Existence of Almost Periodic Solutions of Functional Differential Equations of Neutral Type

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In this paper, we establish the theorems of existence of an almost periodic solution of an almost periodic neutral functional differential equation and its disturbed systems by Liapunov functional.

Hale [1] and Yoshizawa [2] established existence theorems of an almost periodic solution of RFDE with bounded lag by Liapunov functional, resp. Up to now, we have not seen existence theorems of almost periodic solution about Neutral Functional Differential Equation, NFDE. It is our aim that we establish existence theorem of almost periodic NFDE by Liapunov functional. And, our results generalize corresponding results of [1–3].

We consider NFDE

\[
\frac{d}{dt} Dx_i = f(t, x_i)
\]  

and its product systems

\[
\frac{d}{dt} Dx_i = f(t, x_i), \quad \frac{d}{dt} Dy_i = f(t, y_i).
\]  

Consider disturbed systems

\[
\frac{d}{dt} Dx_i = f(t, x_i) + h(t) 
\]

\[
\frac{d}{dt} Dx_i = f(t, x_i) + \eta g(t, x_i)
\]
and its product systems

$$\frac{d}{dt} Dx_i = f(t, x_i) + h(t), \quad \frac{d}{dt} Dy_i = f(t, y_i) + h(t) \quad (2)^*$$

$$\frac{d}{dt} Dx_i = f(t, x_i) + \eta g(t, x_i), \quad \frac{d}{dt} Dy_i = f(t, y_i) + \eta g(t, y_i), \quad (3)^*$$

where $h: \mathbb{R} \to \mathbb{R}^n$ is continuous, $f, g: \mathbb{R} \times C \to \mathbb{R}^n$ is continuous, and $f, g$ are local Lipschitzian in $\varphi$. $C = C([-r, 0], \mathbb{R}^n)$. For $x_i \in C$, we define $x_i(\theta) = x(t + \theta)$, $\theta \in [-r, 0]$. Suppose $|\cdot|$ denotes the norm in $\mathbb{R}^n$. For $\varphi \in C$, we define $|\varphi| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$. $D\varphi = \varphi(0) - \tilde{g}(\varphi)$. $\tilde{g}: C \to \mathbb{R}^n$ is linear continuous, i.e.,

$$\tilde{g}(\varphi) = \int_{-r}^{0} d\mu(\theta) \cdot \varphi(\theta), \quad \forall \varphi \in C,$$

where $\mu(\theta)$ is $n \times n$ bounded variation matrix function. Suppose there is a continuous nondecreasing function $l(s)$, $s \in [0, r]$, $l(0) = 0$, such that

$$\left| \int_{-s}^{0} d\mu(\theta) \cdot \varphi(\theta) \right| \leq l(s) \cdot \sup_{-s \leq \theta \leq 0} |\varphi(\theta)|, \quad \forall \varphi \in C.$$

In this paper, we always suppose as follows: $h: \mathbb{R} \to \mathbb{R}^n$ is an almost periodic function. For $0 < H^* \leq \infty$, $f, g: \mathbb{R} \times C_{H^*} \to \mathbb{R}^n$ are almost periodic in $t$ uniformly for $\varphi \in C_{H^*}$ (about its definitions and properties, see [3-5]), and for $\alpha > 0$, there is $n(\alpha) > 0$, such that we have

$$|f(t, \varphi)| \leq n(\alpha), \quad |\varphi| < \alpha \text{ and } (t, \varphi) \in \mathbb{R} \times C_{H^*}.$$

Under the above condition, it is easy to know that there is a unique solution of (1) through initial value $(\sigma, \varphi) \in \mathbb{R} \times C_{H^*}$, and the solution is continuous in $(t, \sigma, \varphi)$. At the same time, when $|x_i(\sigma, \varphi)| \leq H < H^*$, $x_i(\sigma, \varphi)$ exists on $[0, +\infty)$ (see [6, 7]).

Suppose $C([\tau, +\infty), \mathbb{R}^n)$ denotes the set that consists of continuous function $H: [\tau, +\infty) \to \mathbb{R}^n$, where $\tau$ is a fixed number. For $H \in C([\tau, +\infty), \mathbb{R}^n)$, consider

$$Dx_i = Dx_\sigma + H(t) - H(\sigma), \quad t \geq \sigma \geq \tau \quad (4)$$

$$x_\sigma = \varphi.$$

Let $x(\sigma, \varphi, H)(t)$, $t \geq \sigma$, denotes a solution of (4).

