Topology and its Applications 158 (2011) 204–214

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Domination by second countable spaces and Lindelöf Σ-property

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ARTICLE INFO

Article history:
Received 19 August 2010
Accepted 28 October 2010

Keywords:
(Strong) domination by irrationals
(Strong) domination by a second countable space
Diagonal
Metrization
Orderings by irrationals
Orderings by a second countable space
Compact cover
Function spaces
Cosmic spaces
$\aleph_0$-spaces
Lindelöf Σ-space
Compact space
Metrizable space

ABSTRACT

Given a space M, a family of sets $\mathcal{A}$ of a space X is *ordered by* M if $\mathcal{A} = \{A_K\colon K$ is a compact subset of $M\}$ and $K \subset L$ implies $A_K \subset A_L$. We study the class $\mathcal{M}$ of spaces which have compact covers ordered by a second countable space. We prove that a space $C_p(X)$ belongs to $\mathcal{M}$ if and only if it is a Lindelöf Σ-space. Under MA(ω_1), if X is compact and $(X \times X)\backslash\Delta$ has a compact cover ordered by a Polish space then X is metrizable; here $\Delta = \{(x,x)\colon x \in X\}$ is the diagonal of the space X. Besides, if X is a compact space of countable tightness and $X^2\backslash\Delta$ belongs to $\mathcal{M}$ then X is metrizable in ZFC.
We also consider the class $\mathcal{M}^*$ of spaces X which have a compact cover $\mathcal{F}$ ordered by a second countable space with the additional property that, for every compact set $P \subset X$ there exists $F \in \mathcal{F}$ with $P \subset F$. It is a ZFC result that if X is a compact space and $(X \times X)\backslash\Delta$ belongs to $\mathcal{M}^*$ then X is metrizable. We also establish that, under CH, if X is compact and $C_p(X)$ belongs to $\mathcal{M}^*$ then X is countable.

0. Introduction

Given a space X we denote by $\mathcal{K}(X)$ the family of all compact subsets of X. One of about a dozen equivalent definitions says that X is *a Lindelöf Σ-space* (*or has the Lindelöf Σ-property*) if there exists a second countable space M and a compact-valued upper semicontinuous map $\varphi : M \to X$ such that $\bigcup\{\varphi(x)\colon x \in M\} = X$ (see, e.g., [23, Section 5.1]). It is worth mentioning that in Functional Analysis, the same concept is usually referred to as a *countably K-determined space*.

Suppose that X is a Lindelöf Σ-space and hence we can find a compact-valued upper semicontinuous surjective map $\varphi : M \to X$ for some second countable space M. If we let $F_K = \bigcup\{\varphi(x)\colon x \in K\}$ for any compact set $K \subset M$ then the family $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(M)\}$ consists of compact subsets of X, covers X and $K \subset L$ implies $F_K \subset F_L$. We will say that $\mathcal{F}$ is an M-ordered compact cover of X.

The class $\mathcal{M}$ of spaces with an M-ordered compact cover for some second countable space M, was introduced by Cascales and Orihuela in [9]. They proved, among other things, that a Dieudonné complete space is Lindelöf Σ if and only

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1 Research supported by FEDER and MEC, Project MTM2008-05396.
2 Research supported by Fundación Séneca de la CARM, Project 08848/PI/08.
3 Research supported by Consejo Nacional de Ciencia y Tecnología de México, Grant U48602-F.
4 Research supported by Programa Integral de Fortalecimiento Institucional (PIFI), Grant 34536-55.

0166-8641/$ – see front matter
doi:10.1016/j.topol.2010.10.014

if it belongs to $\mathcal{M}$. We proved in the previous paragraph that any Lindelöf Σ-space belongs to $\mathcal{M}$; however, $X \in \mathcal{M}$ does not even imply that X is Lindelöf (see [9,26]) so $\mathcal{M}$ is a new class which seems to be interesting in itself.

Let $\mathbb{P}$ be the set of the irrationals which we will identify with ω^ω; a family $\mathcal{A}$ of subsets of a space X is $\mathbb{P}$-directed if $\mathcal{A} = \{A_p\colon\ p \in \mathbb{P}\}$ and $p \leqslant q$ implies $A_p \subset A_q$. The spaces which have $\mathbb{P}$-directed compact covers were extensively studied in Functional Analysis (see [5,8,12,13,17,25]). Talagrand proved in [25] that if X is compact then $C_p(X)$ has a $\mathbb{P}$-directed compact cover if and only if $C_p(X)$ is K-analytic. Cascales [5] extended Talagrand's results by proving that, for angelic spaces, to have a $\mathbb{P}$-directed compact cover is equivalent to K-analyticity. Tkachuk [26] studied systematically the topology of the spaces which have a $\mathbb{P}$-directed compact cover (calling the respective spaces $\mathbb{P}$-dominated); it was proved in [26] that compactness can be omitted in the mentioned Talagrand result, i.e., for any Tychonoff X, the space $C_p(X)$ is K-analytic if and only if it is $\mathbb{P}$-dominated.

Following the terminology of [26] we say that a space X is M-dominated (or dominated by space M) if X has an M-ordered compact cover, i.e., there exists a family $\mathcal{F} = \{F_K\colon\ K \in \mathcal{K}(M)\} \subset \mathcal{K}(X)$ such that $\bigcup \mathcal{F} = X$ and $K \subset L$ implies $F_K \subset F_L$ for any $K, L \in \mathcal{K}(M)$. In this paper we study the general topological and categorical properties of the class $\mathcal{M}$ of spaces dominated by a second countable space.

We prove, in particular, that for any Tychonoff X, the space $C_p(X)$ has the Lindelöf Σ-property, if and only if it is dominated by a second countable space. We also show that, if X is a compact space of countable tightness and $(X \times X)\setminus\Delta$ belongs to the class $\mathcal{M}$ then X is metrizable. Here $\Delta = \{(x,x)\colon\ x \in X\}$ is the diagonal of the space X. It turns out that, under $\mathrm{MA}(\omega_1)$, if X is compact and $(X \times X)\setminus\Delta$ is dominated by a Polish space then X is metrizable. As in [26], we introduce the notion of a strong M-domination to prove that if X is compact and the space $(X \times X)\setminus\Delta$ is strongly dominated by a second countable space then X is metrizable. Besides, under the Continuum Hypothesis (CH), if X is compact and $C_p(X)$ is strongly dominated by a second countable space then X is countable. Hopefully, our study of M-dominated spaces will find applications in Functional Analysis the same as $\mathbb{P}$-domination already did.

1. Notation and terminology

All spaces under consideration are assumed to be Tychonoff. If X is a space then $\tau(X)$ is its topology and $\tau^*(X) = \tau(X)\setminus\{\emptyset\}$. If X is a space and $A \subset X$ then $\tau(A, X) = \{U \in \tau(X)\colon\ A \subset U\}$; we will write $\tau(x, X)$ instead of $\tau(\{x\}, X)$. Given a space Z the family $\mathcal{K}(Z)$ consists of all compact subsets of Z; we use the symbol $\mathbb{P}$ to denote the set of the irrational numbers which we identify with ω^ω. Given $p, q \in \mathbb{P}$ we write $p \leqslant q$ if $p(n) \leqslant q(n)$ for any $n \in \omega$; we use the notation $p \leqslant^* q$ (or $p =^* q$) if there exists $m \in \omega$ such that $p(n) \leqslant q(n)$ (or $p(n) = q(n)$ respectively) for all $n \geqslant m$. The symbol $\mathbb{Q}$ stands for the set of the rational numbers with the topology induced from the real line $\mathbb{R}$ and $\mathbb{N} = \omega\setminus\{0\}$.

