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Linearization of graphic toposes via Coxeter groups

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Abstract

In an associative algebra over a field K of characteristic not 2, those idempotent elements a, for which the inner derivation $[-,a]$ is also idempotent, form a monoid M satisfying the graphic identity $aba=ab$. In case K has three elements and M is such a graphic monoid, then the category of K-vector spaces in the topos of M-sets is a full exact subcategory of the vector spaces in the Boolean topos of G-sets, where G is a crystallographic Coxeter group which measures equality of levels in the category of M-sets.

MSC: 05; 08; 12; 16; 18; 20

1. Introduction

By a linearization of a topos E, I mean in this paper the abelian category KE of K-modules in E, where K is a suitable commutative ring. For example, if E is the topos of (finite) right M-sets where M is a (finite) monoid, then KE consists of modules in (finite) sets over the noncommutative associative algebra $K[M]$. It can happen that very different toposes have closely related linearizations. For example, if K contains $1/2$ and if M is a monoid generated by idempotent elements, then $K[M]$ is (alternatively) generated by involutions, suggesting that the extensive knowledge about linearizations of G-sets, for groups G, might be applicable to the study of non-Boolean toposes. In particular, linearization may give partial information concerning the lattice of subtoposes of E, because any geometric morphism from E to E' will induce adjoint functors between the abelian categories KE and KE', with a geometric inclusion (localization) inducing a linear inclusion.

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I consider only "finite" toposes E, i.e. presheaves on a finite category. This restriction is made in view of several combinatorial examples, as well as in view of several simplifications of the lattice structure. Namely, all subtoposes are essential and the lattice itself is finite. It follows that the lattice has a Heyting structure (besides the usual co-Heyting structure, with its attendant boundary operator, which the lattice of subtoposes has for any E); moreover, in the "regular" case (defined below), there are further multiplication and Aufhebung operations defined on the lattice.

2. The system of levels in a topos

The multiplication and Aufhebung relations are defined on the lattice of essential subtoposes and under suitable conditions, these "pro-operations" will be representable by operations. The essential subtoposes may be considered as a refined notion of "dimension" of objects, but I will refer to the elements of this lattice as "levels". An essential subtopos of E corresponds to an adjoint pair $sk, cosk$ of idempotent endofunctors of E; an object has this level (or below) iff it is fixed by sk, but on the other hand it is a "sheaf" of this level iff it is fixed by $cosk$. For each level, there are thus two isomorphic subcategories of E, which constitute a unity and identity of opposites, since there is abstractly only one retraction, $sk(cosk) = sk$ and $(cosk)sk = cosk$. The factorizations of the canonical map $skX \to coskX$ constitute an "interval" into which fall all objects X having the same trace at the given level. There is a lowest level, whose only sheaf is the object 1 (with the object 0 as its skeletal companion). The lowest level for which 0 is a sheaf is the famous "double negation" level, whose sheaves form a Boolean topos of "codiscrete" objects and whose companion skeletal objects may be called "discrete"; this Boolean level will be the unit for the multiplication of levels in the frequent "algebraically closed" cases, but due to the logarithmic nature of dimension, $\log(1) = 0$, the objects of this level are of "dimension 0". (Similarly, since $\log(0) = -\infty$, the object of the lowest level can be said to have dimension $-\infty$). There are many other toposes E besides finite ones for which the Boolean subtopos is essential, i.e. has a skeletal companion to the usual sheaf inclusion.

The Boolean level is the Aufhebung of the lowest level, where in general, the Aufhebung a' of a level a is the lowest level b which absorbs a in the sense that every a-skeletal object (as well as every a-sheaf) is also a b-sheaf. Although there may be many refined levels in between, we call "dimension 1" the Aufhebung of "dimension 0" (i.e. of the Boolean level). There is an alternate computation of dimension 1 if not–not is not only essential, but its skeleton functor has still a further left adjoint; in other words, the inclusion of discrete objects has a reflection π_0, which assigns to every X the discrete object $\pi_0 X$ of "connected components" of X. (X is connected iff $\pi_0 X = 1$.) Then the level corresponding to "dimension 1" is the smallest level such that the skeleton skX at that level of any connected object X suffices to connect X, or more generally, the canonical map $skX \to X$ induces an isomorphism on applying π_0 for all X.