DEFINITION [8]. Suppose $\mathcal{H} \subset C([\tau, +\infty), \mathbb{R}^n)$. We say the operator $D$ is uniformly stable with respect to $\mathcal{H}$ if there are constants $K, M$ such that
for any $\varphi \in C$, $\sigma \in [\tau, +\infty)$ and $H \in \mathcal{H}$, the solution $x(\sigma, \varphi, H)$ of (4) satisfies
\[
|x_1(\sigma, \varphi, H)| \leq K|\varphi| + M \cdot \sup_{\sigma \leq u \leq t} |H(u) - H(\sigma)|. \tag{5}
\]

**Lemma 1** [8]. If $D$ is a uniformly stable operator with respect to $C([\tau, \infty), \mathbb{R}^n)$, then there are positive constants $a$, $b$, $c$, $d$ such that for any $h \in C([\tau, +\infty), \mathbb{R}^n)$, $\sigma \in [\tau, +\infty)$, the solution $x(\sigma, \varphi, h)$ of the equation
\[
 Dx_1 = h(t), \quad t \geq \sigma, \quad x_0 = \varphi
\]
satisfies
\[
|x_1(\sigma, \varphi, h)| \leq e^{-a(t-\sigma)}(b|\varphi| + c \cdot \sup_{\sigma \leq u \leq t} |h(u)|) + d \cdot \sup_{\sigma \leq u \leq t} |h(u)|, \quad t \geq \sigma. \tag{6}
\]
Furthermore, the constants $a$, $b$, $c$, $d$ can be chosen so that for any $s \in [\sigma, \infty)$.
\[
|x_1(\sigma, \varphi, h)| \leq e^{-a(t-s)}(b|\varphi| + c \cdot \sup_{\sigma \leq u \leq t} |h(u)|) + d \cdot \sup_{\sigma \leq u \leq t} |h(u)| \tag{7}
\]
for $t \geq s + r$.

**Lemma 2.** Suppose $D$ is a stable operator (for its definition, see [6] or [7]). If Eq. (1) has solution $u(t)$, $u_{t_0} = \varphi$, $|u_1| \leq H < H_*$, $t \in [t_0, \beta)$, then closure $cl\{u_1; t \in [t_0, \beta]\}$ is a compact set in $C$.

**Proof.** For any $\tau \geq 0$, we have
\[
 Du_{t_0 + \tau} = Du_{t_0} + \int_{t_0 + \tau}^{t_0 + \tau + \tau} f(s, u_s) \, ds, \quad \forall t, t + \tau \in [t_0, \beta)
\]
and
\[
 Du_t = D\varphi + \int_{t_0}^{t} f(s, u_s) \, ds, \quad \forall t \in [t_0, \beta).
\]
Hence, we obtain
\[
 D(u_{t_0 + \tau} - u_t) = D(u_{t_0 + \tau} - \varphi) + \left[ \int_{t}^{t + \tau} f(s, u_s) \, ds - \int_{t_0}^{t_0 + \tau} f(s, u_s) \, ds \right].
\]
From (5), we know
\[
|u_{t_0 + \tau} - u_t| \leq K|u_{t_0 + \tau} - \varphi| + 2n(H) \cdot M \cdot \tau, \quad \text{for} \quad t, t + \tau \in [t_0, \beta).
\]
For any \( \varepsilon > 0 \), there exists \( \delta > 0 \), so that \( |u_{t_0 + \tau} - \varphi| < \varepsilon/2K, \) 
\( 2n(H)\tau < \varepsilon/2M, \) for \( 0 \leq \tau \leq \delta \). Thus, \( |u_{t + \tau} - u_t| < \varepsilon, \) for \( 0 \leq \tau \leq \delta \) and any \( t, \) 
\( t + \tau \in [t_0, \beta) \), i.e., \( u_t \) is uniformly continuous on \([t_0, \beta)\). Since \( u_t \) is uniformly bounded, we know that \( \{u_t; t \in [t_0, \beta)\} \) is compact set in \( C \) by Arzela–Ascoli theorem [9, p. 92]. This is the end.

Let \( AP = \{ \varphi \in C(R, R^n); \varphi(t) \) is an almost periodic function\}. For \( \beta > 0 \) and \( N > 0 \), we define \( B_{\beta, N} = \{ \varphi \in AP \mid \| \varphi(t) \| \leq \beta, \ t \in R; \| \varphi(t_1) - \varphi(t_2) \| \leq N \| t_1 - t_2 \|, \) for \( t_1, t_2 \in R \) and mod(\( \varphi \)) \( \in \) mod(\( f, g \))\}.

From the proof of a lemma in [10], we obtain

**Lemma 3.** For \( \beta > 0 \) and \( N > 0 \), \( B_{\beta, N} \) is a compact set in Banach space \( C_0(R, R^n) \) (where \( C_0(R, R^n) \) is the set that consists of bounded continuous functions from \( R \) to \( R^n \), and its norm is supremum norm \( \| \cdot \|^{\infty} \)). Furthermore, if \( \varphi \in B_{\beta, N} \) and \( t \in R \), then \( g(t, \varphi, t) \in AP \) and is bounded uniformly for \( \varphi \in B_{\beta, N} \) and \( t \in R \).

From the condition that \( g \) is local Lipschitzian in \( \varphi \) and Lemma 2, we know that there is a \( K > 0 \), such that for any \( \varphi, \psi \in B_{\beta, N} \), and \( t \in R \), we have \( |g(t, \varphi) - g(t, \psi)| \leq K \| \varphi - \psi \| \).

