Dual Series Relations Involving Generalized Laguerre Polynomials*

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1. INTRODUCTION.

During the last ten years several workers have devoted considerable attention to the investigation of dual equations involving trigonometric series, the Fourier-Bessel series, the Fourier-Legendre series, the Dini series, and series of Jacobi and Laguerre polynomials [see 2 to 4, 7 to 15, 17, 18]. The object of the present paper is to consider the problem of determining the sequence \( \{A_n\} \) such that:

\[
\sum_{n=0}^{\infty} \frac{A_n}{I(\alpha + n + 1)} L_n^{(\alpha)}(x) = f(x), \quad 0 \leq x < y; \quad (1.1)
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{I(\beta + n + 1)} L_n^{(\alpha)}(x) = g(x), \quad y < x < \infty; \quad (1.2)
\]

where \( L_n^{(\alpha)}(x) \) is the generalized Laguerre polynomial defined by

\[
(1 - t)^{-\alpha-1} \exp \left\{ -\frac{xt}{1 - t} \right\} = \sum_{n=0}^{\infty} L_n^{(\alpha)}(x) t^n, \quad (1.3)
\]

\( f(x) \) and \( g(x) \) are prescribed functions of \( x \), and \( \alpha, \beta, \nu, \sigma > -1 \).

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† In another paper [Pacific J. Math. 30 (1969), 525-527] the present author has investigated an exact solution of the dual equations (1.1) and (1.2) by using a generalization of the multiplying factor technique which was developed by Noble [7] for solving dual equations involving series of Jacobi polynomials and employed subsequently by John S. Lowndes [Pacific J. Math. 25 (1968), 123-127] and Richard Askey [J. Math. Anal. Appl. 24 (1968), 677-685] for solving essentially the same special case of the dual equations (1.1) and (1.2). It may be of interest to remark that the observations of Lowndes and Askey follow immediately on appropriately specializing the solution presented here.
With a view to simplifying the calculations, we shall apply the method developed recently by Sneddon and Srivastav [8] and split the problem posed by the pair of dual equations (1.1) and (1.2) into the following two parts:

**Problem (i).** Determine the constants \( \{A_n\} \) satisfying the dual series relations

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(x+n+1)} L_n^{(\alpha)}(x) = f(x), \quad 0 \leq x < y, \tag{1.4}
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\alpha)}(x) = 0, \quad y < x < \infty, \tag{1.5}
\]

\[\alpha, \beta, \nu, \sigma > -1.\]

**Problem (ii).** Determine the constants \( \{A_n\} \) satisfying the dual series relations

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(x+n+1)} L_n^{(\alpha)}(x) = 0, \quad 0 \leq x < y, \tag{1.6}
\]

\[
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta+n+1)} L_n^{(\alpha)}(x) = g(x), \quad y < x < \infty, \tag{1.7}
\]

\[\alpha, \beta, \nu, \sigma > -1.\]

The solution of the general problem is then obtained by merely adding the solutions of the individual problems (i) and (ii).

In what follows we shall observe that the classes of the functions \( f(x) \) and \( g(x) \) for which the problem under consideration is solvable must be such that

(a) \( F(x) = x^r f(x) \) is continuously differentiable on \( 0 \leq x \leq a < y, \)

(b) \( G(x) = \int_a^x e^{-x} g(x) \, dx \) is continuously differentiable for \( y < 1 \leq x < \infty. \)

It might be worthwhile to remark that our analysis is purely formal and that in the special case when \( \alpha = \beta + \frac{1}{2} = \nu = \sigma \) and \( \lambda = \mu = \frac{1}{2} \) the results presented here will yield the recent observations of Srivastava [12].

2. **List of Known Results.**

For the sake of ready reference we list here the following results that will be required in the course of our investigation:—
(i) The orthogonality property of Laguerre polynomials, viz. [6, p. 292(2), p. 293(3)]

\[
\int_0^\infty x^\sigma e^{-x} L_m^{(\sigma)}(x) L_n^{(\sigma)}(x) \, dx = \frac{\Gamma(\sigma + n + 1)}{n!} \delta_{mn}, \quad \sigma > -1, \tag{2.1}
\]

where \(\delta_{mn}\) is the Kronecker delta.

