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Nonparametric Integral Statistics $\omega_n^k = n^{k/2} \int_{-\infty}^{\infty} [S_n(x) - F(x)]^k \, dF(x)$: Main Properties and Applications

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Abstract—The main characteristics of nonparametric statistics, ω_n^k, which can be represented as integrals of the k^{th} degree of the empirical process, are considered. An algebraic form of these statistics, convenient for practical usage, is derived. Tables of percentage points for $k = 1, 2, \ldots, 5$ and small empirical sample sizes $n = 1, 2, \ldots, 10$ are obtained. Goodness-of-fit criteria based on ω_n^k statistics are constructed, and their main properties are studied. A method for classifying multidimensional data based on these criteria, which was successfully applied in several experiments, is described.

Keywords—Nonparametric statistics, Goodness-of-fit criteria, Multidimensional data analysis, Pattern recognition.

1. INTRODUCTION

In solving problems of experimental data processing in intermediate and high energy physics use is often made of nonparametric goodness-of-fit criteria based on the well-known ω_n^2-statistic [1–3],

$$\omega_n^2 = n \int_{-\infty}^{\infty} [S_n(x) - F(x)]^2 \, dF(x),$$

and on the recently proposed ω_n^3-statistic [4,5],

$$\omega_n^3 = n^{3/2} \int_{-\infty}^{\infty} [S_n(x) - F(x)]^3 \, dF(x).$$

Here $F(x)$ and $S_n(x)$ are the theoretical and empirical distribution functions of the random variable x, n is the sample size.

The distributions of the ω_n^2 and ω_n^3 statistics are independent of F, as they may be represented as functions of the empirical process (see, for instance, [6,7])

$$v_n(t) = \sqrt{n}[F_n(t) - t], \qquad 0 \le t \le 1.$$

The statistic[1]

$$\bar{F}_n = \int_0^1 F_n(t) \, dt = 1 - \frac{1}{n} \sum_{i=1}^{n} t_i$$

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[1]In an earlier paper [8] the statistic $\bar{U}_n = (1/n) \sum_{i=1}^{n} t_i$, proposed by Moses, was investigated.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX considered in [9] pertains to the same class of statistics, since it can be represented in the following form:

$$\omega_n^1 = \int_0^1 v_n(t)\,dt = \sqrt{n}\int_0^1 [F_n(t) - t]\,dt = \sqrt{n}\left(\bar{F}_n - \frac{1}{2}\right).$$

In paper [10], the momentum and correlation properties of the statistics

$$\omega_n^k = \int_0^1 v_n^k(t)\,dt = n^{k/2}\int_{-\infty}^{\infty} [S_n(x) - F(x)]^k\,dF(x), \qquad k = 1,2,3,\dots \tag{1}$$

were examined within the framework of the general form of the integral statistics $\omega_n(\phi)$ represented as

$$\omega_n(\phi) = \int_0^1 \phi[t, v_n(t)]\,dt,$$

where ϕ is a function of the empirical process $v_n(t)$ (see footnote[2]). Clearly, the statistics ω_n^1, ω_n^2, and ω_n^3 pertain to statistics of class (1).

The present paper[3] is devoted to a generalization of the properties of integral statistics of type (1), to an investigation of their distribution functions, to the construction of goodness-of-fit criteria, and to their comparison with known criteria of the type considered. The method for multidimensional data analysis based on the ω_n^k-criteria is described.

2. ALGEBRAIC FORM OF ω_n^k-STATISTICS

For practical purposes it is convenient to use the algebraic form of the ω_n^k-statistics. Dividing the integration region in (1) into intervals (see, for instance, [4]) $(-\infty, x_1)$, $(x_1, x_2), \dots, (x_n, \infty)$ and taking into account that the empirical distribution function $S_n(x)$ of a variable x has the form:

$$S_n(x) = \begin{cases} 0, & \text{for } x < x_1, \\ \dfrac{i}{n}, & \text{for } x_i \le x < x_{i+1} \qquad (i = 1, 2, \dots, n-1), \\ 1, & \text{for } x \ge x_n, \end{cases}$$

where $x_1 < x_2 < \cdots < x_n$ is the respective variational series, and n is the sample size, one gets:

$$\omega_n^k = (-1)^k n^{k/2}\int_{-\infty}^{x_1} F^k(x)\,dF(x) + n^{k/2}\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{i}{n} - F(x)\right]^k dF(x) + n^{k/2}\int_{x_n}^{\infty}[1 - F(x)]^k\,dF(x).$$

Since $\lim_{x\to-\infty} F(x) = 0$ and $\lim_{x\to\infty} F(x) = 1$, we have:

$$(-1)^k n^{k/2}\int_{-\infty}^{x_1} F^k(x)\,dF(x) = (-1)^k\frac{n^{k/2}}{k+1}F^{k+1}(x_1), \tag{2}$$

$$n^{k/2}\int_{x_n}^{\infty}[1 - F(x)]^k\,dF(x) = \frac{n^{k/2}}{k+1}[1 - F(x_n)]^{k+1}, \tag{3}$$

$$n^{k/2}\sum_{i=1}^{n-1}\int_{x_i}^{x_{i+1}}\left[\frac{i}{n} - F(x)\right]^k dF(x) = -\frac{n^{k/2}}{k+1}\left\{\sum_{i=1}^{n}\left[\frac{i-1}{n} - F(x_i)\right]^{k+1} - \sum_{i=1}^{n}\left[\frac{i}{n} - F(x_i)\right]^{k+1} + [1 - F(x_n)]^{k+1} + (-1)^k F^{k+1}(x_1)\right\}. \tag{4}$$

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[2]Authors are grateful to Ya. Yu. Nikitin who called their attention to the fact that ω_n^k type statistics were proposed and investigated in a general form from the viewpoint of its asymptotic efficiency and local optimality in his papers (see, for example, [11]).

[3]This work represents an extended version (from the point of view of applications) of the article [12].

Adding up the right-hand parts of equations (2)–(4), one gets the algebraic form of the ω_n^k-statistics:

$$\omega_n^k = -\frac{n^{k/2}}{k+1}\sum_{i=1}^{n}\left\{\left[\frac{i-1}{n} - F(x_i)\right]^{k+1} - \left[\frac{i}{n} - F(x_i)\right]^{k+1}\right\}. \tag{5}$$

3. DISTRIBUTION FUNCTIONS OF ω_n^k-STATISTICS

Let us introduce the notation $y_i = F(x_i)$, $y_i \in [0,1]$, $i = 1,2,\dots,n$. Then equation (5) assumes the form

$$\omega_n^k = -\frac{n^{k/2}}{k+1}\sum_{i=1}^{n}\left[\left(\frac{i-1}{n} - y_i\right)^{k+1} - \left(\frac{i}{n} - y_i\right)^{k+1}\right]. \tag{6}$$

Now we denote by z_n^k the variable

$$z_n^k = \omega_n^k - R_n^k,$$

where R_n^k is the minimal value of the random variable ω_n^k. With the aid of equation (6), it is possible to show that

$$R_n^k = \begin{cases} -\dfrac{n^{k/2}}{k+1}, & \text{for odd } k, \\[2ex] \dfrac{1}{2^k n^{k/2}(k+1)}, & \text{for even } k, \end{cases}$$

and the maximal value of variable ω_n^k equals

$$\{\omega_n^k\}_{\max} = \frac{n^{k/2}}{k+1}, \qquad \text{for any } k.$$

The random variable z_n^k varies in the interval $[0, \{\omega_n^k\}_{\max} - R_n^k]$, and its distribution function $F(z)$ equals zero on $(-\infty, 0)$. It has been shown in paper [13], that the inequality

$$F(h) \le \frac{h}{\lambda} + \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\sin(hk\pi/\lambda)}{k}\,\mathrm{Re}\left[\Phi\left(\frac{k\pi}{\lambda}\right)\right] \le F(h) + [1 - F(\lambda)], \tag{7}$$

where $0 < h < \lambda$, is valid for such functions. Inequality (7) permits one to determine the distribution function $F(h)$ by calculating the characteristic function $\Phi(t)$ of the variable z_n^k for any current t.

