On the metrizability of cone metric spaces

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Abstract

We have shown in this paper that a (complete) cone metric space \((X, E, P, d)\) is indeed (completely) metrizable for a suitable metric \(D\). Moreover, given any finite number of contractions \(f_1, \ldots, f_n\) on the cone metric space \((X, E, P, d)\), \(D\) can be defined in such a way that these functions become also contractions on \((X, D)\).

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1. Introduction

Starting with the initial paper of Huang and Zhang in 2007 [3], there have been a lot of papers dealing with the theory of cone metric spaces. Cone metric spaces are in crude terms similar to metric spaces with a positive cone in a Banach space replacing positive real numbers. Here a metric is represented for a cone metric space \((X, E, P, d)\) that induces the cone metric topology of \(d\) on \(X\). Recently, there have been several attempts to reduce cone metric spaces to their metric counterparts. (See for example [1,2,6].) In this paper we make another attempt of this kind, giving an explicit construction of a standard metric on a given cone metric space. This gives rise to a feasible uniform way of dealing with cone metric spaces and reproving some fixed point results available for example in [4,5]. It should be noted that, although this construction preserves some basic contractive properties of given maps (mostly linear ones), not all contractive conditions can be reduced to their metric counterparts in this way.

We begin with a short introduction to cone metric spaces and discuss the metrizability of a cone metric space in the sequel.

2. Preliminaries

Let \(E\) be a real Banach space and \(P\) a subset of \(E\). We call \(P\) a cone if

1. \(P\) is closed, \(P \neq \emptyset\) and \(P \neq \{0\}\);
2. \(0 \leq a, b \in \mathbb{R}\) and \(x, y \in P\) \(\Rightarrow ax + by \in P\);
3. \(x \in P\) and \(-x \in P\) \(\Rightarrow x = 0\).

We call \((E, P)\) a cone space and equip it with a partial ordering \(\leq\) as:

\[ x \leq y \iff y - x \in P. \]
Furthermore, we write $x \ll y$ and say $x$ is way-below $y$ whenever $y - x \in \text{Int}P$. The way-below relation is transitive and antisymmetric but not in general reflective.

A cone space $(E, P)$ is called regular if for each bounded increasing sequence

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y$$

in $E$, there exists $x \in E$ such that $\lim x_n = x$ (with the topology of the Banach space $E$). Obviously, one can replace “increasing” by “decreasing” in the above definition.

A cone space $(E, P)$ is called normal if there is a number $K > 0$ in $\mathbb{R}$ such that for all $x, y \in E$,

$$0 \leq x \leq y \implies \|x\| \leq K\|y\|.$$  

Now let $(E, P)$ be a cone space, $X$ a nonempty set and $d : X \times X \to E$ a mapping that satisfies:

1. for all $x, y \in X$, $0 \leq d(x, y)$ and $d(x, y) = 0$ iff $x = y$;
2. $d(x, y) = d(y, x)$ for all $x, y \in X$;
3. $d(x, y) \leq d(x, z) + d(z, y)$ for all $x, y, z \in X$.

Then $d$ is called a cone metric on $X$ and $(X, E, P, d)$ a cone metric space which we often write as $(X, E, P)$ where it brings about no confusion.

A sequence $(x_n)_{n \in \mathbb{N}}$ of elements of $X$ is called to converge to an element $x \in X$ whenever for each $\epsilon > 0$ in $E$, there is $N$ such that $d(x_n, x) \ll \epsilon$ for all $n > N$. In the same way we call $(x_n)_{n \in \mathbb{N}}$ a Cauchy sequence if for each $\epsilon > 0$, there exists $N$ such that for each $m, n > N$, $d(x_n, x_m) \ll \epsilon$. A cone metric space is complete when any Cauchy sequence is convergent.

It is well known that in a cone metric space $(X, E, P, d)$ where $(E, P)$ is normal (we will simply call which: a normal cone metric space), a sequence $(x_n)$ is convergent to $x$ iff $d(x_0, x) \to 0$ ($n \to \infty$) and Cauchy iff $d(x_m, x_n) \to 0$ as $(m, n \to \infty)$. Note that $(x_n)$ is a sequence in $X$ and its convergence is discussed under cone metric topology, whereas $d(x_n, x)$ is a sequence of elements of $E$ with the topology induced by norm.