(ii) The formula (27), p. 190 of [5]:

\[
\frac{d^m}{dx^m} \{x^\alpha L_n^{(\alpha)}(x)\} = (\alpha - m + n + 1)_m x^{\alpha-m} L_n^{(\alpha - m)}(x), \tag{2.2}
\]

which, for \(m = 1\), yields

\[
\frac{dx}{dx} \{x^\alpha L_n^{(\alpha)}(x)\} = (\alpha + n) x^{\alpha-1} L_n^{(\alpha - 1)}(x). \tag{2.3}
\]


\[
\int_0^\infty e^{-y} L_n^{(\alpha)}(y) \, dy = e^{-x} L_n^{(\alpha - 1)}(x), \quad \alpha > -1. \tag{2.4}
\]

(iv) The following forms of the known integrals (20), p. 405 of [6] and (30), p. 191 of [5]:

\[
\int_0^\infty e^{-y} (y - x)^{\alpha - 1} L_n^{(\alpha)}(y) \, dy = \Gamma(\beta) e^{-x} L_n^{(\alpha - \beta)}(x) \tag{2.5}
\]

and

\[
\int_0^\infty y^\gamma (x - y)^{\alpha - 1} L_n^{(\alpha)}(y) \, dy = \frac{\Gamma(\alpha + n + 1) \Gamma(\beta)}{\Gamma(\alpha + \beta + n + 1)} x^{\alpha+\beta} L_n^{(\alpha + \beta)}(x), \tag{2.6}
\]

where \(\alpha > -1\) and \(\beta > 0\).

(v) We also note that if \(f(x)\) and \(f'(x)\) are continuous on \(0 \leq x \leq a\), then the Abel integral equation

\[
\int_0^x \frac{\phi(y)}{(x - y)^\lambda} \, dy = f(x) \quad (0 < x < a, 0 < \lambda < 1), \tag{2.7}
\]

has a continuous solution in the form [1, p. 134]

\[
\phi(y) = \frac{\sin(\pi \lambda)}{\pi} \frac{d}{dy} \int_0^y \frac{f(x)}{(x - y)^{2-\lambda}} \, dx. \tag{2.8}
\]
Furthermore, it can be proved fairly easily by using the techniques illustrated in [16, p. 229] that the integral equation

$$\int_{x}^{\infty} \frac{\phi(y)}{(y-x)^\lambda} \, dy = f(x) \quad (x > 1, 0 < \lambda < 1), \quad (2.9)$$

possesses a continuous solution given by

$$\phi(y) = -\frac{\sin(\pi \lambda)}{\pi} \frac{d}{dy} \int_{y}^{\infty} \frac{f(x)}{(x-y)^{\lambda-1}} \, dx, \quad (2.10)$$

where $f(x)$ is continuously differentiable on the interval $[1, \infty)$.

3. Investigation of the Problem (i).

We begin with the assumption that for $0 < x < y$ and $0 < \lambda < 1$,

$$\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta + n + 1)} L^{(\omega)}_{n}(x) = -e^{x} \frac{d}{dx} \int_{x}^{y} \frac{\phi(\xi)}{(\xi-x)^{\lambda}} \, d\xi. \quad (3.1)$$

Making use of the orthogonality relation (2.1), we have

$$A_n = -\frac{n! \Gamma(\beta + n + 1)}{\Gamma(\sigma + n + 1)} \int_{x}^{y} x^2 L^{(\omega)}_{n}(x) \left( \frac{d}{dx} \int_{x}^{y} \frac{\phi(\xi)}{(\xi-x)^{\lambda}} \, d\xi \right) dx, \quad (3.2)$$

and since

$$\frac{d}{dx} \int_{x}^{y} \frac{\phi(\xi)}{(\xi-x)^{\lambda}} \, d\xi = -\frac{\phi(y)}{(y-x)^{\lambda}} + \int_{x}^{y} \frac{\phi'(\xi)}{(\xi-x)^{\lambda}} \, d\xi, \quad (3.3)$$

an appeal to Dirichlet's formula [16, p. 77], and the known results (2.3) and (2.6), will give us

$$A_n = \frac{n! \Gamma(1 - \lambda) \Gamma(\beta + n + 1)}{\Gamma(\sigma - \lambda + n + 1)} \int_{0}^{y} \xi^{\sigma-\lambda} I^{(\omega-\lambda)}_{n}(\xi) \phi(\xi) \, d\xi, \quad n = 0, 1, 2, ..., \quad (3.4)$$

provided $\sigma > -1$, $0 < \lambda < 1$, and $\sigma - \lambda + 1 > 0$.