A family of sets $\mathcal{A}$ is *$\mathbb{P}$-directed* if $\mathcal{A} = \{A_p\colon\ p \in \mathbb{P}\}$ and $p \leqslant q$ implies $A_p \subset A_q$. A family $\mathcal{B}$ is *M-ordered* for some space M if $\mathcal{B} = \{B_K\colon\ K \in \mathcal{K}(M)\}$ while $K \subset L$ implies $B_K \subset B_L$. A space X is *$\mathbb{P}$-dominated* if it has a $\mathbb{P}$-ordered compact cover; in general, the space X is dominated by a space M if it has an M-ordered compact cover. Say that X is *strongly M-dominated* if it has an M-ordered compact cover $\mathcal{C}$ such that for every compact subset $K \subset X$ there exists $C \in \mathcal{C}$ with $K \subset C$.

If X is a space and $\mathcal{C}$ is a cover of X then a family $\mathcal{F}$ is called *a network modulo $\mathcal{C}$* if for any $C \in \mathcal{C}$ and $U \in \tau(C, X)$ there is $F \in \mathcal{F}$ with $C \subset F \subset U$. A family $\mathcal{N}$ of subsets of a space X is a network in X if it is a network modulo the cover $\{\{x\}\colon\ x \in X\}$. The network weight $nw(X)$ of a space X is the minimal cardinality of a network in X. A space X is *cosmic if* $nw(X) = \omega$.

A cover $\mathcal{C}$ of X is compact if all elements of $\mathcal{C}$ are compact. A space X is *Lindelöf Σ* if it has a countable network modulo a compact cover of X. Say that X is an *$\aleph_0$-space* if it has a countable network modulo $\mathcal{K}(X)$. The space X is *hemicompact* if there exists a countable family $\mathcal{F}$ of compact subsets of X such that every $K \in \mathcal{K}(X)$ is contained in an element of $\mathcal{F}$.

If X is a space then $\Delta = \{(x,x)\colon\ x \in X\}$ is its diagonal. The space X has *a small diagonal* if, for any uncountable set $A \subset (X \times X)\setminus\Delta$ there exists an uncountable $B \subset A$ such that $\overline{B} \cap \Delta = \emptyset$. The *spread* $s(X)$ of a space X is the supremum of cardinalities of discrete subspaces of X and $ext(X) = \sup\{|D|\colon\ D$ is a closed and discrete subset of $X\}$. Now, $hl(X) = \sup\{l(Y)\colon\ Y \subset X\}$ is the hereditary Lindelöf number of X. The cardinal $iw(X) = \min\{\kappa\colon$ the space X has a weaker topology of weight $\kappa\}$ is called *i-weight* of X. Recall that $iw(X) \leqslant nw(X)$ and $hl(X) \leqslant nw(X)$ for any space X.

If X is a space and $A \subset X$ we say that a family $\mathcal{B}$ of subsets of X is an outer network (base) of the set A in X if ($\mathcal{B} \subset \tau(X)$ and) for any $U \in \tau(A, X)$ there exists $B \in \mathcal{B}$ such that $A \subset B \subset U$. Given an infinite cardinal κ, recall that $t(X) \leqslant \kappa$ if $\overline{A} = \bigcup\{\overline{B}\colon\ B \subset A$ and $|B| \leqslant \kappa\}$ for any $A \subset X$. A continuous map $f: X \to Y$ is *compact-covering* if for any $L \in \mathcal{K}(Y)$ there exists $K \in \mathcal{K}(X)$ such that $f(K) = L$. For any spaces X and Y the space $C_p(X, Y)$ consists of continuous functions from X to Y with the topology induced from Y^X. The space $C_p(X, \mathbb{R})$ is denoted by $C_p(X)$.

The rest of our notation is standard and follows [11]; our reference book on C_p-theory is [2].

2. General properties of spaces dominated by second countable ones

Our purpose is to find interesting classes in which domination by a second countable space coincides with the Lindelöf Σ-property. We show that this coincidence takes place for the spaces $C_p(X)$ and sometimes for the complements of the diagonal of compact spaces. The following result summarizes the simplest properties of spaces dominated by second countable ones.

2.1. Theorem.

(a) *Every Lindelöf Σ-space is dominated by a second countable space*;
(b) *If X is dominated by a second countable space then any continuous image of X is also dominated by a second countable space*;
(c) *If X is dominated by a second countable space then any closed subspace of X is also dominated by a second countable space*;
(d) *If $X = \bigcup_{i\in\omega} X_i$ and X_i is dominated by a second countable space for every $i \in \omega$ then X is dominated by a second countable space*;
(e) *If X_i is dominated by a second countable space for each $i \in \omega$ then the space $X = \prod_{i\in\omega} X_i$ is dominated by a second countable space*;
(f) *If X is a space and $Y_i \subset X$ is dominated by a second countable space for every $i \in \omega$ then $Y = \bigcap_{i\in\omega} Y_i$ is also dominated by a second countable space*;
(g) *A space X is Lindelöf Σ if and only if it is Dieudonné complete* (*i.e., homeomorphic to a closed subspace of a product of metrizable spaces*) *and dominated by a second countable space*;
(h) *If X is dominated by a second countable space then $ext(X) = \omega$.*

Proof. The statement of (a) was proved in the first paragraph of Introduction; the proofs of (b) and (c) are straightforward and can be left to the reader. To see that (d) is true suppose that X_i has an M_i-ordered compact cover $\mathcal{F}_i = \{P(K,i)\colon K \in \mathcal{K}(M_i)\}$ for some second countable space M_i for every $i \in \omega$. The space $M = \bigoplus_{i\in\omega} M_i$ is second countable; we identify every M_i with the corresponding clopen subset of M. Given any $K \in \mathcal{K}(M)$ the set $N_K = \{i \in \omega\colon K \cap M_i \neq \emptyset\}$ is finite so the set $F_K = \bigcup\{P(K \cap M_i, i)\colon i \in N_K\}$ is compact. It is immediate that the family $\{F_K\colon K \in \mathcal{K}(M)\}$ is an M-ordered compact cover of X.

(e) For each $i \in \omega$ fix a second countable space M_i and an M_i-ordered compact cover $\mathcal{F}_i = \{Q(K,i)\colon K \in \mathcal{K}(X_i)\}$ of the space X_i. For the space $M = \prod_{i\in\omega} M_i$ let $p_i : M \to M_i$ be the natural projection for every $i \in \omega$. Given any $K \in \mathcal{K}(M)$, the set $F_K = \prod\{Q(p_i(K), i)\colon i \in \omega\}$ belongs to $\mathcal{K}(X)$. It is an easy exercise that the family $\{F_K\colon K \in \mathcal{K}(M)\}$ is an M-ordered compact cover of X.

