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Ranking Fuzzy Numbers with an Area between the Centroid Point and Original Point

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Abstract—To improve the ranking method of Lee and Li [1], Cheng [2] proposed the coefficient of variation (CV index). Shortcomings are also found in the CV index. Cheng [2] also proposed the distance method to improve the ranking method of Murakami *et al.* However, the distance method is not sound either. Moreover, the CV index contradicts the distance method in ranking some fuzzy numbers. Therefore, to overcome the above shortcomings, we propose ranking fuzzy numbers with the area between the centroid point and original point.

Keywords—Ranking, Fuzzy numbers, Area, Centroid point.

1. INTRODUCTION

In a fuzzy environment, ranking fuzzy numbers is a very important decision making procedure. Since Jain [3,4] employed the concept of maximizing set to order the fuzzy numbers in 1976 (1978), many authors have investigated various ranking methods.

Some of these ranking methods have been compared and reviewed by Bortolan and Degani [5], and more recently by Chen and Hwang [6]. Other contributions in this field include: an index for ordering fuzzy numbers defined by Choobineh and Li [7], ranking alternatives using fuzzy numbers studied by Dias [8], automatic ranking of fuzzy numbers using artificial neural networks proposed by Requena *et al.* [9], ranking fuzzy values with satisfaction function investigated by Lee *et al.* [10], ranking and defuzzification methods based on area compensation presented by Fortemps and Roubens [11], and ranking alternatives with fuzzy weights using maximizing set and minimizing set given by Raj and Kumar [12]. However, some of these methods are computationally complex and difficult to implement, and others are counterintuitive and not discriminating. Furthermore, many of them produce different ranking outcomes for the same problem.

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In 1988, Lee and Li [1] proposed a comparison of fuzzy numbers by considering the mean and dispersion (standard deviation) based on the uniform and the proportional probability distributions. Based on this ranking method, in 1998 Cheng [2] proposed two comments:

(1) from the concept of statistics, the standard deviation and mean value cannot be the sole basis for comparing two fuzzy numbers, respectively;
(2) when higher mean value and at the same time higher spread/or lower mean value and at the same time lower spread, it is not easy to compare the orderings clearly.

Therefore, Cheng [2] proposed the coefficient of variance (CV index), i.e., CV $= \sigma$ (standard error)$/|\mu|$ (mean), $\mu \neq 0$, $\sigma > 0$, to improve Lee and Li's ranking method [1]. In the coefficient of variance approach, the fuzzy number with smaller CV index is ranked higher. However, Cheng's CV index also contains shortcomings.

Consider the two triangular fuzzy numbers, $U_1 = (0,1,2)$ and $U_2 = (1/5,1,7/4)$, shown in Figure 1, from Cheng's paper [2]. By Cheng's CV index, $\text{CV}(U_1) = 0.1667$ and $\text{CV}(U_2) = 0.1018$ for uniform distribution, and $\text{CV}(U_1) = 0.1$ and $\text{CV}(U_2) = 0.061$ for proportional distribution, where the ranking order is $U_1 < U_2$ for both. From this, we can logically infer the ranking order of the images of these fuzzy numbers as $-U_1 > -U_2$; that is, $(-2,-1,0) > (-7/4,-1,-1/5)$. However, by the CV index, the ranking order remains $-U_1 < -U_2$ for both the uniform and proportional distributions. Therefore, the CV index has shortcomings.

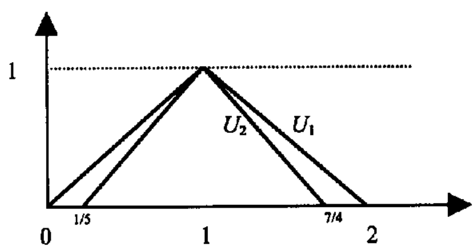

Figure 1. Triangular fuzzy numbers $U_1 = (0,1,2)$ and $U_2 = (1/5,1,7/4)$.

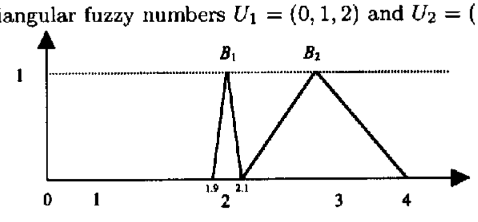

Figure 2. Triangular fuzzy numbers $B_1 = (1.9,2,2.1)$, and $B_2 = (2.1,3,4)$.