**Theorem 1.** Suppose \( D \) is a stable operator, and Eq. (1) has a solution \( \xi(t), |\xi| \leq H < H^*, t \geq 0 \). If \( \xi(t) \) is asymptotic almost periodic (about its definition, see [3, 5]), then Eq. (1) has an almost periodic solution.

**Proof.** According to Lemma 2, \( W = \operatorname{cl}\{\xi_t; t \geq 0\} \) is a compact set in \( C \). Since \( f(t, \varphi) \) is almost periodic in \( t \) uniformly for \( \varphi \in C_{H^*} \), there exists \( \{\alpha_k\}, \alpha_k \to \infty \) as \( k \to \infty \), so that \( f(t + \alpha_k, \varphi) \to f(t, \varphi) \) uniformly on \( R \times W \) as \( k \to \infty \). Because \( \xi(t) \) is asymptotic almost periodic, \( \xi(t) = p(t) + q(t) \), where \( p(t) \) is an almost periodic function, and \( q(t) \to 0 \) as \( t \to \infty \). We may suppose (if necessary, find subsequence) \( p(t + \alpha_k) \to p^*(t) \) uniformly on \( R \) as \( k \to \infty \). Thus, \( p^*(t) \) is an almost periodic function (see [6]). It is easy to know that for any compact set \( K \subset R, q(t + \alpha_k) \to 0 \) uniformly on \( t \in K \) as \( k \to \infty \). Therefore, for fixed \( t \in R \), we have

\[
\lim_{k \to \infty} |\xi_{t + \alpha_k} - p_t^*| = 0, \quad \lim_{k \to \infty} |\xi_{\alpha_k} - p_0^*| = 0.
\]

By continuity of \( D \), we obtain

\[
\lim_{k \to \infty} D\xi_{t + \alpha_k} = DP_t^*, \quad \lim_{k \to \infty} D\xi_{\alpha_k} = DP_0^*.
\]

Since \( f(t + \alpha_k, \varphi) \to f(t, \varphi) \) uniformly on \( R \times W \) as \( k \to \infty \) and \( f \) is continuous, we have

\[
\lim_{k \to \infty} f(t + \alpha_k, \xi_{t + \alpha_k}) = f(t, p_t^*).
\]
Using Lebesgue control convergence theorem, we obtain

\[
Dp^*_t = \lim_{k \to \infty} D_{\tau + \alpha_k} \xi t = \lim_{k \to \infty} \left[ D_{\alpha_k} \xi \tau + \int_{\alpha_k}^{\tau + \alpha_k} f(s, \xi s) \, ds \right]
\]

\[
= \lim_{k \to \infty} \left[ D_{\alpha_k} \xi + \int_{0}^{\tau} f(s + \alpha_k, \xi s + \alpha_k) \, ds \right]
\]

\[
= Dp_0^* + \int_{0}^{\tau} f(s, p^*_t) \, ds
\]

for fixed \( t \in \mathbb{R} \). Thus, \( p^*(t) \) is almost periodic solution of Eq. (1). This is the end.

Suppose \( V: \mathbb{R}^+ \times C_{H^*} \times C_{H^*} \to \mathbb{R}^+ \) continuous. We define the derivation of \( V \) alone solution of (1)* as

\[
V'_{(1),*}(t, \varphi, \psi) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, x_{t+h}(t, \varphi), y_{t+h}(t, \psi)) - V(t, \varphi, \psi) \right]
\]

where \( (x(t, \varphi), y(t, \psi)) \) is a solution of (1)* through \( (t, (\varphi, \psi)) \), \( \varphi, \psi \in C_{H^*} \).

**Theorem 2.** Suppose \( D \) is a stable operator, and there is \( V: \mathbb{R}^+ \times C_{H^*} \times C_{H^*} \to \mathbb{R}^+ \) continuous that satisfies as follows:

(i) \( u(|D\varphi - D\psi|) \leq V(t, \varphi, \psi) \leq v(|\varphi - \psi|) \), \( u, v \in CIP, v(0) = 0 \)

(ii) \( |V(t, \varphi_1, \psi_1) - V(t, \varphi_2, \psi_2)| \leq L[|D\varphi_1 - D\varphi_2| + |D\psi_1 - D\psi_2|] \), \( L > 0 \) for \( (t, \varphi_1, \psi_1) \in \mathbb{R}^+ \times C_{H^*} \times C_{H^*}, i = 1, 2 \)

(iii) \( V_{(1)}(t, \varphi, \psi) \leq -c_0 V(t, \varphi, \psi), c_0 > 0 \) const., \( (t, \varphi, \psi) \in \mathbb{R}^+ \times C_{H^*} \times C_{H^*} \).

If Eq. (1) has a solution \( \xi(t), |\xi| < H < H^*, t \geq t_0 \), then Eq. (1) has a unique almost periodic solution \( p(t) \) which is uniformly asymptotic stable, and \( \text{mod}(p) \subseteq \text{mod}(f) \). Furthermore, if \( f(t+\omega, \varphi) = f(t, \varphi), (t, \varphi) \in \mathbb{R} \times C_{H^*} \), then Eq. (1) has \( \omega \)-periodic solution.