On substituting for the coefficients \{A_n\} in (1.4) from (3.4), if we invert the order of summation and integration, we obtain

$$f(x) = \int_{0}^{y} \xi^{\sigma-\lambda} \phi(\xi) M(\xi, x) \, d\xi, \quad 0 \leq x < y, \quad (3.5)$$

where

$$M(\xi, x) = \sum_{n=0}^{\infty} \frac{n! \Gamma(1 - \lambda) \Gamma(\beta + n + 1)}{\Gamma(\sigma + n + 1) \Gamma(\sigma - \lambda + n + 1)} I^{(\omega-\lambda)}_{n}(\xi) L^{(\omega)}_{n}(x). \quad (3.6)$$
By applying (2.1) and (2.5), the last relation can be shown to imply
\begin{equation}
M(\xi, x) = a_n^* e^{\xi} x^{-\nu} (x - \xi)^{\lambda + \nu - \sigma - 1} H(x - \xi),
\end{equation}
where \( H(t) \) denotes Heaviside's unit function, and
\begin{equation}
a_n^* = \frac{\Gamma(1 - \lambda) \Gamma(\beta + n + 1) \Gamma(\nu + n + 1)}{\Gamma(\lambda + \nu - \sigma) \Gamma(\alpha + n + 1) \Gamma(\sigma - \lambda + n + 1)},
\end{equation}
it being assumed that the parameters \( \alpha, \beta, \lambda, \nu, \) and \( \sigma \) are so constrained that \( a_n^* \) is independent of \( n \). This of course is possible when, for instance, \( \alpha = \nu, \lambda = \sigma - \beta, \) and the parameters \( \beta \) and \( \sigma \) remain free.

It is not difficult to prove that (3.7) does imply (3.6). Indeed if we let
\begin{equation}
M(\xi, x) = \sum_{n=0}^{\infty} c_n(\xi) L_n^{(\nu)}(x),
\end{equation}
then using (3.7) and the orthogonality property (2.1) we have
\begin{align*}
c_n(\xi) &= \frac{n!}{\Gamma(\nu + n + 1)} \int_{0}^{\infty} x^n e^{-x} M(\xi, x) L_n^{(\nu)}(x) \, dx \\
&= \frac{n! a_n^*}{\Gamma(\nu + n + 1)} e^\xi \int_{\xi}^{\infty} e^{-x} (x - \xi)^{\lambda + \nu - \sigma - 1} L_n^{(\nu)}(x) \, dx \\
&= \frac{n! \Gamma(1 - \lambda) \Gamma(\beta + n + 1)}{\Gamma(\alpha + n + 1) \Gamma(\sigma - \lambda + n + 1)} L_n^{(\sigma - \lambda)}(\xi),
\end{align*}
by virtue of (2.5) and (3.8), and (3.6) follows immediately.

In view of (3.7), Eq. (3.5) is equivalent to the Abel integral equation
\begin{equation}
F(x) = x^\nu f(x) = a_n^* \int_{0}^{x} \frac{\xi^{\alpha + \lambda} e^{\xi} \phi(\xi)}{(x - \xi)^{\lambda + \nu - \sigma}} d\xi, \quad 0 \leq x < y,
\end{equation}
and if \( F(x) \) is continuously differentiable for \( 0 \leq x \leq a < y \), then it follows from (2.8) that
\begin{equation}
\xi^{\alpha + \lambda} e^{\xi} \phi(\xi) = \frac{\sin(\lambda + \nu - \sigma) \pi}{\pi a_n^*} \frac{d}{dx} \int_{0}^{x} \frac{F(\xi)}{(\xi - x)^{\lambda + \nu - \sigma}} \, d\xi,
\end{equation}
provided \( 0 < \lambda + \nu - \sigma < 1 \).