The lattice of levels has a further rig structure, whose addition is idempotent, because it is the same as the sup operation of the lattice. But the multiplication, usually non-idempotent, corresponds to the idea of adding dimensions as follows: Given two levels a and b, let $a * b$ denote the smallest level s for which the following holds: for any X, Y with $a\text{-}skX = X$ and $b\text{-}skY = Y$, one has $s\text{-}sk(X \times Y) = X \times Y$ for the cartesian product. The unit for the multiplication is the lowest level for which 1 is skeletal, or equivalently, for which X and $cosk\ X$ have the same rational points (maps from 1) for all objects X. In the examples we will consider, the unit for the multiplication of levels coincides with the Boolean (or "zero dimensional") level which is the Aufhebung of the lowest level; for other toposes E which are not "algebraically closed", this coincidence may not hold, but inspired by Galois theory, we can recover it by restricting to only *those* levels at which the adjointness of sk and $cosk$ is enriched over the Boolean level. Here I will denote by u the unit of the rig of levels, so that by the foregoing $u\text{-}skX = X$ just for discrete objects X, and $u * a = a = a * u$ for all levels.

Very little information seems to be available concerning the relations between the boundary and Aufhebung operators and the "dimension rig" and lattice structure, for the levels even of simple combinatorial or algebraic–geometric toposes. Some easy ones are

$$u \leqslant a \ \&\ u \leqslant b \Rightarrow a' + b' \leqslant (a * b)',$$
$$\partial(a \cap b) = a \cap \partial b + (\partial a) \cap b.$$

Part of the reason why the Aufhebung and multiplication may fail to be representable by operations, is the behavior of infinite meets so that the "smallest" level with a certain reasonable property may not exist. However, for many combinatorial examples, consisting of presheaves on a suitable finite category, this difficulty does not arise. For presheaves on an arbitrary small category C, the levels are determined by idempotent two-sided ideals S in C, $\cap$ and $+$ being the ordinary intersection and union of these ideals, with X being an S-sheaf iff $X(B) = (S(B), X)$ for all B in C. Here $S(B)$ is the presheaf whose value at any A in C is $S(A, B)$. For these examples, the above-mentioned difficulty is due to the fact that an infinite intersection of idempotent two-sided ideals is not necessarily idempotent, so that the meet of levels may be smaller than this intersection. Call a category (von Neumann) "regular" if it satisfies the equivalent conditions in the following

Proposition 1. *For a category C the following are equivalent*:
(1) *all two-sided ideals in C are idempotent*;
(2) *for any map a in C, there exist a reverse map $\bar{a}$ and two endomaps x and y for which*

$$a = ya\,\bar{a}\,ax.$$

For example, any monoid in which every element is idempotent is regular. For any regular category C, the Aufhebung and multiplication of levels of presheaves is well defined, as is the Heyting implication, because the infinite meets and joins of levels are just the set-theoretic operations on two-sided ideals. The basic categories I will consider are regular, but also satisfy another, "graphic", condition.

3. Graphic algebra

The three-element monoid {identity, source, target}, whose right actions are just reflexive graphs, generates a subvariety (of the category of all monoids) whose objects I call "graphic" monoids because it is possible to attach "displays" to them which generalize the one-dimensional pictures attached to graphs. All elements of such a monoid M are idempotent and hence reappear as objects in the category $\bar{M}$ whose presheaves are equivalent to right M-sets. $\bar{M}$ is an example of a "graphic category" C meaning that idempotents split and that if p is any splittable epimorphism, a is any parallel map and b is any reverse map, then

$$aba = abp.$$

It follows that $aba = ab$ for any endomorphisms of a common object, and that between any two objects, there is at most one splittable epimorphism. All endomorphisms are in particular idempotent. Each of the unique "downward maps" p in the poset of splittable epimorphisms has some set $S(p)$ of sections.

For any presheaf ($=$ right C-set) X on any graphic category C and a (the) downward map p from B to A, each A-shaped figure in X has a unique degeneration to a B-shaped figure and for each section b in $S(p)$, each B-shaped figure x has a well-defined bth element xb of shape A. These "boundary pieces" may sometimes be pictured as a boundary, or part of a boundary of x, but in other cases are more like a "core". The fundamental conceptual problem of how to picture (or display) the general X reduces in principle to picturing the maximal objects in C itself, since every presheaf is a direct limit of representable ones.