**Proof.** Suppose \( a, b, c, d \) are the numbers in Lemma 1. The proof is divided into four steps as follows:

(1) Prove that Eq. (1) has an almost periodic solution.

We may suppose \( t_0 = 0 \). Define \( W_1 = \text{cl}\{ \xi, t \geq t_0 \} \), then \( W_1 \) is a compact set by Lemma 2. Let \( \alpha = \{ \alpha_n \}, \alpha_n \to \infty (n \to \infty) \), be any sequence. We may suppose \( \{ \alpha_n \} \) is increase (if necessary, find subsequence). Because \( f(t, \varphi) \) is
almost periodic in $t$ uniformly for $\varphi \in C_{H^*}$, we may suppose $f(t + \alpha_n, \varphi)$ (if necessary, find subsequence) converge uniformly on $R \times W_1$ as $n \to \infty$. Therefore, for any $\varepsilon > 0$ ($\varepsilon < H$), there is $l_0 = l_0(\varepsilon, W_1)$, so that when $m \geq k \geq l_0$, we have $\alpha_k \geq r$ and

$$2He^{-\alpha_k}[b + l(r)e] < \varepsilon/2$$

and

$$|f(t + \alpha_k, \varphi) - f(t + \alpha_m, \varphi)| \leq \frac{u(\varepsilon/2d) \cdot c_0}{2L}, \quad \text{for} \quad (t, \varphi) \in R \times W_1.$$

Consider functional $V(t, \xi_t, \xi_t + \alpha_m - \alpha_k)$, $t \geq 0$, $m \geq k \geq l_0$.

$$V'(t, \xi_t, \xi_t + \alpha_m - \alpha_k)$$

$$= \lim_{h \to 0^+} \frac{1}{h} [V(t + h, \xi_t + h, \xi_t + \alpha_m - \alpha_k + h) - V(t, \xi_t, \xi_t + \alpha_m - \alpha_k)]$$

$$\leq -c_0 V(t, \xi_t, \xi_t + \alpha_m - \alpha_k) + \frac{u(\varepsilon/2d) \cdot c_0}{2}, \quad \text{for} \quad t \geq 0.$$

Using differential inequality, we obtain

$$V(t, \xi_t, \xi_t + \alpha_m - \alpha_k) \leq e^{-c_0 t} \left[ V(0, \xi_0, \xi_0 + \alpha_m - \alpha_k) - \frac{u(\varepsilon/2d)}{2} \right]$$

$$+ \frac{u(\varepsilon/2d)}{2}, \quad \text{for} \quad t \geq 0.$$

For the above $\varepsilon > 0$, there is $T > 0$, so that when $t > T$, we have

$$e^{-c_0 t}v(2H) \leq \frac{u(\varepsilon/2d)}{2}.$$

Thus, when $t \geq T$, we have

$$V(t, \xi_t, \xi_t + \alpha_m - \alpha_k) \leq u(\varepsilon/2d).$$

Hence, we obtain

$$|D(\xi_t - \xi_t + \alpha_m - \alpha_k)| \leq \varepsilon/2d, \quad \forall t \geq T.$$

By (7) in Lemma 1, we have
There is $l_1$, so that for any $t \in R^+$, we have $t + a_{l_1} \geq T$. Thus, we obtain

$$|\xi_{t + a_{l_1}} - \xi_{t + x_m}| \leq e^{-a_{l_1}} [b \left| \xi_0 - \xi_{x_m + a_{l_1}} \right| + c \cdot \sup_{0 \leq \eta \leq l + a_{l_1}} |D(\xi_{\eta + a_{l_1}} - \xi_{\eta + x_m + a_{l_1}})|] + d \cdot \sup_{l \leq u \leq t + a_{l_1}} |D(\xi_u - \xi_{u + x_m - a_{l_1}})|$$

$$\leq e^{-a_{l_1}} [2Hb + 2Hl(r) \cdot c] + d \cdot \sup_{l \leq u \leq t + a_{l_1}} |D(\xi_u - \xi_{u + x_m - a_{l_1}})|$$

$$\leq \varepsilon/2 + \varepsilon/2 = \varepsilon, \quad \forall t \geq T, \quad m \geq k \geq l_0.$$
Thus, we have

$$|Dx_t - Dp_i| \leq \frac{\varepsilon}{2(c + d)}, \quad \forall t \in [t_0, \beta].$$

Using (6) in Lemma 1, we obtain

$$|x_t - p_i| \leq e^{-a(t - t_0)} \left[ b |x_{t_0} - p_{t_0}| + c \cdot \sup_{t_0 \leq u \leq t} |D(x_u - p_u)| \right]$$

$$+ d \cdot \sup_{t_0 \leq u \leq t} |D(x_u - p_u)| < \varepsilon, \quad \forall t \in [t_0, \beta]$$

By the arbitrariness of \( \beta \), we have

$$|x_t - p_i| < \varepsilon, \quad \forall t \geq t_0,$$

i.e., \( p(t) \) is uniformly stable.