The characteristic function $\Phi_n^k(t)$ has the form:

$$\Phi_k^n(t) = n!\int_0^1\int_0^{x_n}\cdots\int_0^{x_2}\exp\left\{-it\left\{\frac{n^{k/2}}{k+1}\sum_{i=1}^{n}\left[\left(\frac{i-1}{n} - t_i\right)^{k+1} - \left(\frac{i}{n} - t_i\right)^{k+1}\right] + R_n^k\right\}\right\} dx_1\dots dx_n.$$

Let us denote

$$I(x,t,l) = l!\int_0^x\int_0^{x_l}\cdots\int_0^{x_2}\exp\left\{-it\left\{\frac{n^{k/2}}{k+1}\sum_{i=1}^{l}\left[\left(\frac{i-1}{n} - t_i\right)^{k+1} - \left(\frac{i}{n} - t_i\right)^{k+1}\right] + R_n^k\right\}\right\} dx_1\dots dx_n.$$

For the value $I(x,t,l)$, one can write the recurrent expression

$$I(x,t,l+1) = (l+1)\int_0^x dy\, I(y,t,l)\cdot \exp\left\{-it\left\{\frac{n^{k/2}}{k+1}\left[\left(\frac{l}{n}-y\right)^{k+1} - \left(\frac{l+1}{n}-y\right)^{k+1}\right]\right\}\right\}, \tag{8}$$

which holds true for $l = 2,3,\dots,n-1$. Setting $I(x,t,0) = 1$ for all x and t, one can represent $I(x,t,1)$ in the form

$$I(x,t,1) = \int_0^x dy \exp\left\{-it\left\{\frac{n^{k/2}}{k+1}\left[(-y)^{k+1} - \left(\frac{1}{n}-y\right)^{k+1}\right]\right\}\right\}.$$

The value $\Phi_n^k(t)$ for any chosen t may be defined by numerical integration. In this case the values $I(x,t,l)$ are determined on the grids $x_j = j/m$ $(j = 1,2,\dots,m)$, where m is sufficiently large. Then, setting

$$\Delta I(x,t,l) = I\left(x+\frac{1}{m},t,l\right) - I(x,t,l) \tag{9}$$

and $I(0,t,l+1) = 0$ for any t, we can write the equality:

$$I\left(\frac{r}{m},t,l+1\right) = \sum_{\rho=1}^{r}\Delta I\left(\frac{\rho-1}{m},t,l+1\right), \qquad r = 1,2,\dots,m, \tag{10}$$

which determines the algorithm for calculation of the characteristic function $\Phi_n^k(t) = I(1,t,n)$. With the aid of equations (8) and (9), the values $\Delta I(x,t,l+1)$ in equations (10) can be represented as

$$\Delta I(x,t,l+1) = (l+1)\int_x^{x+(1/m)} dy\, I(y,t,l)\cdot \exp\left\{-it\left\{\frac{n^{k/2}}{k+1}\left[\left(\frac{l}{n}-y\right)^{k+1} - \left(\frac{l+1}{n}-y\right)^{k+1}\right]\right\}\right\} \tag{11}$$

and can be calculated using high-order interpolation formulae [14].

It can be seen from inequality (7), that the accuracy of the $F(h)$ calculation depends on λ and K. K determines the number of terms of the Fourier series. The value of λ was chosen equal to the interval $[0,\ \{\omega_n^k\}_{\max} - R_n^k]$. Varying λ in the region slightly above this value permits one to control the accuracy of calculation of the function $F(h)$, since it should be close to 1 when the value of h is larger than $\{z_n^k\}_{\max}$. The parameter K was determined from the equation $K = 150 \times \lambda$.

The distribution function $F(h)$ of z_n^k was calculated in variable steps Δh, chosen so that $F(h+\Delta h) - F(h) \approx 0.001$. The calculated values were then transformed into values of the distribution function $F_n^k(x)$ of the ω_n^k-statistic in accordance with the expression $F(h) = 1 - F_n^k(x)$, where $x = R_n^k - h$, and were subsequently used for determining the percentage points Z_p.

4. TABLES OF PERCENTAGE POINTS

The percentage points were calculated for $k = 1(1)5$ and sample sizes of $n = 1(1)10$, which took up much computer CPU-time. Tables 2, 4, and 6 (see the Appendix) show the percentage points (Z_p) of the ω_n^k-statistics for odd k, and Tables 3 and 5 (see the Appendix) for even k. Only points $Z_p > 0$ are presented in Tables 2, 4, and 6 for odd k, since owing to the symmetry of the ω_n^k-distribution about zero, the equality

$$F_n^k(Z_p) = 1 - F_n^k(-Z_p)$$

holds true.

4.1. Analysis of Calculations

It is possible to derive an analytical form for the distribution function $F_n^1(z)$ of the ω_n^1-statistic using the following equation:

$$
\begin{aligned}
F_n^1(z) &= \Pr\{\omega_n^1 \le z\} = \Pr\left\{\frac{\sqrt{n}}{2} - \frac{1}{\sqrt{n}}\sum_{i=1}^{n} t_i \le z\right\} \\
&= \Pr\left\{t_1 + t_2 + \cdots + t_n \ge \frac{n}{2} - \sqrt{n}z\right\} \\
&= 1 - \Pr\left\{t_1 + t_2 + \cdots + t_n < \frac{n}{2} - \sqrt{n}z\right\}.
\end{aligned} \tag{12}
$$

We now denote $a = n/2 - \sqrt{n}z$. The probabilities (12) can be calculated for $n = 1, 2$ using geometrical constructions, also. For instance, in the case of a sample of size $n = 3$ one has:

$$
F_3^1(a) = \begin{cases}
1, & a \le 0, \\
1 - \dfrac{1}{2\cdot 3}a^3, & 0 < a \le 1, \\
1 - \dfrac{1}{2\cdot 3}\left[a^3 - 3(a-1)^3\right], & 1 < a \le 2, \\
1 - \dfrac{1}{2\cdot 3}\left[a^3 - 3(a-1)^3 + 3(a-2)^3\right], & 2 < a \le 3, \\
0, & a > 3.
\end{cases}
$$

This result can be generalized to the case of any n as follows:

$$
F_n^1(a) = \begin{cases}
1, & a < 0, \\
1 - \dfrac{1}{n!}\sum_{m=0}^{k}(-1)^m C_n^m (a-m)^n, & k < a < k+1, \\
 & k = 0, 1, \ldots, n-1, \\
0, & a \ge n,
\end{cases}
$$

or passing from a to z,

$$
F_n^1(z) = \begin{cases}
0, & z \le -\dfrac{\sqrt{n}}{2}, \\
1 - \dfrac{1}{n!}\sum_{m=0}^{k}(-1)^m C_n^m \left(\dfrac{n}{2} - \sqrt{n}z - m\right)^n, & \dfrac{n}{2} - \dfrac{k+1}{\sqrt{n}} < z \le \dfrac{n}{2} - \dfrac{k}{\sqrt{n}}, \\
 & k = 0, 1, \ldots, n-1, \\
1, & z > \dfrac{\sqrt{n}}{2}.
\end{cases}
$$

Table data for $k = 1$ (see Tables 2a and 2b) coincide within an accuracy up to $(1 \div 2)\cdot 10^{-5}$ with calculations by this formula.

Analytical formulae for the distribution functions $F_n^2(z)$ of the ω_n^2-statistics for $n = 1, 2$ were obtained in paper [15]. The function $F_n^2(z)$ for $n = 1$ is determined as:

$$
F_1^2(z) = \begin{cases}
0, & z \le \dfrac{1}{2}, \\
\left(4z - \dfrac{1}{3}\right)^{1/2}, & \dfrac{1}{12} \le z \le \dfrac{1}{3}, \\
1, & z > \dfrac{\sqrt{1}}{3},
\end{cases}
$$

and for $n = 2$ has a more complicated form[4]

$$F_2^2(z) = \begin{cases} 0, & z < \dfrac{1}{24}, \\ 2\pi\left(z - \dfrac{1}{24}\right), & \dfrac{1}{24} \le z \le \dfrac{5}{48}, \\ \left(z - \dfrac{1}{24}\right)\left[2\pi - 4\arccos\dfrac{1}{4}\left(z - \dfrac{1}{24}\right)^{-1/2} + \dfrac{(z-(5/48))^{1/2}}{z-(1/24)}\right], & \dfrac{5}{48} < z \le \dfrac{1}{6}, \\ \left(z + \dfrac{1}{24}\right)\left[\dfrac{3\pi}{2} - 2\arccos\dfrac{(1/4)(1/8)^{1/2} - (z-(1/6))^{1/2}(z-(5/48))^{1/2}}{z-(1/24)}\right] & \\ \quad +\dfrac{1}{8} + \left[\dfrac{1}{2}\left(z - \dfrac{1}{6}\right)\right]^{1/2} + \dfrac{1}{2}\left(z - \dfrac{5}{48}\right)^{1/2}, & \dfrac{1}{6} < z \le \dfrac{2}{3}, \\ 1, & z > \dfrac{\sqrt{2}}{3}. \end{cases}$$

Comparison of tabulated data for ω_n^2 for $n = 1, 2$ (see Tables 3a and 3b) with calculations using analytical formulae has shown that they coincide within an accuracy up to $(1 \div 2) \cdot 10^{-5}$.

The percentage points of the ω_n^2-distribution for $n > 2$ were determined with the same accuracy and, excluding some particular points, coincide with the results obtained by Knott in paper [13].

The calculation accuracy of percentage points for the ω_n^3-distribution was estimated by comparison with the values obtained earlier in paper [17]. The calculation accuracy of ω_n^4- and ω_n^5-distribution functions was evaluated by varying the number K and, in the case of the ω_n^5-distribution, by comparing the absolute values of the percentage points, which are symmetrical about $Z_p = 0$. The results for the ω_n^3-statistic coincide with the figures presented in [17], and for the ω_n^4 and ω_n^5 statistics (except for 2–3 points on the tails of the ω_n^5-distribution) an accuracy of $(1 \div 2) \cdot 10^{-4}$ was achieved.

5. GOODNESS-OF-FIT CRITERIA

A detailed description of integral goodness-of-fit criteria for ω_n^3-statistics is presented in paper [18]. In this paper, a brief description is presented of the general scheme for constructing one- and two-sided goodness-of-fit criteria based on ω_n^k-statistics.