Example 1. Any finite totally ordered set P occurs as the poset of down maps in many graphic categories. Indeed, if P is represented first in any way as splittable epimorphisms in any category, then any set of sections in that category is chosen for each pair of elements of P, a graphic category results. (Note that any poset in itself is graphic by our definition, but not regular.)

Example 2. Ball complexes, also known as reflexive globular sets, form a graphic topos. These are also the underlying combinatorial structures resulting from the multi-dimensional source and target operators in n-categories. Here a basic down map can be pictured as squashing a ball to its equatorial disk, and the two sections interpret a disk as the north and south hemispheres of the ball of one dimension greater.

Any two-sided ideal in a small category is a union of principal ideals (generated by single maps). Thus for a regular category every level is a sum of principal levels.

Proposition 2. *For a regular category in which all endomorphisms are idempotent, every morphism a has a reverse morphism $\bar{a}$ for which*

$$a = a\bar{a}a \quad \text{and} \quad \bar{a} = \bar{a}a\bar{a}$$

In particular, in any category which is both regular and graphic, every $a : B \to A$ can be expressed as

$$a = sq,$$

where q is the splittable epimorphism from B to a certain "image" C, and s is some appropriate section of the unique splittable epimorphism p from A to C so that $ps = 1_C = qj$ for some j. Then $\bar{a}$ can be defined as jp.

Proof. If $a = ya\bar{a}ax$ with idempotents x and y, then $ax = a = ya$; then replacing $\bar{a}$ by $\bar{a}a\bar{a}$ if necessary we find $\bar{a}$ satisfying both equations. The images of the resulting two idempotents $a\bar{a}$ and $\bar{a}a$ are isomorphic. □

Remark. For any presheaf X on a graphic category, the total category of the associated discrete fibration is again graphic. However, the total category may not be regular (even if the base is) because the pseudo-inverses $\bar{a}$ need not lift.

Proposition 3. *The free graphic monoid M on two generators e_1, e_2 has five elements. The two elements e_1e_2 and e_2e_1 both determine the same (discrete) level, but there are nonetheless five levels in all, because there is the ideal $Me_1 \cup Me_2$ corresponding to the sup (sum) of two principal levels.*

Proposition 4. *In a graphic category, any two down maps with the same domain have a pushout. This pushout is preserved by the Yoneda embedding into the graphic topos of presheaves.*

Proof. Suppose that p_1 and p_2 with domain B have sections s_1 and s_2. The idempotents $e_1 = s_1p_1$ and $e_2 = s_2p_2$ generate a graphic monoid of endomorphisms of B, and e_1e_2 and e_2e_1 determine the same level T with the unique down map p from B to T. If we are given any map f from B to another object D which factors across both p_1 and p_2, we must show that there is a single map g from T to D, for which $f = gp$. Uniqueness of g is automatic, so choose any section s of p and define $g = fs$; then

$$gp = fsp = fe_1e_2 = fe_2 = f$$

as required. The same proof works if D is any presheaf. In fact, such a pushout is preserved by any functor whatever. □

The truth-values for any topos of presheaves are of course given by right ideals and any truth-value is a disjunction of principal ones. In a graphic topos the principal truth-values determine their generators. Here for any small category C, the principal truth-value aC, generated by a given map $a : B \to A$, is the presheaf whose value at any D is the set of all those maps $x: D \to A$ which can be expressed as $x = am$ for some map $m: D \to B$.

Proposition 5. *If $a_1C = a_2C$ for graphic C, then $a_1 = a_2$. Moreover, any presheaf X has enough maps to the truth-value presheaf to distinguish figures in X of any shape A.*

Proof. In fact, it suffices that $(a_1C)(B) = (a_2C)(B)$. Then $a_1 = a_2m_1$ and $a_2 = a_1m_2$ for some endomorphisms m_1, m_2 of B. By the graphic identity, $a_1 = a_1m_2m_1 = a_2m_1m_2m_1 = a_2m_1m_2 = a_2$. Much the same calculation applies to any two figures x_1, x_2 of shape B , which if they are different, yields a subpresheaf of X which contains one but not the other. □

By contrast with the situation with right ideals, many distinct maps in C are at the same level in that they generate the same two-sided ideal; for example, source and target in the theory of reflexive graphs, or in general, any two sections of the same splittable epimorphism, are at the same level.