It is standard to deduce (f) from (c) and (e); the statement of (g) was proved in [9]. If X is dominated by a second countable space and D is a closed discrete subspace of X then D is also dominated by a second countable space by (c). Since D is also Dieudonné complete, it must be Lindelöf and hence countable by (g). This shows that $ext(X) = \omega$, i.e., (h) is proved. □

2.2. Proposition. *The following conditions are equivalent for any space X*:

(a) *X has a $\mathbb{P}$-directed compact cover, i.e., X is dominated by the irrationals in the sense of* [26];
(b) *X is $\mathbb{P}$-dominated*;
(c) *X is dominated by a Polish space.*

Proof. (a) ⇒ (b) Fix a $\mathbb{P}$-directed compact cover $\{Q(p)\colon p \in \mathbb{P}\}$ of the space X and let $\pi_i : \mathbb{P} \to \omega$ be the projection of $\mathbb{P}$ onto its i-th factor, i.e., $\pi_i(s) = s(i)$ for any $s \in \mathbb{P}$. If $K \in \mathcal{K}(\mathbb{P})$ then $\pi_i(K)$ is a finite set so the number $s_K(i) = \max(\pi_i(K))$ is well defined for any $i \in \omega$ and hence we have an element $s_K \in \mathbb{P}$ for any $K \in \mathcal{K}(\mathbb{P})$. It is immediate that $K \subset L$ implies that $s_K \leqslant s_L$; let $F_K = Q(s_K)$ for any $K \in \mathcal{K}(\mathbb{P})$. It is straightforward that $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(\mathbb{P})\}$ is a $\mathbb{P}$-ordered family of compact subsets of X. To see that $\mathcal{F}$ is a cover of X fix any point $x \in X$ and $p \in \mathbb{P}$ with $x \in Q(p)$. The set $K = \prod\{\{0,\dots,p(i)\}\colon i \in \omega\}$ is compact and $s_K = p$; as a consequence, $x \in Q(p) = Q(s_K) = F_K$ so $\mathcal{F}$ is a compact $\mathbb{P}$-ordered cover of X, i.e., X is $\mathbb{P}$-dominated.

(b) ⇒ (a) Assume that the space X is $\mathbb{P}$-dominated and fix a respective compact cover $\{F_K\colon K \in \mathcal{K}(\mathbb{P})\}$. For any $p \in \mathbb{P}$ the set $K(p) = \prod\{\{0,\dots,p(i)\}\colon i \in \omega\}$ is compact; let $Q(p) = F_{K(p)}$. It is easy to see that the family $\mathcal{Q} = \{Q(p)\colon p \in \mathbb{P}\}$ is $\mathbb{P}$-directed. To see that $\mathcal{Q}$ is a cover of X take a point $x \in X$; there exists $K \in \mathcal{K}(\mathbb{P})$ with $x \in F_K$. Consider the point $p \in \mathbb{P}$ such that $p(i) = \max(\pi_i(K))$ for every $i \in \omega$. Then $K \subset K(p)$ and hence $x \in F_K \subset F_{K(p)}$ so $\mathcal{Q}$ is a $\mathbb{P}$-directed compact cover of X.

The implication (b) ⇒ (c) being clear, assume that a space X is dominated by a Polish space M and take a respective M-ordered compact cover $\{F(L)\colon L \in \mathcal{K}(M)\}$. There exists an open continuous onto map $\varphi : \mathbb{P} \to M$; observe that the family $\mathcal{F} = \{F(\varphi(K))\colon K \in \mathcal{K}(\mathbb{P})\}$ is $\mathbb{P}$-ordered. To see that $\mathcal{F}$ covers X take any point $x \in X$ and a compact set $L \subset M$ such that $x \in F(L)$. Any open map between Polish spaces is inductively perfect and hence compact-covering (see, e.g., [11, 5.5.8]) so there exists $K \in \mathcal{K}(\mathbb{P})$ such that $\varphi(K) = L$. Therefore $x \in F(\varphi(K)) \in \mathcal{F}$ and hence $\mathcal{F}$ is a $\mathbb{P}$-ordered compact cover of X, i.e., we settled (c) ⇒ (b). □

2.3. Corollary. *A Dieudonné complete space is K-analytic if and only if it is dominated by a Polish space.*

Proof. It was proved in [9] that a Dieudonné complete space is K-analytic if and only if it has a $\mathbb{P}$-directed compact cover; Proposition 2.2 does the rest. □

2.4. Corollary. *For any space X, if $C_p(X)$ is dominated by a Polish space then it is K-analytic.*

Proof. It was proved in [26] that any $\mathbb{P}$-dominated space $C_p(X)$ is K-analytic so we can apply Proposition 2.2 to finish our proof. □

Cascales and Orihuela proved (using a different terminology), that if X is compact and $X^2\backslash\Delta$ is strongly dominated by the irrationals then X is metrizable (see [8, Theorem 1]). It is a very interesting question whether the word "strongly" can be omitted in this statement. Our plan is to show that this is true under $\mathrm{MA}(\omega_1)$. We will use the methods developed in [6] adapted to our situation. For the reader's convenience we avoid citing very general technical results from [6] and give direct short proofs here for some particular cases we need.

2.5. Proposition. *Suppose that X is dominated by a second countable space M and a collection $\{F_K\colon K \in \mathcal{K}(M)\}$ witnesses this. Take a countable base $\mathcal{B}$ in M such that the union and the intersection of any finite subfamily of $\mathcal{B}$ belongs to $\mathcal{B}$. For any $U \in \mathcal{B}$ let $G(U) = \bigcup\{F_K\colon K \in \mathcal{K}(M)$ and $K \subset U\}$. Fix a set $K \in \mathcal{K}(M)$ and a family $\mathcal{B}_K = \{U_n\colon n \in \omega\} \subset \mathcal{B}$ such that $U_{n+1} \subset U_n$ for each $n \in \omega$ and $\mathcal{B}_K$ is an outer base of K in M; then $F_K \subset C_K = \bigcap\{G(U)\colon U \in \mathcal{B}_K\}$. If $S = \{y_n\colon n \in \omega\} \subset X$ is a sequence such that $y_n \in G(U_n)$ for all $n \in \omega$, then*

(a) *the set $\overline{S}$ is compact and hence the set D of cluster points of S is non-empty*;
(b) *there exists a compact set Q_K such that $D \subset Q_K \subset C_K$.*

Proof. Take a set $K_n \in \mathcal{K}(M)$ such that $K_n \subset U_n$ and $y_n \in F_{K_n}$ for any $n \in \omega$. It is straightforward that the set $L_m = K \cup (\bigcup\{K_i\colon i \geqslant m\})$ is compact for any $m \in \omega$. The sequence $\{y_n\}$ is eventually in the compact set F_{L_m} which shows that the set $\overline{S}$ is compact, $D \neq \emptyset$ and $D \subset F_{L_m}$ for any $m \in \omega$. Therefore D is contained in the compact set $Q_K = \bigcap\{F_{L_m}\colon m \in \omega\} \subset C_K$ as promised. □

2.6. Proposition. *Suppose that X is dominated by a second countable space M and a collection $\{F_K\colon K \in \mathcal{K}(M)\}$ witnesses this. Fix a countable base $\mathcal{B}$ in M such that the union and the intersection of any finite subfamily of $\mathcal{B}$ belongs to $\mathcal{B}$. For any $U \in \mathcal{B}$ let $G(U) = \bigcup\{F_K\colon K \in \mathcal{K}(M)$ and $K \subset U\}$. Then there exists a family $\mathcal{C}$ in the space X with the following properties*:

(a) *if $C \in \mathcal{C}$ and $A \subset C$ is a countable set then the set $\overline{A}$ is compact and $\overline{A} \subset C$; in particular, each $C \in \mathcal{C}$ is countably compact*;
(b) *for every $K \in \mathcal{K}(M)$ there exists a set $C_K \in \mathcal{C}$ such that $F_K \subset C_K$ and hence $\mathcal{C}$ is a cover of X*;
(c) *the family $\mathcal{N} = \{G(U)\colon U \in \mathcal{B}\}$ is a network with respect to $\mathcal{C}$.*

Proof. Fix any compact subset K of the space M and observe that we can choose a family $\mathcal{B}_K = \{U_n\colon n \in \omega\} \subset \mathcal{B}$ such that $U_{n+1} \subset U_n$ for each $n \in \omega$ and $\mathcal{B}_K$ is an outer base of K in M. It is evident that $F_K \subset C_K = \bigcap\{G(U)\colon U \in \mathcal{B}_K\}$. Let $\mathcal{C} = \{C_K\colon K \in \mathcal{K}(M)\}$; it is clear that the property (b) holds for $\mathcal{C}_K$.