Defining $B(x, r) = \{y \in X : d(x, y) \ll r\}$ for each $x \in X$ and $r \in \text{Int}P$, one can easily prove that the collection $\{B(x, r) : x \in X, r \gg 0\}$ forms a basis for cone metric topology under which the above definitions of convergent and Cauchy sequences are fully justified.

We mean by a contraction on the cone metric space $X$, a mapping $f : X \to X$ associated with a constant $c \in \mathbb{R}^+$ such that $d(fx, fy) \ll cd(x, y)$ for all $x, y \in X$. It is proved in [1] that each contraction on a complete normal cone metric space $(X, E, P, d)$ has a unique fixed point. The same result is shown in [3] with the condition of normality omitted.

3. A cone metric space is metrizable

In this section we prove the metrizability of cone metric spaces. But before proceeding further, we need a couple of useful lemmas:

**Lemma 3.1.** Let $(X, E)$ be a cone space with $x \in P$ and $y \in \text{Int}P$. Then one can find $n \in \mathbb{N}$ such that $x \ll ny$.

**Proof.** We have $y = \lim_{n \to \infty}(y - x/n) \in \text{Int}P$. So, there exists $n_0$ with $y - x/n_0 \in \text{Int}P$ since $\text{Int}P$ is open. Consequently $ny - x \in \text{Int}P$ which means $x \ll ny$. \(\square\)

**Lemma 3.2.** Let $y \in \text{Int}P$. Then

$$\forall x(x \gg y \implies x \in \text{Int}P).$$

**Proof.** We have $x \in P$ since $x = y + (x - y)$ and $y, x - y \in P$. The mapping $f : E \to E$ ($u \to u + (y - x)$) is continuous, so $f^{-1}(U)$ is open when $U$ is the open neighborhood of $y$ in $P$. We need only to show that $f^{-1}(U) \subseteq P$; then since $x \in f^{-1}(U)$, we have $x \in \text{Int}P$. We have $f^{-1}(U) = \{f^{-1}(t) : t \in U\}$ for $f$ is a bijection. But $f^{-1}(t) = t + (x - y)$ for each $t$. Thus if $t \in U \subseteq P$ then $f^{-1}(t) \in P$. \(\square\)

As an instant result of the above lemmas we have:

**Lemma 3.3.** In a cone space $(E, P)$:

$$x \ll y \ll z \implies x \ll z.$$  

**Proof.** It is the case that $z - y \in \text{Int}P$ and $z - x \gg z - y$. By the previous lemma, $z - x \in \text{Int}P$. \(\square\)
Theorem 3.4. Let \((X, E, P, d)\) be a cone metric space, \(\alpha \in \text{IntP}\) and \(c < 1\) be in \(\mathbb{R}^+\). Then there exists a metric \(D : X \times X \to \mathbb{R}^+\) which induces the same topology on \(X\) as the cone metric topology induced by \(d\). Moreover, a sequence \((x_n)\) is Cauchy in \((X, E, P, d)\) if and only if it is Cauchy in \((X, D)\). In particular, \((X, E, P, d)\) is complete iff \((X, D)\) is complete.

Proof. Set \(d = 1/c\) and define a function \(A : X \times X \to \mathbb{R}^+\) as follows:

\[
A(x, y) = \begin{cases} d_{\min(k: d(x, y) \ll d_k \alpha)} & \text{if } d(x, y) \neq 0, \\ 0 & \text{if } d(x, y) = 0. \end{cases}
\]

Note that \(k\) is in \(\mathbb{Z}\) and \(A\) is well defined by Lemma 3.2. One can easily check that \(A(x, y) = A(y, x)\) and \(A(x, y) = 0 \iff x = y\).

Now we define \(D : X \times X \to \mathbb{R}^+\) in this way:

\[
D(x, y) = \inf \left\{ \sum_{i=1}^{n-1} A(x_i, x_{i+1}) : x_1 = x, \ldots, x_n = y \right\}.
\]

\(D\) is obviously symmetric and \(D(x, y) = 0\) if and only if \(x = y\). So, for \(D\) to be a metric, it suffices to prove that it satisfies the triangle inequality: \(D(x, y) \leq D(x, z) + D(z, y)\). For \(\epsilon > 0\), we show that \(D(x, y) \leq D(x, z) + D(z, y) + \epsilon\). By definition, there exists \(x_1 = x, \ldots, x_n = z\) with \(\sum A(x_i, x_{i+1}) \leq D(x, z) + \epsilon/2\) and \(y_1 = z, \ldots, y_m = y\) with \(\sum A(y_i, y_{i+1}) \leq D(z, y) + \epsilon/2\). Thus \(D(x, y) \leq \sum A(x_i, x_{i+1}) + \sum A(y_i, y_{i+1}) \leq D(x, z) + D(z, y) + \epsilon\). Thus \(D\) is a metric. Now we claim that \(D\) induces the same topology as the cone metric topology of \(d\).

