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Eigenvalue Distribution of Time and Frequency Limiting

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1. INTRODUCTION

Let S and T represent sets, each the union of a finite number (μ and ν, respectively) of fixed disjoint intervals. We shall be interested in the behavior, as the parameter $c \to \infty$, of the eigenvalues $\{\lambda_k(c)\}$ of the integral equation

$$\int_{y \in cS} h_T(x-y)\, \phi_k(y)\, dy = \lambda_k(c)\, \phi_k(x), \qquad x \in cS, \tag{1}$$

in which the Fourier transform of $(2\pi)^{1/2} h_T$ is $\chi_T(\omega)$, the characteristic function of T:

$$h_T(s) \equiv \frac{1}{2\pi} \int_{\omega \in T} e^{is\omega}\, d\omega.$$

The operator in (1) consists of restricting $\phi_k(y)$ to the set cS, restricting the Fourier transform of the function so obtained to the set T, and viewing the result again on cS. We can therefore represent it compactly as $A_c \equiv P(cS)\, Q(T)\, P(cS)$, where $P(\Omega)$ and $Q(\Delta)$ represent orthogonal projections in the Hilbert space $L^2(-\infty, \infty)$ onto the subspaces of those functions which vanish outside of Ω, and those whose Fourier transform vanishes outside of Δ. The interest of the problem lies in the fact that the eigenvalues are useful in describing the geometry of these subspaces, and thereby provide information about how the energy of a function can be distributed over time and frequency. From a comparison of $\sum_k \lambda_k(c)$ and

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$\sum_k \lambda_k^2(c)$ it is easy to see that the eigenvalues (when arranged in non-increasing order) are very close to 1, then very close to 0, the transition occurring in an interval of values of k which is centered at $k = c\mathbf{m}(S)\,\mathbf{m}(T)/2\pi$, with $\mathbf{m}(\cdot)$ Lebesgue measure, and grows in width at the rate of only log c. This fact has found application to certain questions concerning sampling [4], to an extension of Szegö's theorem [5], and to the eigenvalues of Hankel matrices [7]. In a paper [6] devoted to asymptotic expressions for the eigenvalues and for the corresponding eigenfunctions, in the case that S and T are single intervals, Slepian conjectured that when $n = (2\pi)^{-1}\,\mathbf{m}(S)\,\mathbf{m}(T)c + \pi^{-2}b \log c$, $\lambda_n(c) \to (1 + e^b)^{-1}$. Here we prove this result, by showing it to stem from like behavior of Hankel operators [7], and generalize it to sets S and T which are finite unions of intervals. Specifically, letting $N(A, \alpha)$ denote the number of eigenvalues of an operator A which exceed α, we will show that, for $0 < \alpha < 1$,

$$N(A_c, \alpha) = \frac{\mathbf{m}(S)\,\mathbf{m}(T)}{2\pi} c + \left(\frac{\mu\nu}{\pi^2} \log \frac{1-\alpha}{\alpha}\right) \log c + o(\log c). \tag{2}$$

A traditional method for describing the distribution of eigenvalues for a difference kernel relies on finding the trace of iterates of the operator. The fact that the kernel of (1) is not absolutely integrable presents a severe impediment to that approach. We circumvent the difficulty here by considering, instead, the operator $A_c(1 - A_c)$ whose eigenvalues, $\lambda_k(c)[1 - \lambda_k(c)]$, are large precisely when $\lambda_k(c)$ lies in the intermediate range which we wish to study. We will be able to compute the trace of $[A_c(I - A_c)]^n$ and of $A_c[A_c(I - A_c)]^n$. The resulting estimates, combined with the fact that, from (1),

$$\operatorname{tr} A_c = \int_{y \in cS} h_T(0)\, dy = \frac{\mathbf{m}(cS)\,\mathbf{m}(T)}{2\pi}, \tag{3}$$

are sufficient to establish (2).

The discrete analogue of our results for S a single interval, concerning the eigenvalues of Toeplitz matrices, follows easily from a result of Basor [1] on the asymptotics of their determinants. Her methods, although more complicated, also give the next term in the asymptotic expansion. They could undoubtedly be adapted to the present situation.