If $K \in \mathcal{K}(M)$ and $\{G(U_n)\colon n \in \omega\}$ is not an outer network for C_K then we can choose a point $y_n \in G(U_n)\backslash W$ for some $W \in \tau(C_K, X)$. The sequence $\{y_n\}$ must have a cluster point in C_K by Proposition 2.5 which contradicts the fact that $\{y_n\} \subset X\backslash W$ while $C_K \subset W$. Therefore the family $\mathcal{C}$ has the property (c).

Furthermore, if $A \subset C_K$ is countable then we can choose an enumeration $\{y_n\colon n \in \omega\}$ of the set A. It is clear that $y_n \in G(U_n)$ for all $n \in \omega$ and hence we can apply Proposition 2.5 again to see that $\overline{A} = \overline{\{y_n\colon n \in \omega\}}$ is compact. If $x \in \overline{A}\backslash C_K$ then $x \in \overline{A}\backslash A$ and hence x is a cluster point of the sequence $S = \{y_n\}$. However, all cluster points of S belong to C_K by Proposition 2.5. This contradiction shows that $\overline{A} \subset C_K$ so (a) is proved as well. □

Given a space X recall that a set $A \subset X$ is *relatively countably compact* if every sequence in A has a cluster point in X. The following result was implicitly proved in [9,6].

2.7. Corollary. *Suppose that, in a space X, every relatively countably compact set has compact closure. Then X is dominated by a second countable space if and only if it has the Lindelöf Σ-property. In particular, an angelic space X is dominated by a second countable space if and only if X is Lindelöf Σ.*

Proof. It suffices to prove necessity so assume that X is dominated by a second countable space. It follows from Proposition 2.6 that we can find a cover $\mathcal{C}$ of the space X such that every $C \in \mathcal{C}$ is countably compact and there exists a countable network $\mathcal{N}$ with respect to $\mathcal{C}$. The family $\mathcal{F} = \{\overline{C}\colon C \in \mathcal{C}\}$ is a cover of X and all elements of $\mathcal{F}$ are compact. It is standard that $\mathcal{M} = \{\overline{N}\colon N \in \mathcal{N}\}$ is a countable network with respect to $\mathcal{F}$ so X is a Lindelöf Σ-space. □

2.8. Theorem. *Suppose that Z is a compact space of countable tightness. Then a set $X \subset Z$ is dominated by a second countable space if and only if X has the Lindelöf Σ-property.*

Proof. Fix any set $X \subset Z$ and assume that X is dominated by a second countable space. For any set $A \subset X$ we denote by $\mathrm{cl}_X(A)$ (or $\mathrm{cl}_Z(A)$) the closure of the set A in the space X (or in Z respectively). By Proposition 2.6, there exist a cover $\mathcal{C}$ of the space X and a countable network $\mathcal{N}$ with respect to $\mathcal{C}$ such that for every $C \in \mathcal{C}$ and any countable $A \subset C$ the set $\mathrm{cl}_X(A)$ is compact and contained in C.

If $C \in \mathcal{C}$ and C is not closed in Z then we can find a point $x \in \mathrm{cl}_Z(C) \backslash C$. By countable tightness of Z, there exists a countable $A \subset C$ such that $x \in \mathrm{cl}_Z(A)$. The set $F = \mathrm{cl}_X(A) \subset C$ is compact and hence closed in Z; as a consequence, $x \in \mathrm{cl}_Z(A) \subset F \subset C$. This contradiction shows that every $C \in \mathcal{C}$ is compact being closed in X. Thus $\mathcal{N}$ is a countable network with respect to the compact cover $\mathcal{C}$ of the space X, i.e., X has the Lindelöf Σ-property. □

2.9. Theorem. *If X is a compact space with $t(X) \leqslant \omega$ and $X^2 \backslash \Delta$ is dominated by a second countable space then X is metrizable.*

Proof. The space X^2 also has countable tightness [1, Theorem 2.3.3] so we can apply Theorem 2.8 to the set $X^2 \backslash \Delta \subset X \times X$ to conclude that $X^2 \backslash \Delta$ is a Lindelöf Σ-space; this easily implies that the diagonal Δ is a G_δ-subset of $X \times X$ and hence X is metrizable by [11, 3.12.22(e)]. □

2.10. Corollary. *If X is a Corson compact space or a first countable compact space such that $X^2 \backslash \Delta$ is dominated by a second countable space then X is metrizable.*

2.11. Theorem. *If X is a dyadic compact space and $X^2 \backslash \Delta$ is dominated by a second countable space then X is metrizable.*

Proof. If X is first countable then it is metrizable by [11, 3.12.12(e)]. Therefore we can assume that there exists a point $x \in X$ of uncountable character in X. Apply [11, 3.12.12(i)] to find an uncountable one-point compactification A of a discrete space such that $A \subset X$ and x is the unique non-isolated point of A. Then $B = (A \backslash \{x\}) \times \{x\}$ is an uncountable closed discrete subspace of $(X \times X) \backslash \Delta$ while we have $ext(X^2 \backslash \Delta) = \omega$ by Theorem 2.1(h), a contradiction. □

The above results show that, to prove that any compact space X with $X^2 \backslash \Delta$ dominated by a second countable space is metrizable, it suffices to show that any such space has a countable tightness. While we don't know whether this implication is true in general, we do present some partial progress in this direction.

2.12. Theorem. *Assume* MA(ω_1) *and suppose that X is a compact space such that $X^2 \backslash \Delta$ is $\mathbb{P}$-dominated. Then X has a small diagonal and hence $t(X) = \omega$.*

Proof. Suppose that $A = \{z_\alpha\colon \alpha < \omega_1\} \subset X^2 \backslash \Delta$ and $\alpha \neq \beta$ implies $z_\alpha \neq z_\beta$. Fix a $\mathbb{P}$-directed cover $\{K_p\colon p \in \mathbb{P}\}$ of compact subsets of $X^2 \backslash \Delta$. Take $p_\alpha \in \mathbb{P}$ such that $z_\alpha \in K_{p_\alpha}$ for any $\alpha < \omega_1$.

It follows from MA(ω_1) that there exists $p \in \mathbb{P}$ such that $p_\alpha \leqslant^* p$ for any $\alpha < \omega_1$. The set $P = \bigcup\{K_q\colon q \in \mathbb{P}$ and $q =^* p\}$ is σ-compact and $A \subset P$. Consequently, there is $q \in \mathbb{P}$ for which $K_q \cap A$ is uncountable; therefore the set $K_q \cap A$ witnesses the small diagonal property of X. Since no space with a small diagonal can have a convergent ω_1-sequence, it follows from [16, Theorem 1.2] that X has no free sequences of length ω_1, i.e., $t(X) \leqslant \omega$. □

2.13. Corollary. *Under* MA(ω_1), *if X is a compact space such that $X^2 \backslash \Delta$ is dominated by a Polish space then X is metrizable.*

Proof. Apply Proposition 2.2 to see that the space $X^2 \backslash \Delta$ is dominated by $\mathbb{P}$ so $t(X) \leqslant \omega$ by Theorem 2.12 and hence X is metrizable by Theorem 2.9. □

In the rest of this section we study the spaces hereditarily dominated by a second countable space. The motivation here is a result of Hodel established in [14, Corollary 4.13]; it says that any hereditarily Lindelöf Σ-space is cosmic. We will look at this hereditary property in function spaces to show that a somewhat stronger statement is true in a general situation under Martin's Axiom.