Figure 2 presents the two triangular fuzzy numbers, $B_1 = (0.9,1,1.1)$ and $B_2 = (1.1,2,3)$. Intuitively, the ranking order is $B_1 < B_2$. However, by the CV index, the ranking order is $B_1 > B_2$ for both the uniform and proportional distributions, which is unreasonable. This is another shortcoming of the CV index.

To improve Murakami *et al.*'s method [13], Cheng [2] proposed the distance method for ranking fuzzy numbers; i.e., $R(A) = \sqrt{\bar{x}^2 + \bar{y}^2}$. For any two fuzzy numbers, A_i and A_j, if $R(A_i) < R(A_j)$, then $A_i < A_j$; if $R(A_i) = R(A_j)$, then $A_i = A_j$; if $R(A_i) > R(A_j)$, then $A_i > A_j$. However, the distance method is not logically sound either. Moreover, the distance method contradicts the CV index in ranking some fuzzy numbers.

Consider the three triangular fuzzy numbers, $U_1 = (0.2,0.3,0.5)$, $U_2 = (0.17,0.32,0.58)$, and $U_3 = (0.25,0.4,0.7)$ shown in Figure 3, from [2]. By Cheng's distance method, $R(U_1) = 0.590$, $R(U_2) = 0.604$, and $R(U_3) = 0.662$, producing the ranking order $U_1 < U_2 < U_3$. From this result, we can logically infer the ranking order of the images of these fuzzy numbers as $-U_1 > -U_2 > -U_3$, that is $(-0.5,-0.3,-0.2) > (-0.58,-0.32,-0.17) > (-0.7,-0.4,-0.25)$. However,

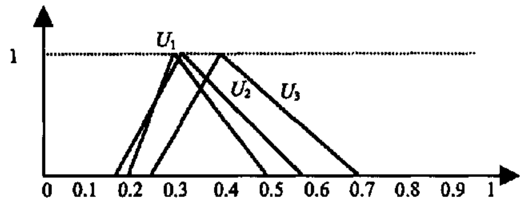

Figure 3. Triangular fuzzy numbers $U_1 = (0.2, 0.3, 0.5)$, $U_2 = (0.17, 0.32, 0.58)$, and $U_3 = (0.25, 0.4, 0.7)$.

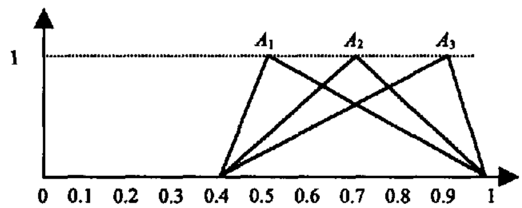

Figure 4. Triangular fuzzy numbers $A_1 = (0.4, 0.5, 1)$, $A_2 = (0.4, 0.7, 1)$, and $A_3 = (0.4, 0.9, 1)$.

by distance method, the ranking order remains $-U_1 < -U_2 < -U_3$. Obviously, the distance method also has shortcomings.

Figure 4 presents the three triangular fuzzy numbers, $A_1 = (0.4, 0.5, 1)$, $A_2 = (0.4, 0.7, 1)$ and $A_3 = (0.4, 0.9, 1)$, from [2]. By the CV index, the ranking order is $A_1 < A_3 < A_2$ for both the uniform and proportional distributions. However, by the distance method, the ranking order is $A_1 < A_2 < A_3$. Therefore, Cheng's CV index and distance method do not consistently rank these fuzzy numbers.

To overcome the above-mentioned problems, we propose ranking fuzzy numbers with an area between the centroid and original points, i.e., $S(A) = \bar{x}\bar{y}$, where $\bar{x}$ and $\bar{y}$ are the geometric center of fuzzy number A. The larger the area $S(A)$, the larger the fuzzy number A.

The rest of this work is organized as follows. Section 2 briefly introduces the fuzzy numbers. Section 3 introduces ranking fuzzy numbers with the area between the centroid point and original point. Comparative examples are presented in Section 4 to illustrate the advantage of the proposed method, and conclusions are made in Section 5.

2. FUZZY NUMBERS

The concept of fuzzy number can be defined as follows [14].

DEFINITION 1. *A real fuzzy number A is described as any fuzzy subset of the real line R with membership function f_A which processes the following properties:*

(a) *f_A is a continuous mapping from R to the closed interval $[0, w]$, $0 \le w \le 1$;*
(b) *$f_A(x) = 0$, for all $x \in (-\infty, a]$;*
(c) *f_A is strictly increasing on $[a, b]$;*
(d) *$f_A(x) = w$, for all $x \in [b, c]$, where w is a constant and $0 < w \le 1$;*
(e) *f_A is strictly decreasing on $[c, d]$;*
(f) *$f_A(x) = 0$, for all $x \in (d, \infty]$,*

where a, b, c, and d are real numbers. We may let $a = -\infty$, or $a = b$, or $b = c$, or $c = d$, or $d = +\infty$.