5.1. Construction of One-Sided Criteria

The structure of ω_n^k-statistics with odd k makes possible their usage for one-sided goodness-of-fit criteria, i.e., for criteria applied in testing the hypothesis $H_0 : F = F_0$ against one of the two possible alternatives: $F < F_0$ (right-sided) or $F > F_0$ (left-sided).

In the case of the hypothesis $F < F_0$, the critical region B_{α_1} of the right-sided criterion for the selected significance level α_1 is determined from the inequality $\omega_n^k < Z_{\alpha_1}$, where Z_{α_1} is a root of the equation $F_n^k(Z_{\alpha_1}) = \alpha_1$ and $F_n^k(Z)$ is the distribution function of the variable ω_n^k. In the case of the alternative hypothesis $F > F_0$, the critical region of the left-sided criterion is determined from the inequality $\omega_n^k > Z_{\alpha_2}$, and Z_{α_2} satisfies the equation $F_n^k(Z_{\alpha_2}) = 1 - \alpha_2$.

5.2. Construction of Two-Sided Criteria

ω_n^k-statistics with even k turn out to be more appropriate in constructing goodness-of-fit criteria for testing the hypothesis $H_0 : F = F_0$ against the two-sided alternatives $F \neq F_0$. However, in the case of certain hypotheses, the usage of two-sided goodness-of-fit criteria based on the ω_n^k-statistics with odd k proves to be justified and effective (see [19]). The critical region for such criteria is determined as follows:

$$B = \left\{|\omega_n^k| > Z_\alpha^*\right\},$$

where Z_α^* is a root of the equation $F_n^k(Z_\alpha) = 1 - (\alpha/2)$.

[4]In paper [15] this formula is given with an error, so we present, here, the correct expression for $F_2^2(z)$ taken from paper [16].

The critical region for ω_n^k-criteria with even k has the form:

$$B = \{\omega_n^k > Z_\alpha^{**}\},$$

where Z_α^{**} is a root of the equation $F_n^k(Z_\alpha) = 1 - \alpha$.

In Table 1, a summary of criteria based on the ω_n^k-statistics is presented.

Table 1. Goodness-of-fit criteria based on the ω_n^k-statistics.

Type of Criterion	Left-Sided	Right-Sided	Two-Sided
Alternative	$F > F_0$	$F < F_0$	$F \neq F_0$
Critical region, odd k	$\omega_n^k > Z_\alpha$	$\omega_n^k < Z_\alpha$	$\mid \omega_n^k \mid > Z_\alpha$
Equation for Z_α, odd k	$F_n^k(Z_\alpha) = 1 - \alpha$	$F_n^k(Z_\alpha) = \alpha$	$F_n^k(Z_\alpha) = 1 - \frac{\alpha}{2}$
Critical region, even k	—	—	$\omega_n^k > Z_\alpha$
Equation for Z_α, even k	—	—	$F_n^k(Z_\alpha) = 1 - \alpha$

In tests of the hypothesis H_0 with the aid of a one- or a two-sided goodness-of-fit criterion, the $H_0 : F = F_0$ hypothesis is accepted with the significance level α, if the variable ω_n^k, calculated by formula (5), happens to be in the admissable region and is rejected otherwise.

5.3. Properties of the ω_n^k-Criteria

The ω_n^k-criteria presented above are stable, and with respect to one-sided alternative hypotheses they are also consistent and unbiased. The proofs of these properties are similar to the proofs considered in paper [8] for $\bar{U}_n$ and ω_n^2 criteria and in paper [18] for the ω_n^3-criterion.

For studying the behaviour of the power functions of integral criteria, a comparison was performed of the ω_n^k-criteria for $k = 1, 2, \ldots, 5$, making use of large empirical samples, $n \geq 50$. The comparison was carried out applying numerical simulation within the framework of the Chapman approach [8], in which among sets of one-sided alternative hypotheses (for which the value

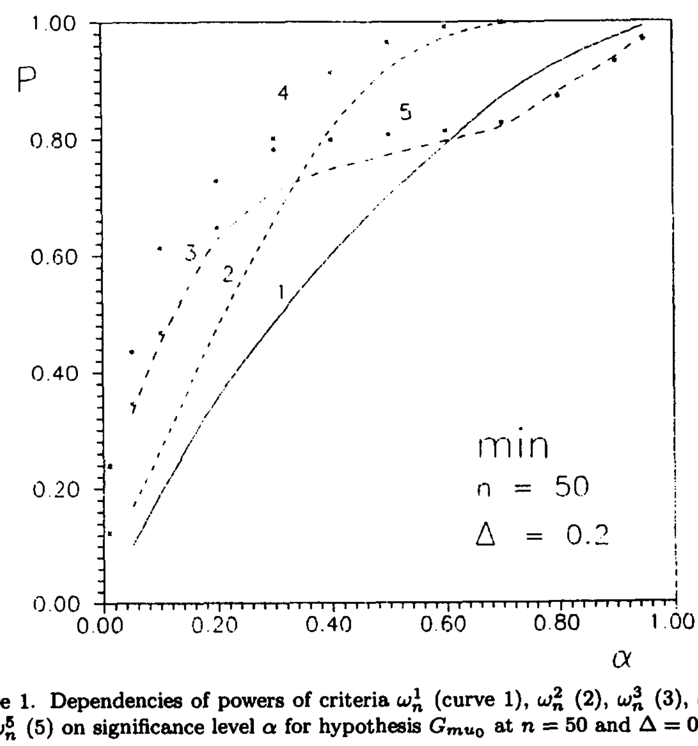

Figure 1. Dependencies of powers of criteria ω_n^1 (curve 1), ω_n^2 (2), ω_n^3 (3), ω_n^4 (4), and ω_n^5 (5) on significance level α for hypothesis G_{mu_0} at $n = 50$ and $\Delta = 0.2$.

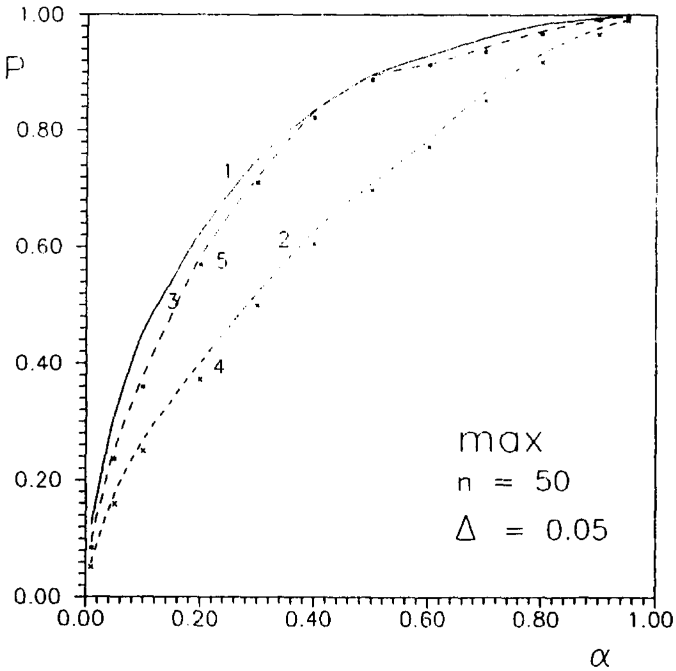

Figure 2. Dependencies of powers of criteria ω_n^1 (curve 1), ω_n^2 (2), ω_n^3 (3), ω_n^4 (4), and ω_n^5 (5) on significance level α for hypothesis G_M at $n = 50$ and $\Delta = 0.05$.

$\sup_{-\infty<x<\infty}[G(x) - F_0(x)] = \Delta$ $(0 < \Delta < 1)$ is fixed) two distribution functions are considered: $G_{mu_0}(x)$ and $G_M(x)$. Here, $G_{mu_0}(x)$ minimizes and $G_M(x)$ maximizes the criterion power. Simulation was done for different values of the significance level α and of the Δ-parameter, in accordance with the procedure described in paper [19].

In Figures 1 and 2 are presented the power curves for the hypothesis G_{mu_0} $(\Delta = 0.20)$ and G_M $(\Delta = 0.05)$, respectively. The size of the empirical samples for both hypotheses was $n = 50$. An analysis of the curves in Figures 1 and 2 reveals that in the case of the G_{mu_0}-hypothesis, ω_n^k-criteria with larger k are more powerful in the region of $\alpha < 0.2$. This is especially important for practical applications. In the case of the G_M-hypothesis, the powers of the criteria are mainly related as follows:

$$P(\omega_n^1) > P(\omega_n^3) > P(\omega_n^5) > P(\omega_n^2) > P(\omega_n^4),$$

and the criteria with odd k $(k = 1,3,5)$ considerably exceed the criteria with even k $(k = 2,4)$, while the difference in power between the criteria ω_n^1, ω_n^3, and ω_n^5 is insignificant (the same applies to the ω_n^2 and ω_n^4 criteria).

6. DATA ANALYSIS BASED ON THE ω_n^k CRITERIA

Goodness-of-fit criteria are usually applied for testing hypotheses, concerning the form of the unknown distribution function, on the basis of an analysis of samples taken from the general set of random values [20] investigated. The ω_n^k-criteria are convenient in that with their aid it is possible to test the correspondence of each individual sample (event) to the distribution known *a priori*.