The following fact is an immediate consequence of [26, Proposition 2.7].

2.14. Proposition. *If X is a space which has a countable network modulo a cover of X by countably compact sets then $C_p(X)$ is Lindelöf Σ-framed, i.e., there is a Lindelöf Σ-space L such that $C_p(X) \subset L \subset \mathbb{R}^X$.*

2.15. Theorem. *A space $C_p(X)$ is dominated by a second countable space if and only if it is Lindelöf Σ.*

Proof. We must only prove necessity. Suppose that $C_p(X)$ is dominated by a second countable space M and fix a family $\{F_K\colon K \in \mathcal{K}(M)\}$ which witnesses this. It follows from Proposition 2.14 and Proposition 2.6 that $C_p(C_p(X))$ is Lindelöf Σ-framed. Applying [21, Theorem 3.5] we conclude that $\upsilon(C_p(X))$ is a Lindelöf Σ-space and hence υX is a Lindelöf Σ-space by [21, Corollary 3.6].

Let $\pi : C_p(\upsilon X) \to C_p(X)$ be the restriction map. If $G_K = \pi^{-1}(F_K)$ then G_K is compact for any $K \in \mathcal{K}(M)$ (see [26, Theorem 2.6]). It is clear that $\mathcal{G} = \{G_K\colon K \in \mathcal{K}(M)\}$ is a cover of $C_p(\upsilon X)$ which shows that $C_p(\upsilon X)$ is dominated by M. By Proposition 2.6 we can find a countable network $\mathcal{N}$ modulo a cover $\mathcal{C}$ of the space $C_p(\upsilon X)$ such that every $C \in \mathcal{C}$ is countably compact. Every countably compact subset of $C_p(\upsilon X)$ is compact by [2, Proposition IV.9.10] (see also [22]) so $\mathcal{C}$ consists of compact subsets of $C_p(\upsilon X)$ and hence $C_p(\upsilon X)$ is a Lindelöf Σ-space. Therefore $C_p(X)$ is also Lindelöf Σ-space being a continuous image of $C_p(\upsilon X)$. □

2.16. Lemma. *If every subspace of a space X is realcompact (i.e., X is hereditarily realcompact) and dominated by a second countable space then X is cosmic.*

Proof. Every subspace of X has to be Lindelöf Σ by Theorem 2.1(g) so we can apply [14, Corollary 4.13] to conclude that X is cosmic. □

2.17. Theorem. *Under Martin's Axiom, the following conditions are equivalent for any space X:*

(a) *every subspace of X is dominated by a second countable space*;
(b) *the space X is cosmic.*

Proof. Every subspace of a cosmic space is cosmic and hence Lindelöf Σ so it is dominated by a second countable space by Theorem 2.1(a). This proves that (b) ⇒ (a); observe that no additional axioms are needed for this conclusion.

Now assume that there exist non-cosmic spaces which are hereditarily dominated by a second countable space and call every such space *a counterexample*. Observe first that a counterexample cannot be hereditarily Lindelöf by Lemma 2.16. Therefore, if X is a counterexample then we can find a right-separated subspace $Y \subset X$ such that $|Y| = \omega_1$. It is immediate that Y is also a counterexample so we can assume, without loss of generality, that $X = Y$, i.e., X is a scattered space. If every countably compact subspace of X is compact and $Y \subset X$ then we can apply Proposition 2.6 to find a cover $\mathcal{C}$ of Y by countably compact (and hence compact) subspaces such that there exists a countable network modulo $\mathcal{C}$. This proves that every $Y \subset X$ is Lindelöf Σ and hence X is cosmic by [14, Corollary 4.13], which is a contradiction.

Therefore we can find an uncountable countably compact subspace $Y \subset X$; it is clear that Y is also a counterexample. Thus we can assume, without loss of generality, that X is countably compact. It follows from Theorem 2.1(h) that $s(X) \leqslant \omega$ and hence X is hereditarily separable (see [15, 2.12]).

If Y is a subspace of X then let $I(Y)$ be the set of isolated points of Y; if $Y \neq \emptyset$ then $I(Y) \neq \emptyset$ because the space X is scattered. Let $X_0 = X$; if α is a countable ordinal and we have X_α then $X_{\alpha+1} = X_\alpha \backslash I(X_\alpha)$. If α is a limit ordinal and we have X_β for every $\beta < \alpha$ then $X_\alpha = \bigcap_{\beta<\alpha} X_\beta$. This gives us a strictly decreasing ω_1-sequence $\{X_\alpha\colon \alpha < \omega_1\}$ of closed subsets of X such that $X \backslash X_\alpha$ is countable and hence $X_\alpha \neq \emptyset$ for any $\alpha < \omega_1$.

The set $Y = \bigcup_{\alpha<\omega_1}(X \backslash X_\alpha)$ is a counterexample because it has cardinality ω_1. The space Y is an increasing union of countable open subsets of X. Therefore every point of Y has a countable countably compact neighbourhood, i.e., Y is locally compact and locally countable. The one-point compactification of Y is an uncountable compact scattered hereditarily separable space. Such spaces do not exist under MA + ¬CH (see [24, Theorem 6.4.1]) so if CH does not hold then our proof is over.

Finally, assume that CH holds and observe that Y is first countable so every countably compact subspace of Y is closed in Y. Therefore every countably compact subset of Y is uniquely determined by its countable dense subset and hence the family $\mathcal{P}$ of uncountable countably compact subspaces of Y has cardinality at most $\omega_1^\omega = \mathfrak{c} = \omega_1$.

It is standard that we can find disjoint subsets A, B of the space Y such that $Y = A \cup B$ and $A \cap P \neq \emptyset \neq B \cap P$ for any $P \in \mathcal{P}$. In particular, every countably compact subset of A as well as every countably compact subspace of B is countable and hence compact. This, together with Proposition 2.6 implies that both A and B are hereditarily Lindelöf Σ so we can apply [14, Corollary 4.13] again to see that $nw(A) = nw(B) = \omega$ and hence $Y = A \cup B$ is cosmic which is a contradiction. □

If a space $C_p(X)$ is hereditarily dominated by a second countable space then no additional axioms are needed to obtain the same conclusion as in Theorem 2.17.

2.18. Proposition. *If every subspace of a space $C_p(X)$ is dominated by a second countable space then $C_p(X)$ is cosmic.*

Proof. We have $s(C_p(X)) = \omega$ by Theorem 2.1(h); besides, $C_p(X)$ is a Lindelöf Σ-space by Theorem 2.15. If $C_p(X)$ is not hereditarily Lindelöf then we can find an uncountable right-separated subspace $Y \subset C_p(X)$ (see [15, Theorem 2.9(b)]). Every right-separated space of countable spread must be hereditarily separable (see [15, Theorem 2.12]) so Y is separable. In the space $C_p(X)$ the closure of every countable subset is cosmic by [3, Theorem 7.21] so we can conclude that $nw(Y) \leqslant \omega$ and, in particular, $hl(Y) \leqslant \omega$ which is a contradiction. This proves that $C_p(X)$ is hereditarily Lindelöf so it follows from Lemma 2.16 that $C_p(X)$ is cosmic. □

3. Strong domination by second countable spaces

Say that X is *strongly dominated by a space M* if there exists an M-ordered compact cover $\mathcal{F}$ of the space X such that the family $\mathcal{F}$ swallows all compact subsets of X in the sense that for any compact $C \subset X$ there is $F \in \mathcal{F}$ such that $C \subset F$. The following two results seem to be a good motivation for a systematic study of the class $\mathcal{M}^*$ of spaces which are strongly dominated by second countable ones.