2. Preliminaries

The main tool we employ is that of the trace norm $\| A \|_1$ of a compact operator A, defined as $\sum_{i=1}^{\infty} s_i(A)$, where $\{s_i^2\}$ are the eigenvalues of A^*A. This norm is finite whenever A is the product $C_1 C_2$ of two operators of

Hilbert–Schmidt type, i.e., ones for which the eigenvalues of $C_i^*C_i$ are summable. We will have occasion to use the following basic properties of the trace norm [3, Chap. 3]:

$$\| A + B \|_1 \leqslant \| A \|_1 + \| B \|_1 . \tag{4}$$

If B is a bounded operator, with (ordinary) norm $\| B \|$,

$$\begin{aligned} \| BA \|_1 &\leqslant \| B \| \, \| A \|_1 \\ \| AB \|_1 &\leqslant \| B \| \, \| A \|_1 . \end{aligned} \tag{5}$$

For any orthonormal basis $\{\phi_i\}$, $\operatorname{tr} A \equiv \sum_{i=1}^{\infty} (A\phi_i, \phi_i)$ is independent of the choice of basis, and

$$| \operatorname{tr} A | \leqslant \| A \|_1 . \tag{6}$$

Now let us turn specifically to the sorts of operators we will consider. The Hilbert space is $L^2(-\infty, \infty)$, in which the Fourier transform

$$F\phi \equiv \frac{1}{(2\pi)^{1/2}} \int_{-\infty}^{\infty} \phi(t) \, e^{-i\omega t} \, dt$$

defines a unitary transformation. Let $P(S)\,\phi$ denote the orthogonal projection of ϕ onto the subspace of functions vanishing outside the set S; explicitly,

$$P(S)\,\phi = \chi_s(t)\,\phi(t).$$

Analogously, let $Q(T)\,\phi = F^{-1}P(T)\,F\phi$ denote the orthogonal projection of ϕ onto the subspace of functions whose Fourier transform vanishes outside T. $Q(T)\,\phi$ can be represented in the form

$$Q(T)\,\phi = \int h_T(x-y)\,\phi(y)\,dy,$$

where

$$Fh_T = \frac{1}{(2\pi)^{1/2}} \chi_T(\omega).$$

If S or T is an interval $[a, b]$, we will often write $P(a, b)$ or $Q(a, b)$ in place of $P(S)$ or $Q(T)$. In this situation,

$$h_T(s) = \frac{e^{ibs} - e^{ias}}{2\pi i s}.$$

Moreover, in an operator of the form $P(J)\,Q(a,b)\,P(K)$, with J and K intervals having disjoint interiors, we can pass to the limit as $a \to -\infty$ or $b \to \infty$, obtaining an integral operator with kernel $\chi_K(y)\,\chi_J(x)(e^{ib(x-y)}/2\pi i(x-y))$ or $\chi_K(y)\,\chi_J(x)(-e^{ia(x-y)}/2\pi i(x-y))$ for $P(J)\,Q(-\infty, b)\,P(K)$ or $P(J)\,Q(a, \infty)\,P(K)$, respectively.

Suppose now that J, K, L, M, N are intervals, finite or semi-infinite, and consider the operator $R = P(J)\,Q(M)\,P(K)\,Q(N)\,P(L)$. Let us say that R_1 and R_2 are unitarily equivalent, written $R_1 \sim R_2$, if $R_1 = UR_2U^{-1}$ for some unitary transformation U. By definition, both $\operatorname{tr} R$ and $\| R \|_1$ are unchanged under this equivalence. By choosing U to be, successively, rescaling $[f(t) \to |\gamma|^{-1/2} f(\gamma t)]$, translation $[f(t) \to f(t+\gamma)]$, shift in frequency $[f(t) \to e^{i\gamma t} f(t)]$, complex conjugation $[f(t) \to \overline{f(t)}]$, and the Fourier transform $[f \to Ff]$, we find, for any scalar γ_j,

$$R \sim P(J/\gamma_1)\,Q(\gamma_1 M)\,P(K/\gamma_1)\,Q(\gamma_1 N)\,P(L/\gamma_1), \tag{7}$$

$$\sim P(J+\gamma_2)\,Q(M+\gamma_3)\,P(K+\gamma_2)\,Q(N+\gamma_3)\,P(L+\gamma_2), \tag{8}$$

$$\sim P(J)\,Q(-M)\,P(K)\,Q(-N)\,P(L), \tag{9}$$

$$\sim Q(J)\,P(M)\,Q(K)\,P(N)\,Q(L). \tag{10}$$

Moreover, if J and K intersect in a set of measure 0, and A and B are complementary, then since $Q(A) = I - Q(B)$, we have

$$P(J)\,Q(A)\,P(K) = -P(J)\,Q(B)\,P(K). \tag{11}$$

We will be interested in conditions under which $\| R \|_1$ is uniformly bounded, independently of the choice of the intervals J, M, K, N, L within a certain class, a property we will denote by $R = O(1)$. To discuss this, let us observe that an integral operator in L^2 whose kernel is $p(x)\,q(y)$ has rank 1 and trace norm $\| p \|\,\| q \|$, where $\| f \|^2 = \int_{-\infty}^{\infty} |f(x)|^2\,dx$. Consequently, by (6) and (4), if an operator A is defined by a kernel $\int p(x,z)\,q(y,z)\,dz$,

$$|\operatorname{tr} A| \leqslant \| A \|_1 \leqslant \int \| p(\cdot, z)\|\,\| q(\cdot, z)\|\,dz. \tag{12}$$

As an immediate consequence we obtain the following criterion.