Denoting by \(B_d(x, \delta)\) the set \(\{y : d(x, y) \ll \delta\}\) and by \(B_D(x, r)\) the set \(\{y \in X : D(x, y) < r\}\) for each \(x \in X, \delta \in \text{IntP}\) and \(r \in \mathbb{R}^+\), we will show that each \(B_d(x, \delta)\) contains some \(B_D(x, r)\) and vice versa.

Consider \(B_D(x, r)\) for \(x \in X\) and \(r \in \mathbb{R}^+\). One can find \(k \in \mathbb{Z}\) with \(d_k < r\). Choosing \(\delta \ll d_k \alpha\), it is the case that if \(d(x, y) \ll \delta\) then \(A(x, y) \ll d_k \alpha\). This readily yields \(B_d(x, \delta) \subseteq B_D(x, r)\) for each \(x \in X\).

For the opposite inclusion consider \(B_d(x, \delta)\) for \(x \in X\) and \(\delta \in \text{IntP}\). For each \(x \in X\) and \(\delta \in \mathbb{R}^+\), it is the case that if \(D(x, y) < r\) then one can find \(x_1 = x, \ldots, x_n = y\) in \(X\) with \(\sum_{i=1}^{n-1} A(x_i, x_{i+1}) < r\). But for each \(i < n\) we have \(d(x_i, x_{i+1}) \ll \Lambda(x_i, x_{i+1})\alpha\) and hence:

\[
d(x, y) \leq d(x_1, x_2) + \cdots + d(x_n, x_{n-1}) \ll \Lambda(x_1, x_2) + \cdots + \Lambda(x_{n-1}, x_n) \ll r\alpha.
\]

Accordingly, choosing \(r\) to satisfy \(r\alpha \ll \delta\), we have \(d(x, y) \ll \delta\) and \(B_d(x, \delta) \subseteq B_D(x, r)\).

Now let \((x_n)\) be a Cauchy sequence in \((X, E, P, d)\). For \(r > 0\) in \(\mathbb{R}\), we find an element \(\delta\) with \(B_d(x_n, \delta) \subseteq B_D(x_n, r)\) for each \(n\) (it is worth noting that in the above argument the choice of \(\delta\) depends only upon \(r\) and not on \(x\)). But there exists \(N\) such that \(x_m \in B_D(x_0, \delta)\) for each \(n, m > N\), since \((x_n)\) is Cauchy. So for \(n, m > N\), \(D(x_m, x_n) < r\). A similar discussion works for showing that a Cauchy sequence in \((X, D)\) is Cauchy in \((X, E, P, d)\).

The last statement of the theorem can be obtained easily by the fact that both \(d\) and \(D\) induce the same topology and therefore the same notion of convergence on \(X\).

As mentioned before, the metric \(D\) can be defined as to have a good behavior towards contractions:

Theorem 3.5. Each finite set of contractions on a cone metric space \((X, E, P, d)\) with \(\alpha \in \text{IntP}\) is in particular a set of contractions on \((X, D)\) for some metric \(D\) representing \((X, E, P, d)\).

Proof. Let \(f_i : X \to X\) \((i = 1, \ldots, n)\) be contractions on a cone metric space \((X, E, P, d)\) with contraction constants \(c_i < 1\) respectively. Consider a constant \(c < 1\) such that \(c < c_i\) for each \(i\) and define a metric \(D\) on \(X\) by setting \(d = 1/c\) in the previous theorem. We show that \(D(f_i(x), f_i(y)) \ll d\alpha\) for all \(x, y \in X\) and \(i = 1, \ldots, n\). For each \(x, y \in X\), \(d(x, y) \ll d_k\alpha\).

Despite the intricacies of their definition, cone metric spaces can in part be dealt with as the familiar metric spaces. However, considering certain topological groups in place of Banach spaces may result in the construction of new spaces which are not in general metrizable. This can serve as a topic for further studies.

References