The following procedure for classifying multidimensional feature events was developed in references [3,5] on the basis of the ω_n^k criteria.

(a) The spectra to be analyzed are transformed ("normalized"), so that the contributions of different dominant distributions (in most cases these are distributions of background events obtained with different detectors) are described by a sole distribution function $F_b(x)$.

(b) Each sample, composed of values pertaining to the different transformed spectra, is tested with the aid of the ω_n^k goodness-of-fit criterion for correspondence to the $F_b(x)$ hypothesis;

in this process the signal events, which do not comply with the null-hypothesis, correspond to large absolute values of the ω_n^k-statistic (see expression (5)), resulting in their clustering in the critical region.

(c) Events that happen to be in the critical region are further subjected to a second test in accordance with items (a) and (b), only with the difference that now it is precisely the signal events that are collected in the admissible region (using the corresponding distribution function $F_s(x)$), and those events that fall into the critical region are rejected; this results in additional suppression of background events in the spectra being studied.

The procedure for data handling described above was applied in analyzing the information obtained in several experiments.

In reference [3], a method was developed for extracting small probability events on the basis of the ω_n^2 (Smirnov-Cramer-von Mises test) goodness-of-fit criterion. It was applied for the identification of the charged particles detected by the MASPIK spectrometer (see details in reference [5]) at the angle of 106 Mrad to the incident beam axis in the collisions of 9 GeV/c deuterons with targets from CD^2, CH^2, and C. The momentum spectrum of secondary particles in the interval 3.5 to 5.5 GeV/c was measured; the main contribution to the spectrum was given by protons, the admixture of deuterons did not exceed 1%. Distributions analyzed were presented as simultaneous time-of-flight measurements performed by two different systems. Analysis of the mass spectra of secondary particles made possible successful extraction of rare events due to secondary deuteron production.

A statistical method based on the ω_n^3 goodness-of-fit criterion for the identification of relativistic charged particles using the measurements of ionization losses in several detectors of the MASPIK spectrometer was suggested in reference [5]. The method was applied for identifying secondary particles (p, d, tritium, ^{3}He and ^{4}He nuclei) produced in collisions of ^{4}He nuclei (4.5 GeV/c per nucleon) with target nuclei at the angle 140 Mrad. It permitted reliable extraction of events due to the production of singly- and doubly-charged particles, the admixture of which did not exceed 0.1%.

Recently, Monte-Carlo simulation has been performed for experimental studies of subthreshold K^+-meson production processes to be carried out at the COSY accelerator (Juelich, Germany). Reliable identification of rare K^+-meson events in conditions of a dominant background of π^+ (by estimation the ratio N_{K^+}/N_{π^+} may amount to 10^{-5}) was shown to be possible applying the traditional statistical method together with a procedure based on the ω_n^3 goodness-of-fit criterion [21].

7. CONCLUSION

The main characteristics of nonparametric statistics, ω_n^k, which can be represented as integrals of the k^{th} degree of the empirical process, are considered. An algebraic form of these statistics, convenient for practical usage, has been obtained. A general method for numerical determination of the distribution functions of ω_n^k statistics is proposed. Tables of percentage points for $k = 1, 2, \ldots, 5$ and small empirical sample sizes $n = 1, 2, \ldots, 10$ have been calculated applying this method. Comparison with known data has been performed. Goodness-of-fit criteria based on these statistics are constructed, and their main properties are considered. A comparative analysis of the powers of ω_n^k criteria for $k = 1, 2, \ldots, 5$ has been performed. The method for classifying multidimensional data based on the ω_n^k criteria, which was successfully applied in several experiments, is described.

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APPENDIX

The tables on the following pages are those referred to in Section 4.

Table 2a. Percentage points Z_p for distribution function of ω_n^1: $F_n^k(Z_p) = \Pr\{\omega_n^1 < Z_p\}$, $n = 1, 2, \ldots, 5$.

Percentage Points Z_p					
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.50	.00000	.00000	.00000	.00000	.00000
.51	.01000	.00711	.00770	.00750	.00747
.52	.02000	.01429	.01540	.01501	.01494
.53	.03000	.02154	.02311	.02252	.02242
.54	.04000	.02887	.03083	.03005	.02991
.55	.05000	.03629	.03857	.03760	.03742
.56	.06000	.04378	.04632	.04518	.04496
.57	.07000	.05136	.05410	.05278	.05252
.58	.08000	.05903	.06190	.06042	.06011
.59	.09000	.06679	.06973	.06810	.06774
.60	.10000	.07465	.07760	.07582	.07541
.61	.11000	.08261	.08551	.08360	.08312
.62	.12000	.09067	.09346	.09142	.09089
.63	.13000	.09883	.10147	.09931	.09872
.64	.14000	.10711	.10952	.10727	.10661
.65	.15000	.11550	.11764	.11530	.11457
.66	.16000	.12401	.12582	.12341	.12261
.67	.17000	.13265	.13408	.13161	.13074
.68	.18000	.14142	.14242	.13990	.13896
.69	.19000	.15033	.15084	.14830	.14727
.70	.20000	.15938	.15936	.15680	.15570
.71	.21000	.16859	.16798	.16543	.16425
.72	.22000	.17796	.17671	.17419	.17294
.73	.23000	.18749	.18558	.18309	.18176
.74	.24000	.19720	.19457	.19214	.19074
.75	.25000	.20711	.20372	.20136	.19989
.76	.26000	.21721	.21304	.21077	.20922
.77	.27000	.22752	.22254	.22036	.21876
.78	.28000	.23807	.23225	.23018	.22852
.79	.29000	.24885	.24218	.24023	.23852
.80	.30000	.25989	.25237	.25054	.24879
.81	.31000	.27122	.26285	.26114	.25935
.82	.32000	.28284	.27366	.27205	.27025
.83	.33000	.29480	.28485	.28332	.28150
.84	.34000	.30711	.29648	.29498	.29317
.85	.35000	.31981	.30860	.30707	.30529
.86	.36000	.33294	.32127	.31967	.31793
.87	.37000	.34655	.33456	.33283	.33116
.88	.38000	.36070	.34856	.34665	.34507
.89	.39000	.37544	.36335	.36123	.35976
.90	.40000	.39088	.37907	.37671	.37537
.91	.41000	.40711	.39587	.39326	.39209
.92	.42000	.42426	.41398	.41113	.41015
.93	.43000	.44253	.43365	.43063	.42988
.94	.44000	.46216	.45531	.45225	.45174
.95	.45000	.48350	.47953	.47668	.47644
.96	.46000	.50711	.50723	.50508	.50513
.97	.47000	.53390	.54004	.53942	.53989
.98	.48000	.56569	.58125	.58382	.58519
.99	.49000	.60711	.64000	.65004	.65421

Table 2b. Percentage points Z_p for distribution function of ω_n^1: $F_n^k(Z_p) = \Pr\{\omega_n^1 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
.50	.00000	.00000	.00000	.00000	.00000
.51	.00742	.00740	.00738	.00736	.00735
.52	.01485	.01480	.01476	.01473	.01470
.53	.02229	.02221	.02215	.02210	.02206
.54	.02974	.02963	.02955	.02949	.02944
.55	.03721	.03707	.03697	.03689	.03683
.56	.04470	.04454	.04441	.04432	.04424
.57	.05222	.05203	.05189	.05178	.05169
.58	.05977	.05955	.05939	.05926	.05916
.59	.06736	.06712	.06693	.06679	.06668
.60	.07499	.07472	.07451	.07436	.07423
.61	.08266	.08237	.08215	.08197	.08184
.62	.09040	.09008	.08983	.08965	.08950
.63	.09819	.09784	.09758	.09738	.09721
.64	.10605	.10567	.10539	.10517	.10500
.65	.11397	.11357	.11327	.11304	.11286
.66	.12198	.12156	.12124	.12099	.12079
.67	.13008	.12962	.12928	.12902	.12882
.68	.13826	.13779	.13743	.13715	.13694
.69	.14656	.14605	.14568	.14539	.14516
.70	.15496	.15443	.15404	.15373	.15349
.71	.16348	.16293	.16252	.16220	.16195
.72	.17214	.17156	.17113	.17081	.17054
.73	.18094	.18034	.17989	.17955	.17928
.74	.18989	.18927	.18881	.18846	.18818
.75	.19902	.19837	.19790	.19754	.19725
.76	.20833	.20766	.20718	.20680	.20651
.77	.21784	.21716	.21666	.21628	.21597
.78	.22758	.22688	.22637	.22597	.22566
.79	.23756	.23685	.23632	.23592	.23560
.80	.24781	.24708	.24655	.24614	.24581
.81	.25835	.25762	.25708	.25666	.25633
.82	.26923	.26848	.26794	.26751	.26718
.83	.28047	.27972	.27917	.27874	.27840
.84	.29211	.29137	.29081	.29038	.29004
.85	.30423	.30348	.30292	.30250	.30216
.86	.31686	.31612	.31557	.31514	.31480
.87	.33009	.32936	.32881	.32839	.32806
.88	.34400	.34329	.34276	.34235	.34202
.89	.35872	.35803	.35751	.35711	.35680
.90	.37437	.37371	.37322	.37285	.37255
.91	.39115	.39054	.39008	.38973	.38945
.92	.40930	.40874	.40833	.40802	.40777
.93	.42915	.42867	.42832	.42806	.42785
.94	.45119	.45082	.45056	.45035	.45019
.95	.47615	.47592	.47577	.47566	.47557
.96	.50519	.50519	.50521	.50522	.50523
.97	.54047	.54082	.54108	.54129	.54146
.98	.58657	.58751	.58819	.58873	.58915
.99	.65717	.65936	.66095	.66218	.66315