(3) Prove that \( p(t) \) is quasi-uniformly asymptotic stable for \( t \geq t_0 \) \((t_0 \in R)\).

Since \( p(t) \) is uniformly stable, there is \( \delta_0 > 0 \), so that when \( |\varphi - p_{t_0}| < \delta_0, |x_t(t_0, \varphi) - p_i| < \overline{H} \left( <H^* - H\right), \ t \geq t_0 \left( \overline{H} > 0 \right. \) const.). Let \( x_t = x_t(t_0, \varphi) \). Thus, \( W_3 = \text{cl}\{x_t; t \in [t_0, \infty)\} \cup \text{cl}\{p_i; t \in R\} \) is compact set in \( C \) by Lemma 2. For any \( \varepsilon > 0 \) \((\varepsilon < \overline{H})\), there is \( \tau_2, \tau_2 + t_0 \geq 0 \), so that

$$|f(t + \tau_2, \varphi) - f(t, \varphi)| < \frac{u(\varepsilon/2d) \cdot c_0}{4L}, \quad \forall (t, \varphi) \in R \times W_3.$$

Using (8), we have

$$V'(t + \tau_2, x_t, p_i) \leq -c_0 V(t + \tau_2, x_t, p_i) + \frac{u(\varepsilon/2d) \cdot c_0}{2}.$$

Using differential inequality, we obtain

$$V(t + \tau_2, x_t, p_i) \leq e^{-\alpha(t - t_0)} \left[ V(t_0 + \tau_2, x_{t_0}, p_{t_0}) - \frac{u(\varepsilon/2d)}{2} \right]$$

$$+ \frac{u(\varepsilon/2d)}{2}, \quad \forall t \geq t_0. \quad (9)$$

Find \( T_1 > 0 \), so that it satisfies

$$e^{-\alpha T_1} \cdot v(\delta_0) < \frac{u(\varepsilon/2d)}{2}.$$
and

$$e^{aT_1[b\delta_0 + c \cdot l(r) \cdot \overline{H}]} < \varepsilon/2.$$  

Thus, when $t \geq t_0 + T_1$, using (9), we obtain

$$|Dx_t - Dp_t| < \varepsilon/2d. \quad (10)$$

Let $T = 2T_1 + r$, then when $t \geq t_0 + T$, using (7) in Lemma 1 and (10), we have

$$|x_t - p_t| \leq e^{-aT_1} (b|_{x_0} - p_{t_0}| + c \cdot \sup_{t_0 \leq u \leq t} |Dx_u - Dp_u|) + d \cdot \sup_{t_0 + T \leq u \leq t} |Dx_u - Dp_u|$$

$$\leq \varepsilon/2 + d \cdot (\varepsilon/2d) = \varepsilon, \quad \forall t \geq t_0 + T,$$

i.e., $p(t)$ is quasi-uniformly asymptotically stable for $t \geq t_0$.

(4) From the third step proof, we may know that for any almost periodic solution $\tilde{p}(t)$ of Eq. (1), $t \in R$, we have

$$|p(t) - \tilde{p}(t)| \to 0, \quad \text{as} \quad t \to \infty.$$  

By the almost periodic property, we obtain

$$p(t) = \tilde{p}(t), \quad \forall t \in R,$$

i.e., the almost periodic solution of Eq. (1) is unique. Finally, we prove $\text{mod}(p) \subseteq \text{mod}(f)$. Suppose $\{\gamma_k\}$ is any sequence and satisfies that for any compact $W$ in $C_{H^*}$, $f(t + \gamma_k, \varphi)$ converge to $f(t, \varphi)$ uniformly on $R \times W$ as $k \to \infty$. Since $p(t)$ is an almost periodic function, there exists $\{\gamma_k\} \subseteq \{\gamma_k\}$, so that $p(t + \gamma_k)$ is uniformly convergent on $R$ as $j \to \infty$. Let $Q(t) = \lim p(t + \gamma_k)$. Thus, $Q(t)$ is an almost periodic function (see [5]). It is easy to prove that $Q(t)$ is a solution of Eq. (1). We have $Q(t) = p(t)$ by the uniqueness of $p(t)$. It implies that $\text{mod}(p) \subseteq \text{mod}(f)$ (see [5]).

When $f(t + \omega, \varphi) = f(t, \varphi)$ for $(t, \varphi) \in R \times C_{H^*}$, it is easy to know that Eq. (1) has a $\omega$-periodic solution. This is the end of the proof of the theorem.