Theorem. *If S is a two-sided ideal generated by a single map a in a category C, which is both graphic and regular, then $S * S = S$ under the multiplication of levels and the S-skeleton functor has a further left adjoint π_s. Thus every level is a sum of multiplicatively idempotent elements.*

Proof. X and Y are S-skeletal iff the action of a is invertible on each; but if so, then the action of a is also invertible on $X \times Y$. The inclusion of the category of these S-skeletal presheaves into all is the same as the inclusion induced by the functor $C \to C[a^{-1}]$ to the fraction category which inverts a; such induced functors always have left adjoints. □

4. Idempotent derivations and graphic involutions

To prepare for the study of linearization of graphic toposes, I first make some very special propositions from linear algebra explicit. For simplicity, let K denote a field containing 1/2.

Definition. An element a in an associative linear algebra is a *graphic idempotent* iff for all elements x, $axa = ax$.

Proposition 6. *The graphic idempotents of an algebra form a graphic monoid. If M is a graphic monoid, then every element of KM is a sum of graphic elements.*

Proof. If a and b are graphic and x is any element, then $abxab = a(bx)ab = abxb = abx$. The defining property of graphic idempotents a is linear in the test variable x (though not in a), which implies the second statement. □

Proposition 7. *In an associative linear algebra consider the derivation $D_a = [-, a]$, the commutator bracket for an idempotent element a. Then $D_a D_a = D_a$ iff a is a graphic idempotent.*

Proof.

$$\begin{aligned}[][x,a],a] &= (xa - ax)a - a(xa - ax)\\ &= xa - axa - axa + ax\\ &= xa - ax - ax + ax\\ &= [x,a].\end{aligned}$$

Conversely, $(D_a D_a - D_a)(x) = 2(ax - axa)$. $\square$

Proposition 8. *If D is a linear endomorphism of a K-linear space, define*

$$T_\lambda = I + (\lambda - 1)D,$$

where I is the identity and $\lambda \varepsilon K$. Then if $D^2 = D$,

$$T_\lambda T_\mu = T_{\lambda\mu}, \qquad T_1 = I$$

so that T_{-1} is an involution. D is recoverable from the family T_λ as $D = (T_\lambda - I)/\lambda - 1$ for any $\lambda \neq 1$, for example $D = \frac{1}{2}(I - T_{-1})$.

Proposition 9. *If D is a derivation of a K-linear associative algebra, then the following are equivalent:*
(1) *$T = I - 2D$ is an algebra homomorphism;*
(2) *D^2 is also a derivation;*
(3) *$(Dx) \cdot (Dy) = 0$ for all x, y in the algebra.*

Proof. (1)

$$\begin{aligned}&T(xy) - (Tx)(Ty)\\ &\quad = xy - 2D(xy) - (x - 2Dx)(y - 2Dy)\\ &\quad = xy - 2(Dx)y - 2xDy - xy + 2(Dx)y + 2xDy - 4DxDy\\ &\quad = -4Dx \cdot Dy.\end{aligned}$$

(Note that the other involutory transformation $2D - I$ does not preserve the constants.)
(2)

$$\begin{aligned}&D^2(xy) - xD^2y - (D^2x)y\\ &\quad = D((Dx)y + xDy) - xD^2y - (D^2x)y\\ &\quad = (D^2x)y + (Dx)\cdot(Dy) + (Dx)Dy + xD^2y - xD^2y - (D^2x)y\\ &\quad = 2(Dx)(Dy). \qquad \square\end{aligned}$$

Corollary. *If a is a graphic idempotent in a linear algebra, then*

$$T(x) = x - 2[x,a]$$

defines an algebra endomorphism with $T^2 = I$. In fact, T is actually conjugation by the involutory element $t = 1-2a$. Moreover $[x,a]\cdot[y,a] = 0$ for all x, y in the algebra.