LEMMA. *Suppose that the interiors of J and L lie in the complement of K. Then $R = O(1)$ under each of the conditions:*

(L1) *If $\mathbf{m}(K)$ is uniformly bounded, and either J and L are on opposite sides of K, or one of J, L is uniformly separated from K.*

(L2) *If $K = (-\infty, 0]$, $\mathbf{m}(J)$ or $\mathbf{m}(L)$ is uniformly bounded, and J or L is uniformly separated from 0.*

(L3) *If $K=(-\infty,0]$, the finite endpoints of M and N are uniformly separated from one another, and either J or L is uniformly separated from* 0.

(L4) *If $K=(-\infty,0]$, $\mathbf{m}(N)$ or $\mathbf{m}(M)$ is uniformly bounded, and $\mathbf{m}(J)$ or $\mathbf{m}(L)$ is uniformly bounded.*

Proof. In L1 we may, by (8) and (7), suppose that $K=[0,1]$. The operator R is then given by a kernel of the form

$$\int_0^1 du\,\frac{e^{in_2(u-y)}-e^{in_1(u-y)}}{2\pi i(u-y)}\,\frac{e^{im_2(x-u)}-e^{im_1(x-u)}}{2\pi i(x-u)},$$

with $x\in J$, $y\in L$; if M or N is semi-infinite, the exponentials corresponding to the infinite endpoint are omitted. Let us write the integrand as a sum of terms of the form

$$\frac{e^{in_j(u-y)}}{u-y}\,\frac{e^{im_k(x-u)}}{x-u}.$$

If $J\subset[1,\infty)$ and $L\subset(-\infty,0]$, we find from (12), with γ an appropriate constant,

$$\|R\|_1\leqslant\gamma\int_0^1(1-u)^{-1/2}\,u^{-1/2}\,du=O(1).$$

Likewise, if J and L are on the same side of 0 or 1, but L is uniformly separated from K, say $L\subset[1+\delta,\infty)$, then

$$\|R\|_1\leqslant\gamma\int_0^1(1-u)^{-1/2}\,(1+\delta-u)^{-1/2}\,du=O(1).$$

For L2, arguing analogously with $J=[a,b]$, $0\leqslant a<b$, and $L\subset[\delta,\infty]$, we find

$$\|R\|_1\leqslant\gamma\int_{-\infty}^0 du[(a-u)^{-1/2}-(b-u)^{-1/2}](\delta-u)^{-1/2}=O(1).$$

In L3, R is given by a sum of at most four kernels of the form $e^{i(\theta_1x-\theta_2y)}\gamma\int_{-\infty}^0 du(e^{i\alpha u}/(u-x)(u-y))$ with $|\alpha|$ bounded uniformly away from 0, $x\in J$, $y\in L$. Integration by parts converts this to

$$\gamma e^{i(\theta_1x-\theta_2y)}\left[\frac{1}{ia}\frac{1}{xy}-\frac{1}{ia}\int_{-\infty}^0 du\,\frac{e^{i\alpha u}}{(u-x)(u-y)}\left(\frac{1}{u-x}+\frac{1}{u-y}\right)\right].$$

If both J and L are uniformly separated from 0, say J, $L\subset[\delta,\infty)$, the first component, of rank 1, has trace norm $\gamma/a\delta$, while the second has uniformly bounded trace norm by (12). If only one of J, L is separated from 0, say $J=[\delta, \infty)$, $L=[0, \infty)$, then by (8)

$$\begin{aligned} P(J)\,Q(M)\,P(K)\,Q(N)\,P(L) &\sim P(2\delta, \infty)\,Q(M)\,P(-\infty, \delta)\,Q(N)\,P(\delta, \infty) \\ &= P(2\delta, \infty)\,Q(M)\,P(-\infty, 0)\,Q(N)\,P(\delta, \infty) \\ &\quad + P(2\delta, \infty)\,Q(M)\,P(0, \delta)\,Q(N)\,P(\delta, \infty). \end{aligned}$$

Now the first operator on the right is $O(1)$ by the preceding reasoning, while the second is $O(1)$ by L1.