Table 3a. Percentage points Z_p for distribution function of ω_n^2: $F_n^k(Z_p) = \Pr\{\omega_n^2 < Z_p\}$, $n = 1, 2, \ldots, 5$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.01	.08362	.04326	.03324	.03008	.02868
.02	.08391	.04485	.03644	.03380	.03265
.03	.08420	.04644	.03902	.03673	.03563
.04	.08450	.04803	.04136	.03925	.03814
.05	.08480	.04962	.04357	.04151	.04036
.06	.08510	.05122	.04565	.04359	.04240
.07	.08542	.05281	.04761	.04555	.04433
.08	.08574	.05440	.04944	.04741	.04617
.09	.08608	.05599	.05119	.04920	.04795
.10	.08643	.05758	.05287	.05092	.04969
.11	.08679	.05917	.05451	.05259	.05139
.12	.08718	.06076	.05613	.05422	.05307
.13	.08760	.06235	.05773	.05582	.05472
.14	.08805	.06395	.05933	.05739	.05637
.15	.08856	.06554	.06092	.05895	.05800
.16	.08913	.06713	.06251	.06052	.05962
.17	.08979	.06873	.06409	.06208	.06124
.18	.09061	.07032	.06568	.06365	.06286
.19	.09166	.07191	.06727	.06523	.06448
.20	.09304	.07350	.06886	.06682	.06610
.21	.09459	.07509	.07046	.06841	.06772
.22	.09599	.07668	.07205	.07002	.06935
.23	.09718	.07827	.07365	.07164	.07098
.24	.09825	.07987	.07524	.07327	.07262
.25	.09926	.08146	.07683	.07493	.07427
.26	.10026	.08305	.07841	.07660	.07593
.27	.10131	.08464	.08000	.07830	.07760
.28	.10247	.08623	.08159	.08001	.07928
.29	.10383	.08781	.08319	.08174	.08097
.30	.10551	.08941	.08480	.08348	.08268
.31	.10749	.09100	.08644	.08525	.08440
.32	.10937	.09260	.08811	.08704	.08614
.33	.11099	.09420	.08981	.08885	.08790
.34	.11245	.09579	.09154	.09069	.08968
.35	.11388	.09737	.09331	.09256	.09149
.36	.11540	.09895	.09512	.09445	.09331
.37	.11714	.10053	.09698	.09636	.09516
.38	.11927	.10213	.09887	.09829	.09704
.39	.12161	.10375	.10081	.10025	.09895
.40	.12372	.10544	.10278	.10225	.10089
.41	.12556	.10721	.10480	.10428	.10286
.42	.12734	.10907	.10687	.10634	.10487
.43	.12923	.11103	.10900	.10842	.10692
.44	.13145	.11308	.11118	.11054	.10901
.45	.13406	.11520	.11342	.11269	.11114
.46	.13657	.11736	.11571	.11488	.11331
.47	.13875	.11957	.11805	.11712	.11554
.48	.14082	.12184	.12045	.11939	.11781
.49	.14304	.12418	.12291	.12171	.12013
.50	.14569	.12659	.12542	.12406	.12251

Table 3b. Percentage points Z_p for distribution function of ω_n^2: $F_n^k(Z_p) = \Pr\{\omega_n^2 < Z_p\}$, $n = 1, 2, \ldots, 5$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.51	.14860	.12905	.12799	.12647	.12495
.52	.15119	.13155	.13060	.12892	.12745
.53	.15353	.13410	.13328	.13144	.13002
.54	.15596	.13672	.13603	.13401	.13265
.55	.15880	.13940	.13884	.13663	.13535
.56	.16196	.14213	.14171	.13932	.13813
.57	.16477	.14491	.14464	.14209	.14098
.58	.16735	.14772	.14764	.14493	.14391
.59	.17009	.15060	.15073	.14786	.14693
.60	.17334	.15355	.15389	.15086	.15004
.61	.17662	.15655	.15711	.15396	.15324
.62	.17949	.15958	.16042	.15717	.15654
.63	.18233	.16265	.16382	.16048	.15994
.64	.18562	.16581	.16731	.16389	.16345
.65	.18919	.16916	.17089	.16742	.16708
.66	.19234	.17277	.17457	.17109	.17083
.67	.19536	.17666	.17836	.17490	.17471
.68	.19882	.18075	.18226	.17884	.17873
.69	.20258	.18501	.18627	.18296	.18289
.70	.20591	.18950	.19041	.18726	.18722
.71	.20915	.19421	.19470	.19173	.19171
.72	.21290	.19910	.19911	.19640	.19639
.73	.21678	.20424	.20370	.20129	.20126
.74	.22020	.20961	.20846	.20638	.20634
.75	.22375	.21521	.21339	.21172	.21165
.76	.22785	.22109	.21854	.21731	.21720
.77	.23170	.22721	.22391	.22316	.22303
.78	.23526	.23365	.22954	.22932	.22914
.79	.23930	.24037	.23546	.23579	.23557
.80	.24354	.24743	.24169	.24261	.24235
.81	.24729	.25482	.24831	.24981	.24952
.82	.25127	.26262	.25535	.25744	.25711
.83	.25575	.27080	.26288	.26554	.26519
.84	.25974	.27942	.27095	.27416	.27381
.85	.26377	.28855	.27963	.28338	.28304
.86	.26839	.29819	.28909	.29327	.29298
.87	.27261	.30841	.29944	.30393	.30373
.88	.27674	.31931	.31094	.31547	.31542
.89	.28150	.33095	.32372	.32805	.32823
.90	.28589	.34343	.33786	.34185	.34238
.91	.29016	.35689	.35352	.35711	.35814
.92	.29509	.37148	.37098	.37417	.37587
.93	.29958	.38744	.39068	.39346	.39606
.94	.30405	.40507	.41318	.41573	.41939
.95	.30917	.42480	.43938	.44203	.44695
.96	.31367	.44732	.47068	.47417	.48054
.97	.31842	.47373	.50951	.51570	.52353
.98	.32369	.50624	.56104	.57414	.58338
.99	.32806	.55058	.63977	.67008	.68348

Table 3c. Percentage points Z_p for distribution function of ω_n^2: $F_n^k(Z_p) = \Pr\{\omega_n^2 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
.01	.02795	.02742	.02703	.02675	.02654
.02	.03189	.03135	.03099	.03074	.03054
.03	.03484	.03433	.03400	.03375	.03354
.04	.03734	.03687	.03655	.03629	.03608
.05	.03959	.03915	.03883	.03857	.03835
.06	.04167	.04125	.04093	.04066	.04044
.07	.04364	.04323	.04290	.04262	.04240
.08	.04552	.04512	.04478	.04449	.04427
.09	.04734	.04693	.04658	.04630	.04608
.10	.04911	.04869	.04833	.04804	.04783
.11	.05083	.05041	.05004	.04975	.04954
.12	.05252	.05208	.05171	.05142	.05121
.13	.05419	.05374	.05335	.05307	.05286
.14	.05584	.05536	.05498	.05470	.05449
.15	.05747	.05698	.05659	.05632	.05611
.16	.05909	.05858	.05819	.05792	.05771
.17	.06069	.06017	.05979	.05952	.05931
.18	.06229	.06176	.06138	.06111	.06090
.19	.06389	.06335	.06297	.06271	.06250
.20	.06548	.06493	.06456	.06430	.06409
.21	.06708	.06653	.06616	.06590	.06569
.22	.06868	.06812	.06776	.06750	.06729
.23	.07028	.06973	.06937	.06911	.06890
.24	.07189	.07134	.07099	.07073	.07052
.25	.07351	.07297	.07262	.07236	.07215
.26	.07514	.07461	.07426	.07400	.07379
.27	.07679	.07626	.07592	.07566	.07544
.28	.07845	.07793	.07759	.07733	.07711
.29	.08012	.07961	.07928	.07902	.07880
.30	.08182	.08132	.08099	.08072	.08050
.31	.08353	.08304	.08271	.08245	.08223
.32	.08527	.08479	.08446	.08419	.08397
.33	.08703	.08656	.08623	.08596	.08574
.34	.08881	.08835	.08802	.08775	.08753
.35	.09062	.09017	.08984	.08957	.08935
.36	.09246	.09202	.09169	.09141	.09119
.37	.09432	.09389	.09356	.09329	.09306
.38	.09622	.09580	.09546	.09519	.09496
.39	.09815	.09773	.09740	.09712	.09689
.40	.10012	.09970	.09936	.09909	.09886
.41	.10212	.10170	.10136	.10108	.10086
.42	.10415	.10374	.10340	.10312	.10290
.43	.10623	.10582	.10548	.10519	.10497
.44	.10834	.10793	.10759	.10731	.10709
.45	.11050	.11009	.10975	.10946	.10924
.46	.11271	.11229	.11195	.11166	.11145
.47	.11496	.11454	.11419	.11391	.11369
.48	.11725	.11684	.11649	.11620	.11599
.49	.11960	.11919	.11883	.11855	.11834
.50	.12200	.12158	.12123	.12095	.12074