Unless elsewhere specified, it is assumed that A is convex and bounded; i.e., $-\infty < a$, $d < \infty$. If $w = 1$ in (d), A is a normal fuzzy number, and if $0 < w < 1$ in (d), A is a nonnormal fuzzy number. For convenience, the fuzzy number in Definition 1 can be denoted by $A = (a, b, c, d; w)$. The image (opposite) of $A = (a, b, c, d; w)$ can be given by $-A = (-d, -c, -b, -a; w)$ (see [15,16]).

The membership function f_A of A can be expressed as

$$f_A(x) = \begin{cases} f_A^L(x), & a \le x \le b, \\ w, & b \le x \le c, \\ f_A^R(x), & c \le x \le d, \\ 0, & \text{otherwise}, \end{cases} \tag{1}$$

where $f_A^L : [a,b] \to [0,w]$ and $f_A^R : [c,d] \to [0,w]$.

Since $f_A^L : [a,b] \to [0,w]$ is continuous and strictly increasing, the inverse function of f_A^L exists. Similarly, since $f_A^R : [c,d] \to [0,w]$ is continuous and strictly decreasing, the inverse function of f_A^R also exists. The inverse functions of f_A^L and f_A^R can be denoted by g_A^L and g_A^R, respectively. Since $f_A^L : [a,b] \to [0,w]$ is continuous and strictly increasing, $g_A^L : [0,w] \to [a,b]$ is also continuous and strictly increasing. Similarly, since $f_A^R : [c,d] \to [0,w]$ is continuous and strictly decreasing, $g_A^R : [0,w] \to [c,d]$ is also continuous and strictly increasing. g_A^L and g_A^R are continuous on $[0,w]$; they are integrable on $[0,w]$. That is, both $\int_0^w g_A^L \, dy$ and $\int_0^w g_A^R \, dy$ exist [17].

3. RANKING FUZZY NUMBERS WITH AN AREA BETWEEN THE CENTROID POINT AND ORIGINAL POINT

This section proposes a novel ranking method with an area between the centroid and original points of a fuzzy number. The centroid point of a fuzzy number corresponds to an $\bar{x}$ value on the horizontal axis and a $\bar{y}$ value on the vertical axis. The centroid point $(\bar{x}, \bar{y})$ for a fuzzy number A in Definition 1 is defined as [2,13]

$$\bar{x}(A) = \frac{\int_a^b \left(x f_A^L\right) dx + \int_b^c x \, dx + \int_c^d \left(x f_A^R\right) dx}{\int_a^b \left(f_A^L\right) dx + \int_b^c dx + \int_c^d \left(f_A^R\right) dx}, \tag{2}$$

$$\bar{y}(A) = \frac{\int_0^w \left(y g_A^L\right) dy + \int_0^w \left(y g_A^R\right) dy}{\int_0^w \left(g_A^L\right) dy + \int_0^w \left(g_A^R\right) dy}, \tag{3}$$

where f_A^L and f_A^R are the left and right membership functions of fuzzy number A, respectively. g_A^L and g_A^R are the inverse functions of f_A^L and f_A^R, respectively.

The area between the centroid point $(\bar{x}, \bar{y})$ and original point $(0,0)$ of the fuzzy number A is then defined as

$$S(A) = \bar{x}\bar{y}, \tag{4}$$

where $\bar{x}$ and $\bar{y}$ are the centroid points of fuzzy number A.

Here, the area $S(A)$ is used to rank fuzzy numbers. The larger the area $S(A)$, the larger the fuzzy number. Therefore, for any two fuzzy numbers A_i and A_j, if $S(A_i) > S(A_j)$, then $A_i > A_j$. If $S(A_i) = S(A_j)$, then $A_i = A_j$. Finally, if $S(A_i) < S(A_j)$, then $A_i < A_j$.

4. COMPARATIVE EXAMPLES

In this section, all the numerical examples (Figures 1, 3–5) of Cheng's paper, one self-designed numerical example (Figure 2), and one numerical example (Figure 6) from Liou and Wang's paper [17] are displayed to illustrate the validity and advantage of the proposed ranking method.