In L4 we may again suppose $N=[0, 1]$, whereupon the kernel is

$$\int_{-\infty}^{0} du\,\frac{e^{i(u-y)}-1}{2\pi i(u-y)}\,\frac{e^{im_2(x-u)}-e^{im_1(x-u)}}{2\pi i(x-u)}.$$

Since $|e^{-is}-1|/|s| \leqslant \gamma/(1+|s|)$ we see that the integrand is bounded by $\gamma/(1+y-u)(x-u)$. If either J or $L \subset [0, \infty)$ is uniformly bounded in length, boundedness of $\|R\|_1$ again follows from (12). This completes the proof of the lemma.

3. Results

We now return to (1), restricting for the moment to the case that S and T consist of a single interval. By (7) and (8) we can renormalize so that $T'=[0, 1]$ and $S'=[0, \mathbf{m}(S)\,\mathbf{m}(T)]$, whereupon $cS'=[0, c\mathbf{m}(S)\,\mathbf{m}(T)]$. On rescaling c by $c'=c\mathbf{m}(S)\,\mathbf{m}(T)$, we see that to establish (2) it is sufficient to prove it for the case $\mathbf{m}(S)=\mathbf{m}(T)=1$. Accordingly, let $A_c \equiv P(0, c)\,Q(0, 1)\,P(0, c)$.

THEOREM 1. *For* $0<\alpha<1$,

$$N(A_c, \alpha)=\frac{c}{2\pi}+\left(\frac{1}{\pi^2}\log\frac{1-\alpha}{\alpha}\right)\log c+o(\log c).$$

Proof. Since P and Q are orthogonal projections, $P^2=P$ and $Q^2=Q$. Thus $A_c^2=P(0, c)\,Q(0, 1)\,P(0, c)\,Q(0, 1)\,P(0, c)$, hence

$$\begin{aligned} A_c-A_c^2 &= P(0, c)\,Q(0, 1)[P(-\infty, 0)+P(c, \infty)]\,Q(0, 1)\,P(0, c) \\ &= P(0, c)\,Q(0, 1)\,P(-\infty, 0)\,Q(0, 1)\,P(0, c) \\ &\quad + P(0, c)\,Q(0, 1)\,P(c, \infty)\,Q(0, 1)\,P(0, c). \end{aligned}$$

The product of the two operators on the right contains the factor $P(-\infty, 0)\, Q(0, 1)\, P(0, c)\, Q(0, 1)\, P(c, \infty)$, which by (7) is equivalent to $P(-\infty, 0)\, Q(0, c)\, P(0, 1)\, Q(0, c)\, P(1, \infty)$, and so is $O(1)$ by L1. The remaining factors, consisting of projections, have norm bounded by 1, so the product is $O(1)$ by (5). Thus for each n,

$$[A_c(I - A_c)]^n = [P(0, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(0, c)]^n + [P(0, c)\, Q(0, 1)\, P(c, \infty)\, Q(0, 1)\, P(0, c)]^n + O(1). \quad (13)$$

If we now replace the first $P(0, c)$ by $P(1, c)$ in $P(0, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(0, c)$, the difference is $P(0, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(0, 1)$, which is $O(1)$ by L4. Thus

$$P(0, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(0, c) = P(1, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(1, c) + O(1).$$

Continuing, let us write $Q(0, 1) = Q(0, \infty) - Q(1, \infty)$ in the right-hand operator and expand. The terms involving both $Q(0, \infty)$ and $Q(1, \infty)$, for example, $P(1, c)\, Q(0, \infty)\, P(-\infty, 0)\, Q(1, \infty)\, P(1, c)$, are each $O(1)$ by L3. Thus

$$P(0, c)\, Q(0, 1)\, P(-\infty, 0)\, Q(0, 1)\, P(0, c) = P(1, c)\, Q(0, \infty)\, P(-\infty, 0)\, Q(0, \infty)\, P(1, c) + P(1, c)\, Q(1, \infty)\, P(-\infty, 0)\, Q(1, \infty)\, P(1, c) + O(1).$$

