( \text{alg} \mathcal{N} \). Let \( \psi \) be a Jordan full-derivable mapping at \( JG^{-1} \) on \( J \text{alg} \mathcal{N}^{-1} \). Define a mapping \( \psi : \text{alg} \mathcal{N} \to \text{alg} \mathcal{N} \) as follows \( \psi(S) = J^{-1}\psi(JS)^{-1}J \). For any \( S, T \in \text{alg} \mathcal{N} \) with \( ST = G \), we have \( \psi(ST + TS) = J^{-1}\psi((ST)^{-1} + (TS)^{-1})J \).

**Theorem 4.2.** \( G \in T_M_n \) is a Jordan full-derivable point in \( T_M_n \).

**Proof.** Let \( e_i \) (\( i = 1, 2, \ldots, n \)) be a normal orthogonal basis on an \( n \)-dimensional Euclidean space \( H \). We may regard \( T_M_n \) as a nest algebra associated with \( \mathcal{N} \), where \( \mathcal{N} = \{N_k : 0 ≤ k ≤ n\} \) and \( N_k = \{e_i : 0 ≤ i ≤ k\} \) (\( e_0 = 0 \)). Thus \( H− = N_{n−1} \). So \( \text{dim}(H−) \perp = \text{dim}(N_{n−1}) \perp = 1 \).

Suppose that \( 0 ≠ G \in T_M_n \). Thus we only need to prove that \( G \) is a Jordan full-derivable point of \( T_M_n \). If we write

\[
G = \begin{bmatrix}
a_{11} & a_{12} & \cdots & a_{1n} \\
0 & a_{22} & \cdots & a_{2n} \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & a_{nn}
\end{bmatrix},
\]

then \( G \) is a Jordan full-derivable point of \( T_M_n \).
we divide the proof of the statement into the following two cases.

Case 1. Suppose that $a_{nn} = 0$. then $G$ can be represented as a $2 \times 2$ block matrix relative to the orthogonal decomposition $H = N_{n-1} \oplus \text{span}\{e_n\}$ as follows $G = \begin{bmatrix} F & D \\ 0 & 0 \end{bmatrix}$. If $F \neq 0$, by hypothesis, $F$ is a Jordan full-derivable point of $P(N_{n-1})T\mathcal{M}_nP(N_{n-1}) = T\mathcal{M}_{n-1}$. It follows from the statement (2) in Theorem 3.1 that $G$ is a Jordan full-derivable point of $T\mathcal{M}_n$. If $F = 0$, then $D \neq 0$. It follows from the statement (4) in Theorem 3.1 that $G$ is a Jordan full-derivable point of $T\mathcal{M}_n$.

Case 2. Suppose that $a_{nn} \neq 0$. If $F \neq 0$, by hypothesis, $F$ is a Jordan full-derivable point of $T\mathcal{M}_{n-1}$. It follows from the statement (1) in Theorem 3.1 that $G$ is a Jordan full-derivable point of $T\mathcal{M}_n$. If $F = 0$, then there exists upper triangular invertible matrix $J$ such that $JGJ^{-1} = \begin{bmatrix} 0 & 0 \\ 0 & a_{nn} \end{bmatrix}$. It follows from the statement (3) in Theorem 3.1 that $JGJ^{-1}$ is a Jordan full-derivable point of $T\mathcal{M}_n$. Notice that $JT\mathcal{M}_nj^{-1} = T\mathcal{M}_n$. Hence $JGJ^{-1}$ is a Jordan full-derivable point of $JT\mathcal{M}_nj^{-1}$. It follows from Theorem 3.1 that $G$ is a Jordan full-derivable point of $T\mathcal{M}_n$. Finally, by Theorem 2.1, we can obtain that $G$ is a Jordan full-derivable point when $G = 0$. □

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