The two triangular fuzzy numbers, $U_1 = (0,1,2)$ and $U_2 = (1/5, 1, 7/4)$ shown in Figure 1 are also ranked by our method. $S(U_1) = \bar{x}_1\bar{y}_1 = 1 \times 0.5 = 0.5$ and $S(U_2) = \bar{x}_2\bar{y}_2 = 0.9833 \times 0.502 = 0.4936$. Obviously, the ranking order is $U_1 > U_2$ (notably, by Cheng's CV index, the ranking order is $U_1 < U_2$). The images of these two fuzzy numbers are $-U_1 = (-2,-1,0)$ and $-U_2 = (-7/4, -1, -1/5)$, respectively. By our method, $S(-U_1) = \bar{x}_1\bar{y}_1 = -1 \times 0.5 = -0.5$ and $S(-U_2) = \bar{x}_2\bar{y}_2 = -0.9833 \times 0.502 = -0.4936$, producing the ranking order $-U_1 < -U_2$. Clearly, our method can overcome the shortcomings of the inconsistency of Cheng's CV index in ranking fuzzy numbers and their images.

The two triangular fuzzy numbers, $B_1 = (1.9, 2, 2.1)$ and $B_2 = (2.1, 3, 4)$, shown in Figure 2 are also ranked by the proposed method. Intuitively, the ranking order is $B_1 < B_2$. But, by Cheng's CV index, $\mathrm{CV}(B_1) = 0.000834$ and $\mathrm{CV}(B_2) = 0.0636$ for uniform distribution, and $\mathrm{CV}(B_1) = 0.0005$ and $\mathrm{CV}(B_2) = 0.0299$ for proportional distribution. The ranking order is $B_1 > B_2$ for both, a result which is unreasonable. By our method, $S(B_1) = \bar{x}_1\bar{y}_1 = 2.00 \times 0.5 = 1.00$, $S(B_2) = \bar{x}_2\bar{y}_2 = 3.033 \times 0.4986 = 1.512$, obtaining the ranking order $B_1 < B_2$. Again, our method can overcome the shortcomings of Cheng's CV index.

The three triangular fuzzy numbers, $U_1 = (0.2, 0.3, 0.5)$, $U_2 = (0.17, 0.32, 0.58)$, and $U_3 = (0.25, 0.4, 0.7)$ shown in Figure 3 are also ranked by the proposed method. $S(U_1) = \bar{x}_1\bar{y}_1 = 0.333 \times 0.4872 = 0.162$, $S(U_2) = \bar{x}_2\bar{y}_2 = 0.357 \times 0.4868 = 0.174$ and $S(U_3) = \bar{x}_3\bar{y}_3 = 0.450 \times 0.4857 = 0.219$, producing the ranking order $U_1 < U_2 < U_3$. The images of the three fuzzy numbers are $-U_1 = (-0.5, -0.3, -0.2)$, $-U_2 = (-0.58, -0.32, -0.17)$, and $-U_3 = (-0.7, -0.4, -0.25)$, respectively. By the proposed method, $S(-U_1) = -0.162$, $S(-U_2) = -0.174$, and $S(-U_3) = -0.219$, obtaining the ranking order $-U_1 > -U_2 > -U_3$. Clearly, our method has consistency in ranking fuzzy numbers and their images.

Consider the three triangular fuzzy numbers, $A_1 = (0.4, 0.5, 1)$, $A_2 = (0.4, 0.7, 1)$, and $A_3 = (0.4, 0.9, 1)$ shown in Figure 4 from [2].

By Cheng's CV index, $\mathrm{CV}(A_1) = 0.0272$, $\mathrm{CV}(A_2) = 0.0214$, and $\mathrm{CV}(A_3) = 0.0224$ for uniform distribution, and $\mathrm{CV}(A_1) = 0.0183$, $\mathrm{CV}(A_2) = 0.0129$, and $\mathrm{CV}(A_3) = 0.01375$ for proportion distribution. For both the uniform and proportional distributions, the ranking order is $A_1 < A_3 < A_2$. However, by Cheng's distance method, $R(A_1) = 0.790$, $R(A_2) = 0.860$, and $R(A_3) = 0.927$. The ranking order is $A_1 < A_2 < A_3$. Cheng's CV index and his distance method cannot consistently rank these fuzzy numbers. By our method, $S(A_1) = \bar{x}_1\bar{y}_1 = 0.633 \times 0.472 = 0.299$, $S(A_2) = \bar{x}_2\bar{y}_2 = 0.7 \times 0.5 = 0.35$, and $S(A_3) = \bar{x}_3\bar{y}_3 = 0.767 \times 0.521 = 0.4$, producing the ranking order $A_1 < A_2 < A_3$. Meanwhile, the areas of the images of these fuzzy numbers are $S(-A_1) = -0.299$, $S(-A_2) = -0.35$, and $S(-A_3) = -0.4$, obtaining the ranking order $-A_1 > -A_2 > -A_3$. Obviously, our method can effectively rank fuzzy numbers and their images.