Proof.

$$\begin{aligned}
t^{-1}xt = txt &= (1-2a)x(1-2a)\\
&= (x-2ax)(1-2a)\\
&= x - 2ax - 2xa + 4axa\\
&= x - 2ax - 2xa + 4ax\\
&= x + 2ax - 2xa\\
&= x - 2[x,a] = T(x). \qquad \square
\end{aligned}$$

Proposition 10. *In any linear associative algebra, $1-2(\ )$ is a bijection between graphic idempotents and "graphic involutions", i.e. elements t satisfying*

$$txt = x + [x,t] \quad \textit{for all } x.$$

Such an involution t is close to the identity in the sense that

$$(txt - x)^2 = 0$$

for all x.

Proof. These statements hold because $[x,t] = 2[a,x]$ has square zero. $\square$

The key ingredient in the presentation of Coxeter groups are the powers of products of pairs of involutions. From this hint, we are led to consider the powers of the products of two graphic involutions t, s:

$$\begin{aligned}
tst &= s + [s,t]\\
(ts)^2 &= s^2 + [s,t]s\\
&= 1 + sts - t\\
&= 1 + [t,s] \quad \text{since } s \text{ is graphic.}
\end{aligned}$$

Thus

Proposition 11. $(ts)^2 = 1$ *iff t commutes with s.*

Proposition 12. *For any two graphic involutions in an associative algebra*

$$(ts)^3 = ts + [t,s],$$

$$(ts)^6 = 1 + 3[t,s].$$

Proof. This follows from Proposition 10. $\square$

To clarify these calculations from an elementary perspective, note that if $x = ts$ is the product of two graphic involutions, then

$$x^2 = 1 - h,$$

where $h = [s, t]$ satisfies $h^2 = 0$ and $hx = h$. Thus we are led to consider the commutative algebra $K[x]$ generated by a single element x satisfying the single cubic equation

$$x^3 = x^2 + x - 1$$

or equivalently

$$(x - 1)^2(x + 1) = 0$$

and to define $h = 1 - x^2$ as an element of this algebra.

Proposition 13. *In $K[x]$ one has*

$$h^2 = 0, \qquad xh = h, \qquad x^2 = 1 - h$$

Moreover the powers of x are

$$x^{2n} = 1 - nh$$

$$x^{2n+1} = x - nh.$$

Proof. The equation $xh = h$ follows from the equation for x,

$$x(1 - x^2) = x - x^3 = x - (x^2 + x - 1) = 1 - x^2. \qquad \square$$

Corollary. *If s and t are graphic involutions in a linear associative algebra and $x = ts$, then the above equations hold.*

Proof.

$$x^3 = x - h = x - (1 - x^2) = x^2 + x - 1.$$

The odd powers of x can be far from 1, for $x = -1$ is a point in the spectrum of $K[x]$. In fact, $K[x] \approx K \times K[h]$. The reciprocal of x^2 is $1 + h$, and therefore the reciprocal of x itself is

$$x(1 + h) = x + xh = x + h.$$

In any homomorphic image of $K[x]$

$$x^3 = 1 \quad \text{iff } x = 1 + h,$$

$$x^6 = 1 \quad \text{iff } 3h = 0. \qquad \square$$

Proposition 14. *If a and b are graphic idempotents and if t and s are the corresponding graphic involutions, let $x = ts$ and $q = (x + x^{-1})/2$. Then*

$x^3 = 1$ *implies* $q = 1$,
$q = 1$ *implies* $x^3 = 1 - 3h/2$,
$q = 1$ *iff* $ab = a$ *and* $ba = b$.