**Theorem 3.** Suppose $D$ is a stable operator, and there is $V: R^+ \times C_{H^*} \times C_{H^*} \to R^+$ continuous that satisfies condition (i) and (iii) of Theorem 2 and

$$(ii)' \quad |V(t, \varphi_1, \psi_1) - V(t, \varphi_2, \psi_2)| \leq L|(D\varphi_1 - D\varphi_2) - (D\psi_1 - D\psi_2)|, \quad L > 0 \quad \text{const. for } (t, \varphi_i, \psi_i) \in R^+ \times C_{H^*} \times C_{H^*}, \quad i = 1, 2.$$
Suppose Eq. (1) has a solution \( \xi(t) \), \( |\xi| < H_1 < H^* \), \( t \geq t_0 \). If \( u^{-1}(LK/c_0) (c + d) + H_1 \leq H < H^* \) (where \( |h(t)| \leq K \), \( t \in R \), \( c \) and \( d \) are the numbers in Lemma 1), then Eq. (2) has a unique almost periodic solution \( p(t) \) which is uniformly asymptotic stable, and \( \text{mod}(p) \subset \text{mod}(f, h) \). Furthermore, if \( f(t, \varphi) \) and \( h(t) \) are \( \omega \)-periodic in \( t \), then Eq. (2) has a \( \omega \)-periodic solution.

**Proof.** Let \( a, b, c, d \) be numbers in Lemma 1. Suppose \( (x_i^{(1)}(t_0, \varphi), y_i^{(1)}(t_0, \varphi)) \) is a solution of Eq. (1)* through \((t_0, (\varphi, \psi)) \) and \( (x_i^{(2)}(t_0, \varphi), y_i^{(2)}(t_0, \psi)) \) is a solution of Eq. (2)* through \((t_0, (\varphi, \psi)) \). Using condition (ii)', we have

\[
V'(t, \xi, \eta) = \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, x_i^{(1)}(t, \varphi), y_i^{(1)}(t, \psi)) - V(t, \varphi, \psi) \right]
\]

\[
= \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, x_i^{(2)}(t, \varphi), y_i^{(2)}(t, \psi)) - V(t, \varphi, \psi) \right]
\]

\[
+ \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, x_i^{(2)}(t, \varphi), y_i^{(2)}(t, \psi)) - V(t + h, x_i^{(1)}(t, \varphi), y_i^{(1)}(t, \psi)) \right]
\]

\[
\leq -c_0 V(t, \varphi, \psi), \quad \forall (t, \varphi, \psi) \in R^+ \times C_{H^*} \times C_{H^*}.
\]

It is only required to prove that Eq. (2) has a solution \( \eta(t) \) on \([t_0, \infty) \) by Theorem 2, \( |\eta| < H < H^* \) for \( t \geq t_0 \geq 0 \). Let \( \varphi = \xi_{t_0}, \eta_i = x_i^{(2)}(t_0, \varphi) \) be the solution of Eq. (2) through \((t_0, \varphi) \) and \([t_0, \beta) \) be its maximal existence interval to the right.

\[
V'(t, \xi, \eta_i) = \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, \xi_{t+h}, \eta_i) - V(t, \xi, \eta_i) \right]
\]

\[
\leq \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, \xi_{t+h}, \eta_i) - V(t, \xi, \eta_i) \right]
\]

\[
+ \lim_{h \to 0^+} -\frac{1}{h} \left[ V(t + h, \xi_{t+h}, \eta_i) - V(t + h, \xi_{t+h}, \eta_i) \right]
\]

\[
\leq -c_0 V(t, \xi, \eta_i) + LK, \quad \forall t \in [t_0, \beta).
\]

Using differential inequality, we obtain

\[
V(t, \xi, \eta_i) \leq e^{-c_0(t - t_0)} \left[ V(t_0, \xi_{t_0}, \eta_{t_0}) - \frac{LK}{c_0} \right] + \frac{LK}{c_0} \leq \frac{LK}{c_0}, \quad \forall t \in [t_0, \beta).
\]

Thus, we have

\[
u(|D\xi_t - D\eta_t|) \leq \frac{LK}{c_0}, \quad \forall t \in [t_0, \beta).
\]
Therefore,
\[ |D\xi_t - D\eta_t| \leq u^{-1} \left( \frac{LK}{c_0} \right), \quad \forall t \in [t_0, \beta). \]

From (6) in Lemma 1, we obtain
\[
|\dot{\xi}_t - \dot{\eta}_t| \leq e^{-\omega(t-t_0)} \left[ b |\xi_{t_0} - \eta_{t_0}| + c \cdot \sup_{t_0 \leq u \leq t} |D(\xi_u - \eta_u)| \right] + d \cdot \sup_{t_0 \leq u \leq t} |D(\xi_u - \eta_u)| \\
\leq (c + d) \sup_{t_0 \leq u \leq t} |D(\xi_u - \eta_u)| \\
\leq u^{-1} \left( \frac{LK}{c_0} \right) \cdot (c + d), \quad \forall t \in [t_0, \beta).
\]

It implies
\[ |\eta_t| \leq u^{-1} \left( \frac{LK}{c_0} \right) \cdot (c + d) + H_1 \leq H < H^*, \quad \forall t \in [t_0, \beta). \]

Thus, \( \beta = +\infty \). This is the end of proof of Theorem 3.