Further consider the two triangular fuzzy numbers, $A_1 = (3, 5, 7; 1)$, $A_2 = (3, 5, 7; 0.8)$, and the three trapezoidal fuzzy numbers, $B_1 = (5, 7, 9, 10; 1)$, $B_2 = (6, 7, 9, 10; 0.6)$, $B_3 = (7, 8, 9, 10; 0.4)$, shown in Figure 5, from [2].

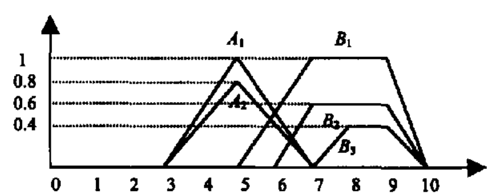

Figure 5. Triangular fuzzy numbers $A_1 = (3, 5, 7; 1)$ and $A_2 = (3, 5, 7; 0.8)$, and trapezoidal fuzzy numbers $B_1 = (5, 7, 9, 10; 1)$, $B_2 = (6, 7, 9, 10; 0.6)$, and $B_3 = (7, 8, 9, 10; 0.4)$.

By the proposed method, $S(A_1) = 5 \times 0.5 = 2.5$, $S(A_2) = 5 \times 0.4 = 2$, $S(B_1) = 7.714 \times 0.505 = 3.896$, $S(B_2) = 8 \times 0.3 = 2.4$, $S(B_3) = 8.5 \times 0.2 = 1.7$, obtaining the ranking order $B_3 = A_2 < B_2 < A_1 < B_1$. Clearly, the proposed method can also rank normal/nonnormal triangular and trapezoidal fuzzy numbers.

Finally, consider the triangular fuzzy number, $A = (1, 2, 5; 1)$, and the general fuzzy number, $B = [1, 2, 2, 4; 1]$, shown in Figure 6, from paper [17]. The membership function of B is defined

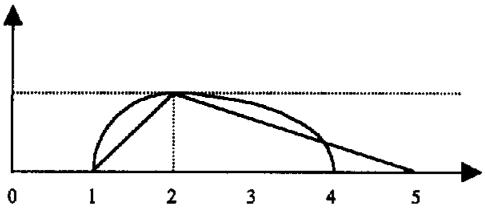

Figure 6. The triangular fuzzy numbers $A = (1, 2, 5; 1)$ and the general fuzzy number $B = [1, 2, 2, 4; 1]$.

as

$$f_B(x) = \begin{cases} \left[1 - (x-2)^2\right]^{1/2}, & 1 \le x \le 2, \\ \left[1 - \frac{1}{4}(x-2)^2\right]^{1/2}, & 2 \le x \le 4, \\ 0, & \text{otherwise.} \end{cases}$$

Since $f_B^L(x) = [1-(x-2)^2]^{1/2}$ and $f_B^R(x) = [1-(1/4)(x-2)^2]^{1/2}$, we have $g_B^L(y) = 2-(1-y^2)^{1/2}$, and $g_B^R(y) = 2 + 2(1-y^2)^{1/2}$, $y \in [0,1]$.

By Liou and Wang's ranking method, the total integral value of A is $I_T^\alpha(A) = 1.5 + 2\alpha$, and the total integral value of B is $I_T^\alpha(B) = 1.2 + 2.4\alpha$. For an optimistic decision maker, with $\alpha = 1$: $A < B$; for a moderate decision maker, with $\alpha = 0.5$: $A > B$; and for a pessimistic decision maker, with $\alpha = 0$: $A > B$. Liou and Wang's method produces different rankings for the same problem when applying different indices of optimism (α). By our method, $S(A) = \bar{x}_1\bar{y}_1 = 2.6667 \times 0.4667 = 1.2445$, $S(B) = \bar{x}_2\bar{y}_2 = 2.4244 \times 0.4876 = 1.1821$, obtaining the ranking order $A > B$. Obviously, our method can also rank fuzzy numbers other than triangular and trapezoidal. Compared to the method of Liou and Wang, our method produces a simpler ranking result.

5. CONCLUSIONS

Shortcomings are found in Cheng's CV index and distance method for ranking fuzzy numbers. To overcome those shortcomings, we proposed a novel ranking method with the area between the centroid point and original point of the fuzzy number. The proposed method can effectively rank various fuzzy numbers and their images (normal/nonnormal, triangular/trapezoidal, and general). Finally, comparative examples are also presented to illustrate the advantage of the proposed method.

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