The product of the two operators on the right contains the factor $P(-\infty, 0)$ $Q(0, \infty)\, P(1, c)\, Q(1, \infty)\, P(-\infty, 0)$, which is equivalent to $P(1, \infty)\, Q(-\infty, 0)$ $P(-c, 0)\, Q(-\infty, -1)\, P(1, \infty)$ by (8) and (7), and so is $O(1)$ by L3. The second operator on the right-hand side of (13) is unitarily equivalent to the first; on applying the analogous chain of argument to it, we finally obtain

$$\begin{aligned}[A_c(I - A_c)]^n &= [P(1, c)\, Q(0, \infty)\, P(-\infty, 0)\, Q(0, \infty)\, P(1, c)]^n \\ &\quad + [P(1, c)\, Q(1, \infty)\, P(-\infty, 0)\, Q(1, \infty)\, P(1, c)]^n \\ &\quad + [P(0, c-1)\, Q(0, \infty)\, P(c, \infty)\, Q(0, \infty)\, P(0, c-1)]^n \\ &\quad + [P(0, c-1)\, Q(1, \infty)\, P(c, \infty)\, Q(1, \infty)\, P(0, c-1)]^n + O(1).\end{aligned} \quad (14)$$

Now by suitably applying (7)–(9) we see that each of the operators on the right-hand side of (14) is unitarily equivalent to the first. Thus, setting

$$K_c \equiv P(1, c)\, Q(0, \infty)\, P(-\infty, 0)\, Q(0, \infty)\, P(1, c)$$

we find, by (6),

$$\mathrm{tr}[A_c(I-A_c)]^n = 4\,\mathrm{tr}\,K_c^n + O(1). \tag{15}$$

Next, let us apply A_c to both sides of (14), obtaining $A_c[A_c(I-A_c)]^n$ as the sum of four operators plus $O(1)$. By (8) with $\gamma_2 = 0$, $\gamma_3 = 1$, (9), and (11) we see that the second, $P(0,c)\,Q(0,1)\,[P(1,c)\,Q(1,\infty)\,P(-\infty,0)\,Q(1,\infty)\,P(1,c)]^n$, is unitarily equivalent to the first, and by analogous arguments, so are the third and fourth. We conclude that

$$\mathrm{tr}\,A_c\,[A_c(I-A_c)]^n = 4\,\mathrm{tr}\,P(0,c)\,Q(0,1)\,K_c^n + O(1). \tag{16}$$

Further, on replacing $Q(0,1)$ in (16) by $Q(0,\infty)$, the difference contains as factor the operator $Q(1,\infty)\,P(1,c)\,Q(0,\infty)\,P(-\infty,0)\,Q(0,\infty) = -Q(1,\infty)\,P(1,c)\,Q(-\infty,0)\,P(-\infty,0)\,Q(0,\infty)$ by (11). But by (10) the latter is equivalent to $-P(1,\infty)\,Q(1,c)\,P(-\infty,0)\,Q(-\infty,0)\,P(0,\infty)$, which is $O(1)$ by L3. It follows that

$$\mathrm{tr}\,A_c[A_c(I-A_c)]^n = 4\,\mathrm{tr}\,P(0,c)\,Q(0,\infty)\,K_c^n + O(1). \tag{17}$$

Finally, by (11), $K_c = P(1,c)\,Q(-\infty,0)\,P(-\infty,0)Q(-\infty,0)\,P(1,c)$ so that, by (9),

$$\mathrm{tr}\,P(0,c)\,Q(0,\infty)\,K_c^n = \mathrm{tr}\,P(0,c)\,Q(-\infty,0)\,K_c^n.$$

But as $Q(0,\infty) + Q(-\infty,0) = I$, this shows that

$$\mathrm{tr}\,P(0,c)\,Q(0,\infty)\,K_c^n = \tfrac{1}{2}\,\mathrm{tr}\,P(0,c)\,K_c^n = \tfrac{1}{2}\,\mathrm{tr}\,K_c^n,$$

so that by (17)

$$\mathrm{tr}\,A_c[A_c(I-A_c)]^n = 2\,\mathrm{tr}\,K_c^n + O(1) = \tfrac{1}{2}\,\mathrm{tr}[A_c(I-A_c)]^n + O(1). \tag{18}$$