**THEOREM 4.** Suppose \( D \) is a stable operator, and there is \( V: \mathbb{R}^+ \times C_{H^*} \times C_{H^*} \rightarrow \mathbb{R}^+ \) continuous that satisfies condition (ii)' of Theorem 3, condition (iii) of Theorem 2, and

(i) \( M_0 |D\varphi - D\psi| \leq V(t, \varphi, \psi) \leq v(|\varphi - \psi|), \quad v \in CIP, \quad v(0) = 0. \)

If Eq. (1) has a solution \( \xi(t), \quad |\xi_t| \leq H < H^*, \quad t \geq t_0, \) then for any \( \beta: H^* > \beta > H \) and \( N > n(x) \cdot M \) (where \( M \) is the number in (4)), there is \( \eta_0 > 0 \) such that when \( 0 \leq \eta < \eta_0, \) Eq. (3) has a unique solution in \( B_{\beta, N}. \) Furthermore, if \( f(t, \varphi), \ g(t, \varphi) \) is \( \omega \)-periodic in \( t, \) then when \( 0 \leq \eta < \eta_0, \) Eq. (3) has a \( \omega \)-periodic solution.

**Proof.** For \( \beta > H, \ N > n(H)M, \) we first prove that there is \( \eta_2 > 0, \) such that when \( 0 \leq \eta < \eta_2, \) for any \( \varphi \in B_{\beta, N}, \) system
\[
\frac{d}{dt} Dx_t = f(t, x_t) + \eta \cdot g(t, \varphi_t) \tag{11}
\]
has a unique solution in \( B_{\beta, N}. \)

Let \( C_1 = \sup \{|g(t, \varphi_t)|; \quad t \in \mathbb{R}, \ \varphi \in B_{\beta, N}\}. \) By Lemma 3, \( C_1 < +\infty. \)
Suppose $x(t)$ is a solution of Eq. (11) that satisfies $x_{t_0} = \xi_{t_0}$, and its maximal existence interval to the right is $[t_0, \alpha)$. We have

$$V'(t, x_t, \xi_t) = \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, x_{t+h}, \xi_{t+h}) - V(t, x_t, \xi_t) \right]$$

$$\quad \leq \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, x_t^{(1)}(t, x_t), \xi_{t+h}) - V(t, x_t, \xi_t) \right]$$

$$\quad + \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, x_{t+h}, \xi_{t+h}) - V(t + h, x_t^{(1)}(t, x_t), \xi_{t+h}) \right]$$

$$\leq -c_0 V(t, x_t, \xi_t) + \lim_{h \to 0^+} \frac{L}{h} |Dx_{t+h} - Dx_t^{(1)}(t, x_t)|$$

$$\leq -c_0 V(t, x_t, \xi_t) + L\eta |g(t, \phi_t)|$$

$$\leq -c_0 V(t, x_t, \xi_t) + L\eta C_1, \quad t \in [t_0, \alpha).$$

By differential inequality, we obtain

$$V(t, x_t, \xi_t) \leq e^{-c_0(t-t_0)} \left[ V(t_0, x_{t_0}, \xi_{t_0}) - \frac{L\eta C_1}{c_0} \right] + \frac{L\eta C_1}{c_0}$$

$$\leq \frac{L\eta C_1}{c_0}, \quad t \in [t_0, \alpha).$$

From condition (i)', we have

$$|Dx_t - D\xi_t| \leq \frac{L\eta C_1}{c_0 M_0}, \quad t \in [t_0, \alpha).$$

Using Lemma 1, we obtain

$$|x_{t} - \xi_{t}| \leq e^{-d(t-t_0)} \left[ b |x_{t_0} - \xi_{t_0}| + c \cdot \sup_{t_0 \leq u \leq t} |D(x_u - \xi_u)| \right]$$

$$\quad + d \cdot \sup_{t_0 \leq u \leq t} |D(x_u - \xi_u)|$$

$$\leq (c + d) \cdot \sup_{t_0 \leq u \leq t} |D(x_u - \xi_u)| \leq (c + d) \frac{\eta LC_1}{c_0 M_0}, \quad t \in [t_0, \alpha).$$

Thus, we have

$$|x_{t}| \leq (c + d) \frac{\eta LC_1}{c_0 M_0} + |\xi_{t}| \leq (c + d) \frac{\eta LC_1}{c_0 M_0} + H.$$
By the supposition, we know that there is \( \eta_1 > 0 \), such that when \( 0 \leq \eta < \eta_1 \), \( (c + d)(\eta L C_1/c_0 M_0) + H < \beta < H^* \). Hence, \( x(t) \) is infinite continuable to the right, i.e., \( x = +\infty \). Suppose \((11)^*\) denotes a product system of Eq. (11). As the proof of (12), it is easy to see

\[
V_{(11)^*}(t, \varphi, \psi) \leq -c_0 V(t, \varphi, \psi).
\]

By Theorem 2, we know that Eq. (11) has a unique almost periodic solution \( p(t) \) that is uniformly asymptotically stable, \( \text{mod}(p) \subset \text{mod}(f, g) \), and

\[
|p| \leq \frac{\eta LC_1}{c_0 M_0} (c + d) + H.
\]