The constants \( \{A_n\} \) can now be calculated by using the relations (3.4), (3.8), and (3.10).
4. INVESTIGATION OF THE PROBLEM (ii).

Suppose that for $y < x < \infty$ and $0 < \mu < 1$,

$$
\sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\alpha + n + 1)} L_n^{(\omega)}(x) = x^{-\eta} \int_{y}^{x} \frac{\psi(y)}{(x-\eta)^{\mu}} \, d\eta. \tag{4.1}
$$

Then using (2.1) and (2.5) it can be shown that

$$
A_n = \frac{n! \, \Gamma(1 - \mu) \Gamma(\alpha + n + 1)}{\Gamma(\nu + n + 1)} \int_{y}^{\infty} e^{-\gamma} L_n^{(\omega + \nu - 1)}(\gamma) \psi(\gamma) \, d\gamma, \quad n = 0, 1, 2, \ldots. \tag{4.2}
$$

Next we multiply both sides of Eq. (1.7) by $e^{-x}$ and integrate with respect to $x$ from $x$ to $\infty$, $y < x < \infty$. Applying the known result (2.4) we thus find that

$$
G(x) = \int_{x}^{\infty} e^{-x} g(x) \, dx = \sum_{n=0}^{\infty} \frac{A_n}{\Gamma(\beta + n + 1)} e^{-x} L_n^{(\omega - 1)}(x). \tag{4.3}
$$

From (4.2) and (4.3) we have

$$
e^{x} G(x) = \int_{y}^{\infty} e^{-y} \psi(y) \, N(\eta, x) \, d\eta, \tag{4.4}
$$

where

$$
N(\eta, x) = \sum_{n=0}^{\infty} \frac{n! \, \Gamma(1 - \mu) \Gamma(\alpha + n + 1) \Gamma(\omega + \nu - 1)}{\Gamma(\beta + n + 1) \Gamma(\nu + n + 1)} L_n^{(\omega + \nu - 1)}(\eta) L_n^{(\omega - 1)}(x). \tag{4.5}
$$

By virtue of (2.1) and (2.6), the relation (4.5) implies

$$
N(\eta, x) = b_n^* e^{\eta - \mu - \nu - (\eta - x)^{\mu + \nu - 1}} H(\eta - x), \tag{4.6}
$$

provided the parameters $\alpha, \beta, \nu$ and $\sigma$ are so constrained that

$$
b_n^* = \frac{\Gamma(1 - \mu) \Gamma(\alpha - n + 1) \Gamma(\mu + \nu + n)}{\Gamma(\mu + \nu - \sigma) \Gamma(\beta + n + 1) \Gamma(\nu + n + 1)} \tag{4.7}
$$

is independent of $n$. Indeed one such possibility exists when $\alpha = \nu$, $\mu = \beta - \nu + 1$, and $\beta$ and $\sigma$ continue to be free.

Conversely (4.6) implies (4.5). The proof runs parallel to that of the derivation of (3.6) from (3.7).
In view of (4.6), Eq. (4.4) reduces to the integral equation

$$G(x) = b_n^* \int_0^\infty \frac{\eta^{1-\mu-v} e^{-\eta \psi(\eta)}}{(\eta - x)^{\mu+v+\sigma}} d\eta, \quad y < x < \infty,$$

which is of the type (2.9).

Hence, if $G(x)$ is continuously differentiable on the interval $y < 1 < x < \infty$, we readily have

$$\eta^{1-\mu-v} e^{-\eta \psi(\eta)} = -\frac{\sin(\mu + v - \sigma) \pi}{\pi b_n^*} \int_0^\infty \frac{G(x)}{(x - \eta)^{\mu+v+\sigma}} d\eta,$$

provided $0 < \mu + v - \sigma < 1$.

The relations (4.2), (4.7), and (4.9) can now be combined to compute the coefficients $\{A_n\}$. And, as we pointed out at the outset, the final result will follow when we merely add the solutions of the problems (i) and (ii).

REFERENCES