Now K_c is given by the kernel $(1/4\pi^2)\int_0^\infty du/(u+x)(u+y)$, with $1 \leqslant x$, $y \leqslant c$, and so resembles a Hankel operator. Proceeding then as in [7], we apply the change of variable $x = e^{2\sigma}$, $y = e^{2\tau}$, $u = e^{2\zeta}$, which transforms K_c unitarily into the integral operator on $L^2[0, (\log c)/2]$ defined by the difference kernel $k(\tau - \sigma)$, with

$$k(s) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty} \operatorname{sech} r \operatorname{sech}(s-r)\,dr,$$

a rapidly decreasing function of s. In accordance with Szegö's theorem

[2, 5], the eigenvalue distribution of K_c can be determined from the Fourier transform of k. Specifically, since

$$\int_{-\infty}^{\infty} dv e^{-ivs} \operatorname{sech} v = \pi \operatorname{sech} \frac{\pi}{2} s,$$

$$\begin{aligned} N(K_c, \alpha) &= \frac{\log c}{2} \frac{1}{2\pi} \mathbf{m} \left\{ s \mid \operatorname{sech}^2 \frac{\pi}{2} s > 4\alpha \right\} + o(\log c) \\ &= \frac{\log c}{4\pi} \cdot 2 \cdot \frac{2}{\pi} \operatorname{sech}^{-1} (4\alpha)^{1/2} + o(\log c). \end{aligned}$$

Thus, by definition of N,

$$\begin{aligned} \operatorname{tr} K_c^n &= \int_{0+}^{1/4} x^n \, d_x[-N(K_c, x)] \\ &= \frac{1}{2\pi^2} \log c \int_{0+}^{1/4} x^n \frac{dx}{x(1-4x)^{1/2}} + o(\log c) \end{aligned}$$

and letting $x = t(1-t)$, $0 < t \leqslant \frac{1}{2}$, we find

$$\begin{aligned} \operatorname{tr} K_c^n &= \frac{1}{2\pi^2} \log c \int_0^{1/2} [t(1-t)]^n \frac{dt}{t(1-t)} + o(\log c) \\ &= \frac{1}{4\pi^2} \log c \int_0^1 [t(1-t)]^n \frac{dt}{t(1-t)} + o(\log c). \end{aligned} \tag{19}$$

We observe also that, by symmetry of the integrand in this expression,

$$\int_0^1 t[t(1-t)]^n \frac{dt}{t(1-t)} = \int_0^1 (1-t)[t(1-t)]^n \frac{dt}{t(1-t)}$$

so that

$$\int_0^1 t[t(1-t)]^n \frac{dt}{t(1-t)} = \frac{1}{2} \int_0^1 [t(1-t)]^n \frac{dt}{t(1-t)}. \tag{20}$$

It is now easy to convert (15) and (18) into information concerning $N(A_c, \alpha)$. For by definition of N,

$$\operatorname{tr} [A_c(I - A_c)]^n = \int_{0+}^1 [t(1-t)]^n \, d_t[-N(A_c, t)],$$

$$\operatorname{tr} A_c[A_c(I - A_c)]^n = \int_{0+}^1 t[t(1-t)]^n \, d_t[-N(A_c, t)],$$

and so by (15), (19), (18), and (20)

$$\int_0^1 [t(1-t)]^n \, d_t[-N(A_c, t)] = \frac{\log c}{\pi^2} \int_0^1 [t(1-t)]^n \frac{dt}{t(1-t)} + o(\log c),$$

$$\int_0^1 t[t(1-t)]^n \, d_t[-N(A_c, t)] = \frac{\log c}{\pi^2} \int_0^1 t[t(1-t)]^n \frac{dt}{t(1-t)} + o(\log c).$$

In consequence, for each fixed polynomial P which vanishes at 0 and at 1,

$$\int_0^1 P(t) \, d_t[-N(A_c, t)] = \frac{\log c}{\pi^2} \int_0^1 P(t) \frac{dt}{t(1-t)} + o(\log c).$$

We can eliminate the second restriction by writing $P(t) = tP(1) + [P(t) - tP(1)]$ and applying (3) to the first component. We obtain, for every P vanishing at 0,

$$\int_0^1 P(t) \, d_t[-N(A_c, t)]$$

$$= P(1) \frac{c}{2\pi} + \frac{\log c}{\pi^2} \int_0^1 [P(t) - tP(1)] \frac{dt}{t(1-t)} + o(\log c),$$

and this relation can be extended by approximation from polynomials to any function $F(t)$ for which $[F(t) - tF(1)]/t(1-t)$ is Riemann-integrable on $[0, 1]$. In particular, taking $F(t)$ to be the characteristic function of the interval $0 < \alpha < t < 1$, we find

$$N(A_c, \alpha) = \int_\alpha^1 d_t[-N(A_c, t)]$$

$$= \frac{c}{2\pi} + \frac{\log c}{\pi^2} \left[-\int_0^\alpha \frac{dt}{1-t} + \int_\alpha^1 \frac{dt}{t} \right] + o(\log c)$$

$$= \frac{c}{2\pi} + \frac{\log c}{\pi^2} \log \frac{1-\alpha}{\alpha} + o(\log c).$$

This completes the proof of Theorem 1.