For any \( t_1, t_2 \in \mathbb{R} \)

\[
|Dp_{t_2} - Dp_{t_1}| = \left| \int_{t_1}^{t_2} \left[ f(s, p_s) + \eta \cdot g(s, \varphi_s) \right] ds \right|
\]

\[
\leq \left[ n \left( \frac{\eta LC_1}{c_0 M_0} (c + d) + H \right) + \eta C_1 \right] |t_1 - t_2|.
\]

By (5), we obtain

\[
|p_{t_2} - p_{t_1}| \leq M \left[ n \left( \frac{\eta LC_1}{c_0 M_0} (c + d) + H \right) + \eta C_1 \right] |t_1 - t_2|.
\]

Since \( \beta > H, N > M_0 n(H) \), we may find \( \eta_2 > 0 \), such that when \( 0 \leq \eta < \eta_2 \), Eq. (11) has a unique solution in \( B_{\beta, N} \).

For any \( \varphi \in B_{\beta, N} \), let \( T\varphi \) be a unique solution of Eq. (11) in \( B_{\beta, N} \). Thus, \( T \) is a mapping from \( B_{\beta, N} \) to \( B_{\beta, N} \). For any \( \varphi, \psi \in B_{\beta, N}, t \geq 0 \), as (12), we have

\[
V'(t, (T\varphi)_t, (T\psi)_t)
\]

\[
= \lim_{h \to 0^+} \frac{1}{h} \left[ V(t + h, (T\varphi)_{t+h}, (T\psi)_{t+h}) - V(t, (T\varphi)_t, (T\psi)_t) \right]
\]

\[
\leq -c_0 V(t, (T\varphi)_t, (T\psi)_t) + L \eta \| g(t, \varphi_t) - g(t, \psi_t) \|
\]

\[
\leq -c_0 V(t, (T\varphi)_t, (T\psi)_t) + L \eta K \| \varphi - \psi \|^\infty.
\]

Using differential inequality, we obtain

\[
V(t, (T\varphi)_t, (T\psi)_t)
\]

\[
\leq e^{-c_0 t} \left( V(0, (T\varphi)_0, (T\psi)_0) \frac{L \eta K}{c_0} \| \varphi \| \| \psi \|^\infty \right) - \frac{L \eta K}{c_0} \| \varphi \| \| \psi \|^\infty.
\]
From condition (i)', we have
\[|D(T\varphi)_t - D(T\psi)_t|\]
\[\leq e^{-\omega t}v(\|(T\varphi)_0 - (T\psi)_0\|/M_0 + \frac{L\eta K}{c_0 M_0} \|\varphi - \psi\|\infty), \quad \text{for } t \geq 0 \quad (13)\]

Since \(T\varphi, T\psi \in B_{\beta,N}\), there is a sequence \(\{\alpha_n\}\), such that \(T\varphi(t + \alpha_n) - T\psi(t + \alpha_n)\rightarrow T\varphi(t) - T\psi(t)\) uniformly on \(R\) as \(n \rightarrow \infty\). \(t + \alpha_n\) is used instead of \(t\) in (13). Using the continuity of operator \(D\), we have
\[|D(T\varphi)_t - D(T\psi)_t| \leq \frac{L\eta K}{c_0 M_0} \|\varphi - \psi\|\infty, \quad \text{for } t \in R. \quad (14)\]

By Lemma 1, when \(t \geq 0\), we have
\[|(T\varphi)_t - (T\psi)_t|\]
\[\leq e^{-\alpha t}[b \cdot |D(T\varphi)_0 - D(T\psi)_0| + c \cdot \sup_{0 \leq u \leq t} |D(T\psi)_u - D(T\psi)_u|]
+ d \cdot \sup_{0 \leq u \leq t} |D(T\varphi)_u - D(T\psi)_u|
\[\leq e^{-\alpha t}\left[b \cdot |D(T\varphi)_0 - D(T\psi)_0| + c \cdot \frac{L\eta K}{c_0 M_0} \|\varphi - \psi\|\infty\right]
+ d \cdot \frac{L\eta K}{c_0 M_0} \|\varphi - \psi\|\infty.\]

If \(t + \alpha_n\) is used instead of \(t\) on the above, and \(n \rightarrow \infty\), we obtain, for any \(t \in R\),
\[|(T\varphi)_t - (T\psi)_t| \leq d \frac{L\eta K}{c_0 M_0} \|\varphi - \psi\|\infty.\]

Let \(\eta_0 = \min\{\eta_1, \eta_2, c_0 M_0 / dL\tilde{K}\}\), then when \(0 \leq \eta < \eta_0\), \(T\) is a contraction mapping from \(B_{\beta,N}\) to \(B_{\beta,N}\). Hence, \(T\) has a unique fixed point in \(B_{\beta,N}\).

If \(f, g\) is \(\omega\)-periodic in \(t\), it is easy to see that there is \(\eta_0\), such that when \(0 \leq \eta < \eta_0\), Eq. (3) has a \(\omega\)-periodic solution. This is the end.

**Remark.** It is easy to see that the results in [1–3] are the special cases of the above theorems when \(D\varphi = \varphi(0)\) for \(\varphi \in C\).

**REFERENCES**