3.1. Theorem. *(Christensen [10, Theorem 3.3]) A second countable space is strongly $\mathbb{P}$-dominated if and only if it is completely metrizable.*

3.2. Theorem. *(Cascales and Orihuela [8, Theorem 1]) If X is a compact space such that $(X \times X)\backslash\Delta$ is strongly $\mathbb{P}$-dominated then X is metrizable. Here $\Delta = \{(x, x)\colon x \in X\}$ is the diagonal of the space X.*

3.3. Proposition.

(a) *If X is strongly dominated by a second countable space and Y is a compact-covering image of X then Y is strongly dominated by a second countable space*;
(b) *Every $\aleph_0$-space is strongly dominated by a second countable space*;
(c) *If X is strongly dominated by a second countable space then every closed subspace of X is also strongly dominated by a second countable space*;
(d) *If X_i is strongly dominated by a second countable space for every $i \in \omega$ then $\prod_{i\in\omega} X_i$ is strongly dominated by a second countable space*;
(e) *If X is a space and $Y_i \subset X$ is strongly dominated by a second countable space for each $i \in \omega$ then $Y = \bigcap_{i\in\omega} Y_i$ is also strongly dominated by a second countable space.*

Proof. Suppose that X is strongly dominated by a second countable space M and $f : X \to Y$ is a compact-covering map. Let $\{F_K\colon K \in \mathcal{K}(M)\}$ be the family which witnesses that X is strongly dominated by M and consider the family $\mathcal{F} = \{f(F_K)\colon K \in \mathcal{K}(M)\}$. It is clear that $\mathcal{F}$ consists of compact subsets of Y and $K \subset L$ implies $f(F_K) \subset f(F_L)$. If P is a compact subset of Y then there exists a compact subset $Q \subset X$ such that $f(Q) = P$. Pick a set $K \in \mathcal{K}(M)$ such that $Q \subset F_K$ and observe that $P = f(Q) \subset f(F_K)$. Therefore the family $\mathcal{F}$ witnesses that Y is strongly dominated by M, i.e., we proved (a).

The item (b) follows from (a) and the fact that every $\aleph_0$-space is a compact-covering image of a second countable space [18, Theorem 11.4]. The proof of (c) is straightforward and can be left to the reader.

Next assume that X_i is strongly dominated by a second countable space M_i and fix a respective family $\mathcal{F}_i = \{F_i(K)\colon K \in \mathcal{K}(M_i)\}$ for any $i \in \omega$. The space $M = \prod_{i\in\omega} M_i$ is second countable; let $\pi_i : M \to M_i$ be the natural projection for each $i \in \omega$. If $K \in \mathcal{K}(M)$ then $F_K = \prod_{i\in\omega} F_i(\pi_i(K))$ is easily seen to be a compact subset of $X = \prod_{i\in\omega} X_i$. Let $p_i : X \to X_i$ be the natural projection for every $i \in \omega$.

The family $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(M)\}$ witnesses that X is strongly dominated by M. Indeed, if Q is a compact subset of X then we can choose $K_i \in \mathcal{K}(M_i)$ such that $p_i(Q) \subset F_i(K_i)$ for each $i \in \omega$; for the set $K = \prod_{i\in\omega} K_i$ we have $Q \subset F_K$. It is immediate that $K \subset L$ implies $F_K \subset F_L$ so we settled (d). As to (e), observe that Y is homeomorphic to a closed subspace of $\prod_{i\in\omega} Y_i$ so we can apply (c) and (d) to finish the proof. □

3.4. Proposition. *The space ω_1 with its interval topology is strongly dominated by the space of rational numbers.*

Proof. Given a compact set $K \subset \mathbb{Q}$, let $\alpha_K \in \omega_1$ be the minimal ordinal such that $F_K = \{\beta\colon \beta < \alpha_K\}$, as a subspace of ω_1, is homeomorphic to K. Such an ordinal α_K exists by [19, Theorem 1]. It is clear that the family $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(\mathbb{Q})\}$ is $\mathbb{Q}$-ordered.

Suppose that L is a compact subset of ω_1 and choose an ordinal $\alpha < \omega_1$ such that $L \subset \{\beta\colon \beta < \alpha\}$. It is easy to see that there exists a countable ordinal $\gamma > \alpha$ such that $Q = \{\beta\colon \beta < \gamma\}$ is a compact subset of ω_1 and no initial segment of Q is homeomorphic to Q. The space $\mathbb{Q}$ is also universal for all countable compact spaces so there exists $K \subset \mathbb{Q}$ with $K \simeq Q$. It is clear that $\alpha_K = \gamma$ and hence $L \subset \{\beta\colon \beta < \alpha\} \subset Q = F_K$. This shows that $\mathcal{F}$ is a $\mathbb{Q}$-ordered compact cover of ω_1 which swallows all compact subsets of ω_1, i.e., ω_1 is strongly $\mathbb{Q}$-dominated. □

3.5. Corollary. *Under* MA + ¬CH *there exists a strongly $\mathbb{Q}$-dominated space which is not $\mathbb{P}$-dominated.*

Proof. The space ω_1 is not $\mathbb{P}$-dominated under MA + ¬CH (see [26, Theorem 3.6]) so apply Proposition 3.4 to see that ω_1 is as promised. □

Proposition 3.3(b) and Proposition 3.4 show that $\mathcal{M}^*$ is strictly larger than the class of $\aleph_0$-spaces. Therefore it is natural to ask when strong domination by a second countable space must imply the $\aleph_0$-property. Recall that a space is called *submetrizable* if it has a weaker metrizable topology.

3.6. Theorem. *The following conditions are equivalent for any space X:*

(a) *X is an $\aleph_0$-space;*
(b) *X is strongly dominated by a second countable space and $iw(X) \leqslant \omega$;*
(c) *X is submetrizable and strongly dominated by a second countable space.*

Proof. Every $\aleph_0$-space X is cosmic and hence $iw(X) \leqslant \omega$; this, together with Proposition 3.3(b), shows that (a) $\Rightarrow$ (b). The implication (b) $\Rightarrow$ (c) being trivial assume that X is submetrizable and strongly dominated by a second countable space. It follows from [9, Theorem 4] that X is a Lindelöf Σ-space so its weaker metrizable topology must be second countable, i.e., $iw(X) \leqslant \omega$.

Fix an M-ordered family $\{F_K \colon K \in \mathcal{K}(M)\}$ of compact subsets of X such that every $L \in \mathcal{K}(X)$ is contained in some F_K. Apply Proposition 2.6 to find a family $\mathcal{C}$ of countably compact (and hence compact) subsets of X such that some countable family $\mathcal{N}$ is a network modulo $\mathcal{C}$ and, for every $K \in \mathcal{K}(M)$ there exists $C_K \in \mathcal{C}$ such that $F_K \subset C_K$. In particular, the family $\mathcal{C}$ swallows all compact subsets of X.

Taking the closures of the elements of $\mathcal{N}$ we will still have a network modulo $\mathcal{C}$ so we can assume, without loss of generality, that $\mathcal{N}$ consists of closed subsets of X. Fix a second countable topology μ on the set X such that $\mu \subset \tau(X)$. The space (X, μ) has a countable closed network $\mathcal{P}$ modulo all compact subsets of (X, μ). Observe that the identity map $\mathrm{id} : X \to (X, \mu)$ is continuous and hence any compact subset of X is also compact in (X, μ). Consider the family $\mathcal{Q}$ of all finite unions and finite intersections of the elements of the family $\mathcal{P} \cup \mathcal{N}$; we claim that $\mathcal{Q}$ is an outer network for all compact subsets of X.