The same argument is sufficient to prove the more general case.

THEOREM 2. *Let S and T each be the union of a finite number, μ and ν, respectively, of fixed disjoint closed intervals. Let $A_c \equiv P(cS)\, Q(T)\, P(cS)$ denotre the operator of* (1). *Then, for* $0 < \alpha < 1$,

$$N(A_c, \alpha) = \frac{\mathbf{m}(S)\, \mathbf{m}(T)}{2\pi} c + \left(\frac{\mu\nu}{\pi^2} \log \frac{1-\alpha}{\alpha} \right) \log c + o(\log c).$$

Proof. We will show how to reduce the proof to that of Theorem 1. Let γ_i: $g_i \leqslant y \leqslant G_i$, with $g_1 < G_1 < \cdots < g_\mu < G_\mu$, denote the constituent intervals of S. The complement S' of S now consists of ε_0: $-\infty < y < g_1$, ε_i: $G_i < y < g_{i+1}$, $i = 1,..., \mu - 1$, and ε_μ: $G_\mu < y < \infty$. Let δ_i: $d_i \leqslant t \leqslant D_i$, with $d_1 < D_1 \cdots < d_\nu < D_\nu$, denote the constitutent intervals of T. As before, we consider

$$A_c(I - A_c) = P(cS)\, Q(T)\, P(cS')\, Q(T)\, P(cS)$$
$$= \sum_{n,m,l,k,j} P(c\gamma_n)\, Q(\delta_m)\, P(c\varepsilon_l)\, Q(\delta_k)\, P(c\gamma_j). \tag{21}$$

Now by (7),

$$P(c\gamma_n)\, Q(\delta_m)\, P(c\varepsilon_l)\, Q(\delta_k)\, P(c\gamma_j) \sim P(\gamma_n)\, Q(c\delta_m)\, P(\varepsilon_l)\, Q(c\delta_k)\, P(\gamma_j)$$

and, by L1 and L2, this is $O(1)$ unless $n = j$ and γ_j is adjacent to ε_l. If ε_l is a finite interval with γ_j adjoining it one one side, and we extend ε_l to ∞ on the other side, the difference, for example, $P(g_2, G_2)\, Q(c\delta_m)\, P(-\infty, G_1)\, Q(c\delta_k)$ $P(g_2, G_2)$, is $O(1)$ by L2. We conclude that we can reduce the sum in (21) to

$$A_c(I - A_c)$$
$$= \sum_{j,m,k} P(c\gamma_j)\, Q(\delta_m)[P(-\infty, cg_j) + P(cG_j, \infty)]\, Q(\delta_k)\, P(c\gamma_j) + O(1). \tag{22}$$

Moreover, by L4, as in Theorem 1

$$P(c\gamma_j)\, Q(\delta_m)\, P(-\infty, cg_j)\, Q(\delta_k)\, P(c\gamma_j)$$
$$= P(cg_j + 1, cG_j)\, Q(\delta_m)\, P(-\infty, cg_j)\, Q(\delta_k)\, P(cg_j + 1, cG_j) + O(1)$$

and by L3 the latter operator is $O(1)$ unless $m = k$. The same applies to the remaining operators of (22) and so we find

$$A_c(I - A_c)$$
$$= \sum_{j,m} P(cg_j + 1, cG_j)\, Q(\delta_m)\, P(-\infty, cg_j)\, Q(\delta_m)\, P(cg_j + 1, cG_j)$$
$$+ P(cg_j, cG_j - 1)\, Q(\delta_m)\, P(cG_j, \infty)\, Q(\delta_m)\, P(cg_j, cG_j - 1) + O(1).$$

By taking powers on both sides and applying similar considerations to show that the cross-product terms are $O(1)$ we see that

$$[A_c(I-A_c)]^n$$

$$= \sum_{j,m} [P(cg_j+1, cG_j)\, Q(\delta_m)\, P(-\infty, cg_j)\, Q(\delta_m)\, P(cg_j+1, cG_j)]^n$$
$$+ [P(cg_j, cG_j-1)\, Q(\delta_m)\, P(cG_j, \infty)\, Q(\delta_m)\, P(cg_j, cG_j-1)]^n + O(1)$$

$$= \sum_{j,m} [P(cg_j+1, cG_j)\, Q(d_m, \infty)\, P(-\infty, cg_j)\, Q(d_m, \infty)\, P(cg_j+1, cG_j)]^n$$
$$+ [P(cg_j+1, cG_j)\, Q(D_m, \infty)\, P(-\infty, cg_j)\, Q(D_m, \infty)\, P(cg_j+1, cG_j)]^n$$
$$+ [P(cg_j, cG_j-1)\, Q(d_m, \infty)\, P(cG_j, \infty) Q(d_m, \infty)\, P(cg_j, cG_j-1)]^n$$
$$+ [P(cg_j, cG_j-1)\, Q(D_m, \infty)\, P(cG_j, \infty)\, Q(D_m, \infty)\, P(cg_j, cG_j-1)]^n$$
$$+ O(1). \tag{23}$$