Proof. The first two statements are just properties of the commutative algebra $K[x]$. If $q = 1$, then the graphic idempotents satisfy

$$ab + ba = a + b$$

which implies that $ab = a$ and $ba = b$. Conversely, if $ab = a$ and $ba = b$, then $ts + st = 2$, so $q = 1$. Thus the condition $q = 1$ is an indicator for the coincidence of principal levels. □

5. Coxeter groups and linearized graphic toposes

In plane geometry, a rotation can be expressed as a product of mirror reflections t, s and if the rotation is through an angle which divides 360°, it will have finite order, so that $s^2 = 1 = t^2$, $(ts)^n = 1$, where n is the number of sides of the resulting regular polygon. If we consider several such rotations of various orders in various planes, in a higher dimensional space, very interesting combinatorial configurations result. Sometimes the group generated is finite and sometimes the configurations deserve the name of "crystals". Coxeter investigated these matters nearly 70 years ago, and they were further studied by Dynkin, Gabriel, Tits, and others, in a more general linear algebra context. Call a "Coxeter system" any finite set T equipped with a symmetric map $T \times T \xrightarrow{n} \mathbb{N}_1$ to the positive integers which is 1 on the diagonal; a morphism $f : T \to T'$ of Coxeter systems should satisfy that $n'(f(t_1), f(t_2))$ divides $n(t_1, t_2)$. The obvious functor to groups is the Coxeter presentation. Coxeter systems (especially the special Dynkin ones) are also important in Lie algebra and are an ingredient in the presentation of Hecke algebras. If all occurring values of n are among $1, 2, 3, 4, 6$, then the Coxeter system is called "crystallographic".

Now, if E is a graphic topos, for simplicity say presheaves on a graphic monoid (finite) M, then for any commutative ring K, KE is the category of modules over the algebra KM. If T parameterizes a set of generators for M, then T also parameterizes a set of graphic involutions which generate KM. Can their products be of finite order? In terms of the indicator-involution q and the nilpotent $h = [s, t]$ of Section 4,

$(ts)^2 = 1 - h$,
$(ts)^3 = q - 3h$,
$(ts)^6 = 1 - 3h$.

It might be useful to broaden the ideas of Coxeter group and Hecke algebra by requiring that the defining relations hold only up to a first-order infinitesimal difference. But more precisely, in our case, if we restrict to the coefficient rings K in which $3 = 0$, we obtain the following result, under the above hypotheses.

Theorem. *Suppose* $3=0$ *in* K *and* E *is the topos of right* M*-sets for a graphic monoid* M. *Then* KE *is an abelian subcategory of* $K\bar{E}$, *where* $\bar{E}$ *is the Boolean topos of* G*-sets for a crystallographic Coxeter group* G, *wherein a product of two generating involutions has order* 3 *if and only if they determine the same corresponding principal levels in the topos.*

Indeed, the surjective homomorphism $p: KG \to KM$ induces a full inclusion p^* with adjoints $p!$ and p_*.

Remark. The obvious functorial way to obtain such a group is to take all elements of the given monoid as Coxeter generators. As with “standard resolutions”, much smaller groups can be obtained in almost all particular cases, but probably not functorially. Since the Coxeter relations reflect only, for each pair of monoid elements, whether they commute and whether they are at the same level, it seems natural to ask if the more detailed relations in the monoids may correspond to significant subcategories of Coxeter representations. One could also ask if the exclusion of the exponent $n=4$ among the crystallographic possibilities has a geometric significance which could be related to these graphical constructions.

Although the significance of rotations in characteristic 3 is unclear, the idea of M as a squashed crystal seems to be a step toward systematically understanding the pictures which many examples of M suggest. These pictures result from regarding the elements of M as geometric figures, where a is part of b iff $aM \subseteq bM$, but a “looks like” b iff $Ma=Mb$; moreover, the geometric dimension of these elements is given by the length of chains of Aufhebungen. For example, not only the basic reflexive graph example (three element monoid wherein source and target are at the same discrete level), but also any free graphic monoid has associated to it a one-dimensional picture (I do not know the homological dimension of KM); other non-free graphic monoids may correspond to two-dimensional pictures, or to three-dimensional pictures such as the “taco”. The taco has two two-dimensional sides, but no top, with the two rims and the crease all at the one-dimensional level, with the two endpoints constituting the zero-dimensional level which completes the eight-element monoid. This example itself arises when considering a pair of levels in an arbitrary category, the bigger of which includes the Aufhebung of the other (although the ambient category may have nothing directly to do with graphics, its general objects can be approached through these levels via the pattern described by the 2-category structure of the taco). Such pictures may permit the display of the organizational structure of a hierarchical system, such as a document in hypertext, insofar as that structure can be reflected as an object in a graphical topos.

Uncited References

[1–7]

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