Table 3d. Percentage points Z_p for distribution function of ω_n^2: $F_n^k(Z_p) = \Pr\{\omega_n^2 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
.51	.12446	.12404	.12368	.12340	.12320
.52	.12698	.12655	.12619	.12591	.12571
.53	.12955	.12912	.12876	.12849	.12829
.54	.13219	.13176	.13140	.13113	.13093
.55	.13490	.13447	.13410	.13384	.13365
.56	.13768	.13724	.13688	.13662	.13643
.57	.14054	.14009	.13973	.13948	.13929
.58	.14347	.14302	.14266	.14242	.14223
.59	.14648	.14603	.14568	.14544	.14525
.60	.14958	.14913	.14878	.14855	.14836
.61	.15278	.15232	.15198	.15175	.15156
.62	.15607	.15560	.15528	.15506	.15487
.63	.15946	.15900	.15868	.15847	.15828
.64	.16297	.16250	.16220	.16199	.16180
.65	.16658	.16612	.16583	.16562	.16544
.66	.17033	.16986	.16959	.16939	.16921
.67	.17420	.17374	.17348	.17328	.17311
.68	.17821	.17776	.17752	.17732	.17715
.69	.18236	.18193	.18171	.18151	.18135
.70	.18668	.18627	.18606	.18587	.18571
.71	.19117	.19078	.19059	.19040	.19024
.72	.19584	.19548	.19530	.19511	.19497
.73	.20070	.20039	.20021	.20003	.19990
.74	.20579	.20551	.20534	.20517	.20504
.75	.21110	.21087	.21071	.21055	.21043
.76	.21667	.21648	.21633	.21618	.21608
.77	.22251	.22237	.22223	.22210	.22200
.78	.22866	.22857	.22843	.22831	.22824
.79	.23514	.23509	.23497	.23487	.23481
.80	.24198	.24198	.24187	.24179	.24175
.81	.24923	.24927	.24918	.24913	.24910
.82	.25694	.25701	.25694	.25692	.25691
.83	.26514	.26524	.26521	.26521	.26522
.84	.27391	.27403	.27404	.27408	.27411
.85	.28331	.28346	.28351	.28359	.28364
.86	.29342	.29360	.29371	.29383	.29391
.87	.30436	.30457	.30475	.30491	.30502
.88	.31624	.31651	.31676	.31698	.31713
.89	.32922	.32957	.32992	.33019	.33038
.90	.34352	.34398	.34444	.34477	.34502
.91	.35941	.36002	.36060	.36100	.36133
.92	.37726	.37808	.37879	.37929	.37970
.93	.39759	.39867	.39954	.40017	.40068
.94	.42115	.42258	.42364	.42444	.42509
.95	.44911	.45099	.45230	.45333	.45415
.96	.48342	.48586	.48755	.48889	.48995
.97	.52770	.53087	.53315	.53495	.53637
.98	.58987	.59420	.59752	.60005	.60208
.99	.69441	.70159	.70702	.71118	.71450

Table 4a. Percentage points Z_p for distribution function of ω_n^3: $F_n^k(Z_p) = \Pr\{\omega_n^3 < Z_p\}$, $n = 1, 2, \ldots, 5$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n = 1$	$n = 2$	$n = 3$	$n = 4$	$n = 5$
.50	.00000	.00000	.00000	.00000	.00000
.51	.00250	.00147	.00131	.00126	.00124
.52	.00501	.00294	.00262	.00253	.00249
.53	.00753	.00442	.00394	.00382	.00377
.54	.01006	.00592	.00529	.00514	.00507
.55	.01262	.00744	.00667	.00649	.00641
.56	.01522	.00898	.00808	.00790	.00781
.57	.01784	.01056	.00954	.00936	.00926
.58	.02051	.01217	.01104	.01088	.01076
.59	.02323	.01381	.01261	.01247	.01234
.60	.02600	.01550	.01424	.01413	.01398
.61	.02883	.01724	.01595	.01588	.01570
.62	.03173	.01902	.01775	.01770	.01749
.63	.03470	.02086	.01965	.01962	.01936
.64	.03774	.02276	.02166	.02163	.02133
.65	.04087	.02472	.02378	.02375	.02340
.66	.04410	.02676	.02604	.02598	.02557
.67	.04741	.02887	.02842	.02832	.02785
.68	.05083	.03107	.03095	.03078	.03025
.69	.05436	.03337	.03362	.03337	.03277
.70	.05800	.03578	.03646	.03611	.03544
.71	.06176	.03833	.03947	.03900	.03827
.72	.06565	.04104	.04266	.04206	.04125
.73	.06967	.04396	.04604	.04528	.04442
.74	.07382	.04713	.04964	.04871	.04778
.75	.07812	.05061	.05346	.05234	.05136
.76	.08258	.05443	.05753	.05619	.05519
.77	.08718	.05862	.06186	.06030	.05929
.78	.09195	.06320	.06648	.06468	.06369
.79	.09689	.06822	.07142	.06937	.06844
.80	.10200	.07369	.07670	.07440	.07357
.81	.10729	.07966	.08236	.07981	.07914
.82	.11277	.08617	.08844	.08566	.08517
.83	.11844	.09329	.09498	.09201	.09173
.84	.12430	.10107	.10205	.09894	.09889
.85	.13038	.10959	.10972	.10656	.10673
.86	.13666	.11895	.11805	.11499	.11533
.87	.14315	.12924	.12716	.12442	.12485
.88	.14987	.14061	.13718	.13500	.13542
.89	.15682	.15321	.14827	.14691	.14724
.90	.16400	.16726	.16066	.16040	.16058
.91	.17142	.18300	.17465	.17578	.17577
.92	.17909	.20078	.19071	.19351	.19330
.93	.18701	.22105	.20953	.21423	.21387
.94	.19518	.24443	.23236	.23886	.23852
.95	.20363	.27183	.26159	.26887	.26891
.96	.21234	.30464	.30006	.30676	.30793
.97	.22132	.34524	.35175	.35725	.36109
.98	.23059	.39830	.42636	.43142	.44024
.99	.24015	.47592	.55265	.56721	.58339

Table 4b. Percentage points Z_p for distribution function of ω_n^3: $F_n^k(Z_p) = \Pr\{\omega_n^3 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
.50	.00000	.00000	.00000	.00000	.00000
.51	.00123	.00122	.00121	.00120	.00119
.52	.00247	.00244	.00243	.00241	.00240
.53	.00372	.00369	.00366	.00364	.00362
.54	.00501	.00497	.00493	.00490	.00488
.55	.00634	.00628	.00624	.00620	.00617
.56	.00772	.00765	.00759	.00754	.00751
.57	.00915	.00906	.00899	.00894	.00890
.58	.01064	.01054	.01046	.01040	.01036
.59	.01220	.01208	.01199	.01193	.01188
.60	.01382	.01368	.01358	.01352	.01347
.61	.01550	.01535	.01525	.01518	.01512
.62	.01727	.01710	.01699	.01692	.01686
.63	.01911	.01894	.01882	.01874	.01868
.64	.02105	.02086	.02074	.02066	.02059
.65	.02309	.02289	.02277	.02268	.02261
.66	.02524	.02503	.02491	.02482	.02474
.67	.02749	.02728	.02716	.02706	.02698
.68	.02987	.02966	.02954	.02943	.02934
.69	.03237	.03217	.03204	.03193	.03183
.70	.03503	.03484	.03470	.03458	.03447
.71	.03785	.03766	.03752	.03739	.03727
.72	.04084	.04067	.04051	.04037	.04025
.73	.04402	.04385	.04368	.04352	.04341
.74	.04741	.04723	.04705	.04689	.04676
.75	.05104	.05085	.05065	.05048	.05036
.76	.05491	.05471	.05449	.05432	.05419
.77	.05905	.05882	.05859	.05842	.05829
.78	.06349	.06324	.06300	.06282	.06270
.79	.06827	.06799	.06774	.06757	.06744
.80	.07340	.07309	.07283	.07267	.07255
.81	.07895	.07862	.07836	.07820	.07807
.82	.08495	.08459	.08433	.08418	.08406
.83	.09147	.09109	.09085	.09070	.09057
.84	.09857	.09817	.09796	.09782	.09769
.85	.10635	.10595	.10577	.10563	.10550
.86	.11489	.11452	.11436	.11423	.11410
.87	.12434	.12401	.12390	.12376	.12364
.88	.13487	.13462	.13453	.13440	.13429
.89	.14668	.14653	.14646	.14635	.14626
.90	.16005	.16003	.15998	.15989	.15984
.91	.17536	.17548	.17545	.17541	.17540
.92	.19313	.19338	.19339	.19341	.19346
.93	.21410	.21446	.21455	.21468	.21479
.94	.23931	.23977	.24001	.24027	.24047
.95	.27039	.27101	.27149	.27193	.27225
.96	.31014	.31109	.31198	.31267	.31318
.97	.36394	.36563	.36713	.36819	.36906
.98	.44420	.44755	.44998	.45184	.45337
.99	.59220	.59907	.60388	.60773	.61082