Indeed, take any $L \in \mathcal{K}(X)$ and $U \in \tau(L, X)$. There exists $C \in \mathcal{C}$ such that $L \subset C$. The set $C \backslash U$ does not meet L so there exists $P \in \mathcal{P}$ such that $L \subset P$ and $P \cap (C \backslash U) = \emptyset$. The set $P' = P \backslash U$ does not meet C so we can find a set $N \in \mathcal{N}$ such that $C \subset N \subset X \backslash P'$. The set $Q = N \cap P$ belongs to $\mathcal{Q}$ and $L \subset Q \subset U$ so the family $\mathcal{Q}$ witnesses that X is an $\aleph_0$-space. $\square$

3.7. Remark. Adapting to our situation the proof of the implication (ii) $\Rightarrow$ (i) in Theorem 6 of [9] gives another direct (and somewhat shorter) way to establish the implication (c) $\Rightarrow$ (a) in Theorem 3.6.

3.8. Corollary. *Under Martin's Axiom, every subspace of a space X is strongly dominated by a second countable space if and only if X is an $\aleph_0$-space.*

Proof. If X is an $\aleph_0$-space then every subspace of X is also $\aleph_0$-space so X is hereditarily strongly dominated by a second countable space by Proposition 3.3(b); this proves sufficiency.

If X is hereditarily strongly dominated by a second countable space then we can apply Theorem 2.17 to convince ourselves that X is cosmic and hence $iw(X) \leqslant \omega$. Now it follows from Theorem 3.6 that X is an $\aleph_0$-space. $\square$

Given an infinite cardinal κ say that a space X is *κ-hemicompact* if there exists a family $\mathcal{F}$ of compact subsets of X such that $|\mathcal{F}| \leqslant \kappa$ and $\mathcal{F}$ swallows all compact subsets of X, i.e., for any $K \in \mathcal{K}(X)$ there exists $F \in \mathcal{F}$ such that $K \subset F$. Observe that a space is hemicompact if and only if it is ω-hemicompact.

3.9. Theorem. *The σ-product $S_\kappa = \{x \in \mathbb{D}^\kappa \colon |x^{-1}(1)| < \omega\}$ of the space $\mathbb{D}^\kappa$ is not κ-hemicompact for any infinite cardinal κ.*

Proof. Denote by u the point of $\mathbb{D}^\kappa$ which is identically zero on κ and hence $u^{-1}(1) = \emptyset$. Take any family $\mathcal{F} = \{F_\alpha \colon \alpha < \kappa\}$ of compact subsets of S_κ. The set S_κ is not compact so we can pick a point $x_0 \in S_\kappa \backslash F_0$. Proceeding inductively assume that $\alpha < \kappa$ and we have chosen a set $\{x_\beta \colon \beta < \alpha\}$ with the following properties:

(1) $x_\beta \in S_\kappa \backslash F_\beta$ for any $\beta < \alpha$;
(2) the family $\{x_\beta^{-1}(1) \colon \beta < \alpha\}$ is disjoint.

Observe that the set $A = \bigcup\{x_\beta^{-1}(1) \colon \beta < \alpha\}$ has cardinality strictly less than κ. Therefore the subspace $Y = \{x \in S_\kappa \colon x(A) = 0\}$ is not compact so we can choose a point $x_\alpha \in Y \backslash F_\alpha$; it is immediate that the conditions (1) and (2) are still satisfied for the set $\{x_\beta \colon \beta \leqslant \alpha\}$. Thus we can construct a set $\{x_\alpha \colon \alpha < \kappa\}$ for which the properties (1) and (2) hold for any $\alpha < \kappa$.

It follows from (2) that the set $K = \{x_\beta \colon \beta < \kappa\} \cup \{u\}$ is compact; the property (1) shows that $x_\beta \in K \backslash F_\beta$ for any $\beta < \kappa$ and therefore no element of the family $\mathcal{F}$ swallows the set K. $\square$

3.10. Theorem. *Under the Continuum Hypothesis* (CH) *if a space X is compact and $C_p(X)$ is strongly dominated by a second countable space then X is countable and hence $C_p(X)$ is second countable.*

Proof. Apply Theorem 2.15 to see that $C_p(X)$ is a Lindelöf Σ-space and hence X is Gul'ko compact. If the space X is not scattered then we can find a countable dense-in-itself set $A \subset X$. The space $K = \overline{A}$ is compact, second countable and metrizable [3, Theorem 7.21] so $C_p(K)$ embeds in $C_p(X)$ as a closed subspace [3, Theorem 4.1]. This implies, by Proposition 3.3(c), that $C_p(K)$ is strongly dominated by a second countable space. Since $iw(C_p(K)) \leqslant nw(C_p(K)) = \omega$, we can apply Theorem 3.6 to convince ourselves that $C_p(K)$ is an $\aleph_0$-space so K is countable by [18, Proposition 10.7]. However, K has no isolated points; this contradiction shows that X has to be scattered.

The set D of isolated points of the space X is dense in X; if D is countable then X is second countable so we can apply Theorem 3.6 again to see that $C_p(X)$ is an $\aleph_0$-space and hence X is countable by [18, Proposition 10.7]. Therefore we can assume that $\kappa = |D| \geqslant \omega_1$; consider the space Y which is obtained from X by contracting the set $F = X\backslash D$ to a point. It is evident that Y is a compact space with a unique non-isolated point, i.e., Y is homeomorphic to the one-point compactification A_κ of a discrete space of cardinality κ. The space Y is a continuous closed image of X so $C_p(Y)$ is homeomorphic to a closed subspace of $C_p(X)$. Thus $C_p(Y) \simeq C_p(A_\kappa)$ is strongly dominated by a second countable space.

It is an easy exercise that the space $C_p(A_\kappa)$ is homeomorphic to the Σ_*-product $\Omega = \{x \in \mathbb{R}^\kappa\colon$ the set $\{\alpha < \kappa\colon |x(\alpha)| \geqslant \varepsilon\}$ is finite for any $\varepsilon > 0\}$ of the space $\mathbb{R}^\kappa$. Furthermore, $\Omega \cap \mathbb{D}^\kappa = S_\kappa = \{x \in \mathbb{D}^\kappa\colon x^{-1}(1)$ is finite$\}$ so S_κ is a closed subset of Ω; in particular, S_κ is strongly dominated by a second countable space M. Let $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(M)\}$ be a family of compact subsets of S_κ which witnesses this. However, $|\mathcal{K}(M)| \leqslant \mathfrak{c} = \omega_1$ so $|\mathcal{F}| \leqslant \omega_1$ and hence S_κ is ω_1-hemicompact; since $\kappa \geqslant \omega_1$, we have obtained a contradiction with Theorem 3.9. □

It is not difficult to deduce the following theorem from a general result proved by M. Muñoz in her PhD thesis (see [20, Theorem 2.10.1]). This result was also published in [7, Proposition 5.1]. For the reader's convenience we chose to avoid dealing with uniformities and give a direct topological proof here.

3.11. Theorem. *A compact space X is metrizable if and only if $X^2\backslash\Delta$ is strongly dominated by a second countable space.*

Proof. The necessity being evident fix a second countable space E and a family $\mathcal{F} = \{F(Q)\colon Q \in \mathcal{K}(E)\}$ of compact subsets of $X^2\backslash\Delta$ which witnesses that $X^2\backslash\Delta$ is strongly E-dominated. Denote by C the subspace $C_p(X, [0,1])$ of the space $C_p(X)$ and let $I = [0,1]$. For the space $M = E^{\mathbb{N}}$ let $\pi_n : M \to E$ be the natural projection onto the n-th factor of M.