As in Theorem 1, for given j and m, each of these four operators is now unitarily equivalent to

$$K_{c\mathrm{m}(\gamma_j)} = P(1, c\mathrm{m}(\gamma_j))\, Q(0, \infty)\, P(-\infty, 0)\, Q(0, \infty)\, P(1, c\mathrm{m}(\gamma_j)),$$

so that $\mathrm{tr}[A_c(I-A_c)]^n = \sum_{j,m} 4\ \mathrm{tr}\ K^n_{c\mathrm{m}(\gamma_j)} + O(1)$. But from (19), $\mathrm{tr}\ K^n_{c\mathrm{m}(\gamma_j)} = \mathrm{tr}\ K^n_c + O(1)$, consequently

$$\mathrm{tr}[A_c(I-A_c)]^n = 4\mu\nu\, \mathrm{tr}\, K^n_c + O(1). \tag{24}$$

Now let us apply $A_c = \sum_{a,b,d} P(c\gamma_a)\, Q(\delta_b)\, P(c\gamma_d)$ to (23). For the first component of (23), which we denote by $B(j, m)$, we have

$$A_c B(j, m) = \sum_{a,b,d} P(c\gamma_a)\, Q(\delta_b)\, P(c\gamma_d)\, B(j, m).$$

We now argue that the only significant contribution from this sum comes when $a = d = j$ and $b = m$. For if $d \neq j$, the corresponding operator vanishes. If $a \neq j$, the operator contains the factor

$$P(c\gamma_a)\, Q(\delta_b)\, P(cg_j+1, cG_j)\, Q(d_m, \infty)\, P(-\infty, cg_j)$$
$$\sim P(\gamma_a)\, Q(c\delta_b)\, P(g_j + c^{-1}, G_j)\, Q(cd_m, \infty)\, P(-\infty, g_j)$$

and the latter operator is $O(1)$ by L1. Finally, $P(c\gamma_j)\ Q(\delta_b)\ P(c\gamma_j)\ B(j, m)$ contains the factor

$$Q(\delta_b)\, P(cg_j+1, cG_j)\, Q(d_m, \infty)\, P(-\infty, cg_j)\, Q(d_m, \infty)\, P(cg_j+1, cG_j). \tag{25}$$

If $b < m$, δ_b is in the complement of $[d_m, \infty)$. Then by (11), the operator of (25) coincides with

$$-Q(\delta_b)\, P(cg_j + 1, cG_j)\, Q(d_m, \infty)\, P(-\infty, cg_j)\, Q(-\infty, d_m)\, P(cg_j + 1, cG_j)$$

and, by (10), this contains a factor unitarily equivalent to $-P(\delta_b)$ $Q(cg_j + 1, cG_j)\, P(d_m, \infty)\, Q(-\infty, cg_j)\, P(-\infty, d_m)$ which is $O(1)$ by L3. Similarly, if $b > m$, (25) coincides with $-Q(\delta_b)$ $P(cg_j + 1, cG_j)$ $Q(-\infty, d_m)$ $P(-\infty, cg_j)$ $Q(d_m, \infty)$ $P(cg_j + 1, cG_j)$, and again contains a factor equivalent to $-P(\delta_b)$ $Q(cg_j + 1, cG_j)$ $P(-\infty, d_m)$ $Q(-\infty, cg_j)$ $P(d_m, \infty)$; as δ_b is now in the complement of $(-\infty, d_m]$, this is $O(1)$ by L3. We conclude that

$$A_c B(j, m) = P(c\gamma_j)\, Q(\delta_m)\, B(j, m) + O(1),$$

and similarly for each of the remaining terms of (23). Continuing as in Theorem 1, we find

$$\operatorname{tr} A_c[A_c(I - A_c)]^n = \tfrac{1}{2} \operatorname{tr}[A_c(I - A_c)]^n + O(1). \tag{26}$$

The remaining argument of Theorem 1 now applies without change to (24) and (26), and yields Theorem 2.

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