Table 5a. Percentage points Z_p for distribution function of ω_n^4: $F_n^k(Z_p) = \Pr\{\omega_n^4 < Z_p\}$, $n = 1, 2, \ldots, 5$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.01	.01271	.00355	.00198	.00142	.00116
.02	.01291	.00398	.00256	.00206	.00182
.03	.01312	.00440	.00313	.00267	.00245
.04	.01333	.00483	.00369	.00326	.00306
.05	.01354	.00527	.00423	.00382	.00364
.06	.01375	.00570	.00475	.00437	.00419
.07	.01396	.00614	.00526	.00488	.00472
.08	.01418	.00659	.00576	.00538	.00523
.09	.01440	.00704	.00624	.00586	.00572
.10	.01462	.00750	.00671	.00633	.00619
.11	.01485	.00796	.00717	.00678	.00666
.12	.01509	.00843	.00762	.00722	.00711
.13	.01533	.00891	.00806	.00766	.00755
.14	.01558	.00940	.00850	.00809	.00800
.15	.01584	.00990	.00893	.00852	.00843
.16	.01611	.01041	.00936	.00895	.00887
.17	.01640	.01093	.00980	.00938	.00931
.18	.01670	.01146	.01023	.00981	.00975
.19	.01701	.01200	.01067	.01025	.01020
.20	.01735	.01255	.01111	.01070	.01066
.21	.01772	.01311	.01156	.01116	.01113
.22	.01812	.01368	.01202	.01164	.01161
.23	.01857	.01426	.01249	.01213	.01211
.24	.01909	.01484	.01298	.01264	.01263
.25	.01969	.01543	.01348	.01318	.01318
.26	.02042	.01603	.01400	.01374	.01375
.27	.02134	.01663	.01454	.01434	.01435
.28	.02247	.01723	.01511	.01498	.01499
.29	.02368	.01784	.01570	.01566	.01566
.30	.02478	.01845	.01633	.01639	.01636
.31	.02575	.01907	.01699	.01715	.01710
.32	.02662	.01969	.01769	.01794	.01785
.33	.02741	.02031	.01842	.01876	.01862
.34	.02818	.02094	.01918	.01958	.01940
.35	.02893	.02158	.01996	.02041	.02018
.36	.02970	.02223	.02077	.02123	.02095
.37	.03050	.02289	.02159	.02204	.02173
.38	.03138	.02356	.02244	.02286	.02251
.39	.03237	.02424	.02330	.02368	.02330
.40	.03354	.02495	.02418	.02453	.02411
.41	.03499	.02568	.02509	.02540	.02495
.42	.03664	.02644	.02604	.02631	.02583
.43	.03821	.02723	.02705	.02729	.02676
.44	.03957	.02805	.02811	.02833	.02776
.45	.04079	.02893	.02925	.02946	.02884
.46	.04193	.02985	.03047	.03066	.02999
.47	.04307	.03083	.03176	.03192	.03120
.48	.04427	.03187	.03309	.03318	.03244
.49	.04562	.03298	.03445	.03443	.03369
.50	.04725	.03415	.03580	.03567	.03492

Table 5b. Percentage points Z_p for distribution function of ω_n^4: $F_n^k(Z_p) = \Pr\{\omega_n^4 < Z_p\}$, $n = 1, 2, \ldots, 5$.

Percentage Points Z_p					
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.51	.04922	.03539	.03717	.03691	.03615
.52	.05124	.03669	.03857	.03818	.03740
.53	.05300	.03805	.04004	.03951	.03871
.54	.05455	.03945	.04161	.04094	.04012
.55	.05602	.04091	.04331	.04253	.04166
.56	.05754	.04240	.04513	.04426	.04336
.57	.05927	.04396	.04702	.04606	.04517
.58	.06138	.04558	.04890	.04784	.04699
.59	.06382	.04728	.05078	.04958	.04878
.60	.06602	.04908	.05271	.05133	.05058
.61	.06791	.05100	.05477	.05320	.05246
.62	.06970	.05306	.05705	.05528	.05456
.63	.07159	.05529	.05949	.05763	.05696
.64	.07385	.05769	.06194	.06007	.05952
.65	.07660	.06027	.06437	.06243	.06200
.66	.07920	.06302	.06688	.06476	.06444
.67	.08141	.06593	.06966	.06728	.06708
.68	.08350	.06902	.07271	.07021	.07019
.69	.08580	.07231	.07579	.07340	.07352
.70	.08866	.07583	.07883	.07643	.07667
.71	.09174	.07956	.08211	.07949	.07992
.72	.09434	.08353	.08584	.08309	.08382
.73	.09672	.08776	.08960	.08715	.08796
.74	.09933	.09228	.09335	.09094	.09184
.75	.10260	.09708	.09762	.09508	.09637
.76	.10591	.10221	.10222	.10017	.10143
.77	.10867	.10770	.10672	.10491	.10615
.78	.11139	.11356	.11194	.11044	.11202
.79	.11467	.11985	.11738	.11648	.11770
.80	.11842	.12658	.12307	.12262	.12420
.81	.12156	.13383	.12955	.12982	.13097
.82	.12454	.14162	.13609	.13726	.13869
.83	.12811	.15002	.14362	.14533	.14640
.84	.13214	.15911	.15156	.15478	.15571
.85	.13549	.16895	.16013	.16447	.16553
.86	.13882	.17965	.17000	.17506	.17597
.87	.14298	.19132	.18079	.18712	.18781
.88	.14705	.20410	.19276	.20063	.20121
.89	.15054	.21815	.20651	.21595	.21658
.90	.15457	.23368	.22247	.23371	.23424
.91	.15918	.25096	.24111	.25267	.25342
.92	.16302	.27033	.26432	.27600	.27708
.93	.16710	.29226	.29154	.30278	.30470
.94	.17203	.31737	.32459	.33427	.33937
.95	.17620	.34660	.36537	.37396	.38193
.96	.18052	.38137	.41671	.42525	.43699
.97	.18573	.42414	.48623	.49533	.51416
.98	.19016	.47969	.58504	.60376	.62859
.99	.19473	.56052	.75560	.81141	.84804

Table 5c. Percentage points Z_p for distribution function of ω_n^4: $F_n^k(Z_p) = \Pr\{\omega_n^4 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n = 6$	$n = 7$	$n = 8$	$n = 9$	$n = 10$
.01	.00104	.00097	.00092	.00088	.00085
.02	.00172	.00166	.00162	.00159	.00155
.03	.00237	.00233	.00230	.00226	.00223
.04	.00300	.00296	.00293	.00289	.00286
.05	.00359	.00356	.00353	.00349	.00346
.06	.00415	.00412	.00409	.00405	.00401
.07	.00468	.00465	.00462	.00458	.00454
.08	.00519	.00516	.00512	.00508	.00505
.09	.00568	.00565	.00560	.00556	.00553
.10	.00616	.00612	.00607	.00602	.00599
.11	.00662	.00657	.00652	.00647	.00644
.12	.00707	.00702	.00696	.00691	.00687
.13	.00751	.00745	.00739	.00734	.00730
.14	.00795	.00788	.00782	.00776	.00773
.15	.00838	.00831	.00824	.00819	.00815
.16	.00881	.00873	.00866	.00860	.00856
.17	.00925	.00916	.00908	.00902	.00898
.18	.00968	.00959	.00951	.00945	.00941
.19	.01013	.01002	.00994	.00988	.00983
.20	.01058	.01047	.01037	.01031	.01027
.21	.01104	.01092	.01082	.01076	.01072
.22	.01151	.01138	.01128	.01122	.01117
.23	.01200	.01186	.01176	.01169	.01165
.24	.01251	.01236	.01225	.01218	.01214
.25	.01304	.01288	.01277	.01270	.01266
.26	.01359	.01342	.01331	.01324	.01320
.27	.01417	.01400	.01388	.01382	.01377
.28	.01479	.01460	.01449	.01442	.01438
.29	.01543	.01524	.01513	.01507	.01502
.30	.01611	.01592	.01581	.01575	.01571
.31	.01682	.01663	.01652	.01647	.01642
.32	.01756	.01736	.01727	.01722	.01717
.33	.01831	.01812	.01803	.01799	.01794
.34	.01908	.01889	.01881	.01877	.01872
.35	.01985	.01967	.01960	.01956	.01951
.36	.02062	.02045	.02039	.02034	.02029
.37	.02140	.02123	.02117	.02113	.02108
.38	.02218	.02202	.02196	.02192	.02186
.39	.02297	.02282	.02276	.02271	.02265
.40	.02378	.02363	.02357	.02352	.02346
.41	.02461	.02447	.02441	.02435	.02429
.42	.02549	.02535	.02529	.02523	.02516
.43	.02641	.02628	.02622	.02615	.02608
.44	.02740	.02729	.02722	.02714	.02707
.45	.02847	.02837	.02830	.02822	.02814
.46	.02962	.02953	.02947	.02938	.02930
.47	.03083	.03077	.03071	.03062	.03053
.48	.03209	.03204	.03198	.03189	.03181
.49	.03336	.03332	.03326	.03317	.03309
.50	.03462	.03459	.03452	.03443	.03435