For every $K \in \mathcal{K}(M)$ consider the set $H_n = \{f \in I^X\colon |f(x) - f(y)| \leqslant \frac{1}{n}$ for any $(x, y) \in X^2\backslash F(\pi_n(K))\}$ for each $n \in \mathbb{N}$ and let $G_K = \bigcap\{H_n\colon n \in \mathbb{N}\}$. It is immediate that $K \subset L$ implies $G_K \subset G_L$ for any $K, L \in \mathcal{K}(M)$. We omit a simple proof of the fact that the set G_K is closed in I^X and hence compact. To see that $G_K \subset C$ take any $f \in G_K$, $x \in X$ and $\varepsilon > 0$. If $n \in \mathbb{N}$ and $\frac{1}{n} < \varepsilon$ then the set $U = \{y \in X\colon (x, y) \notin F(\pi_n(K))\}$ is an open neighbourhood of x in X and we have the inclusions

$$f(U) \subset \big[f(x) - 1/n, f(x) + 1/n\big] \subset \big(f(x) - \varepsilon, f(x) + \varepsilon\big)$$

which show that f is continuous at the point x. Thus G_K is a compact subset of C for any $K \in \mathcal{K}(M)$.

To see that $\mathcal{G} = \{G_K\colon K \in \mathcal{K}(M)\}$ is a cover of C, take any $f \in C$. Then $O_n = \{(x, y) \in X^2\colon |f(x) - f(y)| < 1/n\}$ is an open neighbourhood of Δ so the set $P_n = X^2\backslash O_n \subset X^2\backslash\Delta$ is compact for any $n \in \mathbb{N}$. The family $\mathcal{F}$ swallows all compact subsets of $X^2\backslash\Delta$ and hence we can find a set $K_n \in \mathcal{K}(M)$ such that $P_n \subset F(K_n)$ for all $n \in \mathbb{N}$. It is straightforward that $f \in G_K$ for the compact set $K = \prod\{K_n\colon n \in \mathbb{N}\}$ of the space M.

This proves that C is dominated by M; since countably compact subsets of C are compact, we can apply Proposition 2.6 to see that there exists a countable network modulo a compact cover of C, i.e., the space C is Lindelöf Σ. The space X being compact, $C_p(X)$ is also Lindelöf Σ being the countable union of subspaces homeomorphic to C. It is easy to see that the space X^2 embeds in $C_p(C_p(X))$ whence $l(X^2\backslash\Delta) = ext(X^2\backslash\Delta) = \omega$ (see Theorem 2.1(h) and [4, Theorem 1′]). Therefore $X^2\backslash\Delta$ is Lindelöf; this easily implies that Δ is a G_δ-subset of $X \times X$ so X is metrizable by [11, 4.2.B]. □

3.12. Corollary. *Suppose that X is a compact space, M is a second countable space and we have a family $\mathcal{G} = \{U_K\colon K \in \mathcal{K}(M)\}$ of neighbourhoods of the diagonal Δ in the space $X \times X$ such that $U_K \subset U_L$ whenever $L \subset K$. If, additionally, $\bigcap\{\overline{G}\colon G \in \mathcal{G} = \Delta\}$ then X is metrizable.*

Proof. Let $F_K = (X \times X)\backslash \mathrm{Int}(U_K)$ for any $K \in \mathcal{K}(M)$. It is immediate that $F_K \subset F_L$ if $K \subset L$, i.e., the family $\mathcal{F} = \{F_K\colon K \in \mathcal{K}(M)\}$ is ordered by M. The equality $\bigcap\{\overline{G}\colon G \in \mathcal{G}\} = \Delta$ shows that $\bigcup\{\mathrm{Int}(F_K)\colon K \in \mathcal{K}(M)\} = X^2\backslash\Delta$. Given a compact set $F \subset X^2\backslash\Delta$, the family $\{\mathrm{Int}(F_K)\colon K \in \mathcal{K}(M)\}$ is an open cover of F so we can find $K_1, \ldots, K_n \in \mathcal{K}(M)$ such that $F \subset \mathrm{Int}(F_{K_1}) \cup \cdots \cup \mathrm{Int}(F_{K_n}) \subset F_K$ for $K = K_1 \cup \cdots \cup K_n \in \mathcal{K}(M)$. Therefore the family $\mathcal{F}$ witnesses that $X^2\backslash\Delta$ is strongly dominated by the second countable space M and hence X is metrizable by Theorem 3.11. □

4. Open problems

One of the niceties of the concept of domination by a second countable space is a possibility to obtain new metrization theorems for compact spaces. We already saw that if X compact and $(X \times X)\backslash\Delta$ is strongly dominated by a second countable space then X is metrizable. The most interesting question here is whether we can omit the word "strongly" in the above statement.

4.1. Problem. Let X be a compact space such that $X^2\backslash\Delta$ is $\mathbb{P}$-dominated. Is it true in ZFC that X must be metrizable?

4.2. Problem. Let X be a compact space such that $X^2\backslash\Delta$ is $\mathbb{Q}$-dominated. Is it true in ZFC that X must be metrizable?

4.3. Problem. Let X be a compact space such that $X^2\backslash\Delta$ is M-dominated for some separable metrizable space M. Is it true in ZFC that X must be metrizable?

4.4. Problem. Suppose that X is a K-analytic space such that $X^2\backslash\Delta$ is strongly $\mathbb{P}$-dominated. Must X be cosmic?

4.5. Problem. Let X be a K-analytic space such that $X^2\backslash\Delta$ is $\mathbb{P}$-dominated. Must X be cosmic?

4.6. Problem. Suppose that X is a Lindelöf Σ-space such that $X^2\backslash\Delta$ is strongly $\mathbb{P}$-dominated. Must X be cosmic?

4.7. Problem. Let X be a Lindelöf Σ-space such that $X^2\backslash\Delta$ is $\mathbb{P}$-dominated. Must X be cosmic?

4.8. Problem. Let X be a Lindelöf Σ-space such that $X^2\backslash\Delta$ is $\mathbb{Q}$-dominated. Must X be cosmic?

4.9. Problem. Suppose that $C_p(X)$ is strongly $\mathbb{Q}$-dominated. Must the space X be countable?

4.10. Problem. Suppose that $C_p(X)$ is strongly M-dominated for some separable metric space M. Must X be countable?

4.11. Problem. Suppose that X is compact and $C_p(X)$ is strongly dominated by a second countable space. Is it true in ZFC that X must be countable?

4.12. Problem. Suppose that X is a compact space and $X^2\backslash\Delta$ is $\mathbb{P}$-dominated. Is it true in ZFC that X must have a small diagonal?

4.13. Problem. Suppose that a separable metrizable space X is $\mathbb{Q}$-dominated. Must X be analytic?

4.14. Problem. Suppose that every subspace of a space X is dominated by a second countable space. Is it true in ZFC that X must be cosmic?

4.15. Problem. Suppose that every subspace of a space X is $\mathbb{Q}$-dominated. Is it true in ZFC that X must be cosmic?

4.16. Problem. Suppose that every subspace of a space X is strongly dominated by a second countable space. Is it true in ZFC that X must be an $\aleph_0$-space?

4.17. Problem. Suppose that every subspace of a compact space X is dominated by a second countable space. It is true in ZFC that X must be metrizable?

4.18. Problem. Suppose that X is a compact space and every subspace of X is $\mathbb{Q}$-dominated. It is true in ZFC that X must be metrizable?

4.19. Problem. Suppose that every subspace of a compact space X is strongly dominated by a second countable space. Is it true in ZFC that X must be metrizable?

4.20. Problem. Suppose that X is a compact space and every subspace of X is strongly $\mathbb{Q}$-dominated. It is true in ZFC that X must be metrizable?

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