Table 5d. Percentage points Z_p for distribution function of ω_n^4: $F_n^k(Z_p) = \Pr\{\omega_n^4$ $Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
.51	.03588	.03585	.03578	.03568	.03560
.52	.03717	.03713	.03705	.03695	.03687
.53	.03851	.03847	.03838	.03826	.03818
.54	.03995	.03991	.03980	.03968	.03960
.55	.04153	.04149	.04137	.04125	.04116
.56	.04327	.04324	.04312	.04299	.04291
.57	.04513	.04510	.04497	.04485	.04478
.58	.04699	.04695	.04683	.04671	.04665
.59	.04881	.04876	.04863	.04852	.04846
.60	.05064	.05057	.05043	.05032	.05027
.61	.05256	.05248	.05232	.05221	.05216
.62	.05471	.05462	.05444	.05433	.05428
.63	.05714	.05704	.05686	.05676	.05672
.64	.05969	.05958	.05942	.05933	.05930
.65	.06215	.06202	.06187	.06180	.06177
.66	.06459	.06444	.06428	.06421	.06418
.67	.06725	.06708	.06691	.06685	.06682
.68	.07037	.07018	.07002	.06998	.06996
.69	.07366	.07347	.07334	.07333	.07331
.70	.07677	.07657	.07645	.07644	.07642
.71	.08002	.07979	.07968	.07968	.07965
.72	.08392	.08368	.08360	.08363	.08361
.73	.08799	.08778	.08774	.08777	.08776
.74	.09184	.09162	.09160	.09162	.09160
.75	.09635	.09613	.09616	.09621	.09620
.76	.10135	.10118	.10125	.10130	.10130
.77	.10604	.10589	.10598	.10602	.10602
.78	.11185	.11177	.11195	.11202	.11203
.79	.11752	.11748	.11764	.11769	.11770
.80	.12398	.12405	.12432	.12441	.12445
.81	.13076	.13088	.13110	.13117	.13121
.82	.13847	.13879	.13911	.13922	.13931
.83	.14624	.14662	.14690	.14701	.14713
.84	.15562	.15610	.15638	.15650	.15664
.85	.16560	.16635	.16670	.16691	.16711
.86	.17627	.17725	.17770	.17800	.17830
.87	.18838	.18949	.18998	.19037	.19076
.88	.20220	.20348	.20406	.20457	.20503
.89	.21820	.21951	.22012	.22065	.22109
.90	.23596	.23703	.23762	.23821	.23869
.91	.25631	.25801	.25907	.25999	.26062
.92	.28002	.28168	.28315	.28452	.28545
.93	.30873	.31105	.31288	.31416	.31499
.94	.34357	.34547	.34752	.34956	.35120
.95	.38636	.39005	.39338	.39513	.39635
.96	.44324	.44811	.45089	.45313	.45589
.97	.52044	.52835	.53193	.53653	.54031
.98	.64024	.65095	.65994	.66382	.66839
.99	.87429	.88984	.90310	.91491	.92022

Table 6a. Percentage points Z_p for distribution function of ω_n^5: $F_n^k(Z_p) = \Pr\{\omega_n^5 < Z_p\}$, $n = 1, 2, \ldots, 5$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$
.50	.00000	.00000	.00000	.00000	.00000
.51	.00064	.00035	.00032	.00032	.00032
.52	.00127	.00071	.00064	.00065	.00065
.53	.00192	.00107	.00096	.00097	.00097
.54	.00258	.00143	.00128	.00130	.00130
.55	.00325	.00179	.00161	.00164	.00164
.56	.00395	.00217	.00195	.00199	.00199
.57	.00467	.00256	.00230	.00234	.00234
.58	.00543	.00295	.00265	.00270	.00270
.59	.00623	.00337	.00303	.00308	.00308
.60	.00707	.00380	.00342	.00348	.00348
.61	.00797	.00425	.00383	.00390	.00390
.62	.00894	.00473	.00426	.00436	.00434
.63	.00997	.00524	.00474	.00484	.00483
.64	.01109	.00579	.00526	.00538	.00536
.65	.01228	.00639	.00584	.00598	.00595
.66	.01355	.00705	.00651	.00668	.00662
.67	.01489	.00778	.00730	.00751	.00742
.68	.01632	.00859	.00829	.00854	.00839
.69	.01783	.00949	.00957	.00985	.00960
.70	.01946	.01048	.01117	.01139	.01105
.71	.02122	.01153	.01282	.01294	.01256
.72	.02313	.01264	.01434	.01436	.01399
.73	.02518	.01379	.01577	.01572	.01534
.74	.02735	.01501	.01719	.01707	.01668
.75	.02963	.01635	.01876	.01856	.01811
.76	.03206	.01793	.02072	.02036	.01979
.77	.03469	.01999	.02354	.02288	.02205
.78	.03752	.02288	.02670	.02593	.02507
.79	.04053	.02594	.02931	.02854	.02787
.80	.04367	.02874	.03189	.03096	.03036
.81	.04703	.03198	.03544	.03391	.03321
.82	.05067	.03671	.04025	.03844	.03771
.83	.05451	.04107	.04402	.04247	.04217
.84	.05852	.04610	.04870	.04633	.04622
.85	.06286	.05257	.05501	.05264	.05291
.86	.06745	.05910	.06029	.05789	.05840
.87	.07224	.06708	.06837	.06569	.06681
.88	.07742	.07678	.07577	.07278	.07423
.89	.08283	.08749	.08498	.08326	.08455
.90	.08856	.09999	.09634	.09547	.09674
.91	.09466	.11473	.10921	.10955	.11071
.92	.10104	.13148	.12399	.12604	.12741
.93	.10785	.15327	.14240	.14881	.14957
.94	.11497	.17894	.16554	.17590	.17644
.95	.12250	.21055	.19795	.21173	.21258
.96	.13050	.25229	.24339	.25819	.25893
.97	.13878	.30585	.31265	.32683	.33437
.98	.14762	.38240	.42757	.44272	.45065
.99	.15703	.50382	.64897	.67313	.70763

Table 6b. Percentage points Z_p for distribution function of ω_n^5: $F_n^k(Z_p) = \Pr\{\omega_n^5 < Z_p\}$, $n = 6, 7, \ldots, 10$.

	Percentage Points Z_p				
$F_n^k(Z_p)$	$n=6$	$n=7$	$n=8$	$n=9$	$n=10$
.50	.00000	.00000	.00000	.00000	.00000
.51	.00032	.00032	.00031	.00031	.00031
.52	.00064	.00064	.00063	.00063	.00063
.53	.00097	.00096	.00095	.00095	.00094
.54	.00129	.00128	.00127	.00127	.00126
.55	.00163	.00161	.00160	.00159	.00159
.56	.00197	.00195	.00193	.00193	.00192
.57	.00232	.00229	.00228	.00227	.00226
.58	.00268	.00265	.00263	.00262	.00261
.59	.00305	.00302	.00300	.00298	.00298
.60	.00344	.00341	.00338	.00337	.00336
.61	.00386	.00381	.00378	.00377	.00376
.62	.00429	.00425	.00421	.00420	.00419
.63	.00477	.00471	.00468	.00466	.00465
.64	.00528	.00522	.00518	.00516	.00515
.65	.00586	.00578	.00574	.00572	.00570
.66	.00651	.00642	.00638	.00635	.00633
.67	.00728	.00717	.00712	.00709	.00707
.68	.00820	.00808	.00803	.00800	.00797
.69	.00936	.00922	.00918	.00914	.00911
.70	.01076	.01064	.01061	.01058	.01053
.71	.01228	.01219	.01218	.01216	.01211
.72	.01374	.01369	.01369	.01366	.01362
.73	.01512	.01510	.01509	.01506	.01501
.74	.01648	.01647	.01646	.01641	.01637
.75	.01792	.01793	.01789	.01784	.01779
.76	.01960	.01962	.01957	.01949	.01943
.77	.02187	.02190	.02181	.02170	.02163
.78	.02501	.02503	.02493	.02483	.02475
.79	.02791	.02791	.02783	.02775	.02770
.80	.03046	.03044	.03035	.03027	.03023
.81	.03339	.03334	.03320	.03310	.03305
.82	.03802	.03791	.03772	.03762	.03759
.83	.04237	.04226	.04215	.04210	.04209
.84	.04643	.04628	.04613	.04607	.04606
.85	.05309	.05289	.05276	.05276	.05276
.86	.05850	.05832	.05822	.05823	.05822
.87	.06684	.06660	.06658	.06665	.06665
.88	.07418	.07393	.07394	.07401	.07399
.89	.08443	.08430	.08440	.08446	.08446
.90	.09657	.09654	.09673	.09680	.09682
.91	.11052	.11065	.11091	.11098	.11103
.92	.12721	.12777	.12826	.12846	.12865
.93	.14964	.15029	.15063	.15081	.15100
.94	.17696	.17793	.17834	.17868	.17902
.95	.21450	.21574	.21634	.21694	.21730
.96	.26515	.26743	.26894	.26898	.26978
.97	.33819	.34191	.34446	.34704	.34895
.98	.46166	.47256	.47476	.47838	.47685
.99	.72513	.74115	.75267	.76172	.76901