Available online at www.sciencedirect.com
ScienceDirect
Journal of Computational and Applied Mathematics 221 (2008) 234–260
JOURNAL OF COMPUTATIONAL AND APPLIED MATHEMATICS
www.elsevier.com/locate/cam

Dual mixed finite element methods for the elasticity problem with Lagrange multipliers

L. Boulaajine[a], S. Nicaise[b,*], L. Paquet[b], Rafilipojaona[c]

[a] *INRIA - Projet Gamma, Domaine de Voluceau, Rocquencourt - B.P. 105, 78153 Le Chesnay Cedex, France*
[b] *Université de Valenciennes et du Hainaut Cambrésis, LAMAV, ISTV, F-59313 - Valenciennes Cedex 9, France*
[c] *Université de Fianarantsoa, Faculté des Sciences, B.P. 1486, Fianarantsoa 301, Madagascar*

Received 2 October 2006; received in revised form 7 September 2007

Abstract

We study a dual mixed formulation of the elasticity system in a polygonal domain of the plane with mixed boundary conditions and its numerical approximation. The (essential) Neumann boundary conditions (or traction boundary condition) are imposed using a discontinuous Lagrange multiplier corresponding to the trace of the displacement field. Moreover, a strain tensor is introduced as a new unknown and its symmetry is relaxed, also by the use of a Lagrange multiplier (the rotation). The singular behaviour of the solution requires us to use refined meshes to restore optimal rates of convergence. Uniform error estimates in the Lamé coefficient λ are obtained for large λ. The hybridization of the problem is performed and numerical tests are presented confirming our theoretical results.

Keywords: Mixed FEM; Elasticity system; Lagrange multipliers; Mesh refinements; Hybrid method

1. Introduction

The analysis of classical finite element methods with Lagrange multipliers, originally developed in [6] has been considered for diverse problems, like the Laplace problem, the biharmonic equation or the Stokes system. On the other hand, the dual mixed finite element method (see [10,27,28]) has the advantage of introducing new unknowns like strain tensors, quantities of physical interest, which are then computed directly with a good accuracy, avoiding the use of numerical postprocessing. Many papers are devoted to the elasticity system; here let us quote [2–5,10,14,16,15,18,29,31]. For the elasticity system, this method has, furthermore, the advantage of avoiding any locking effect for large Lamé coefficients λ.

Recently Babuska and Gatica [7] have introduced a dual mixed finite element method for the Laplace equation with a Lagrange multiplier in order to impose nonhomogeneous Neumann boundary conditions.

Accordingly, the goal of our paper is to extend the analysis carried out for the Laplace equation in [7] to the elasticity system, but with discontinuous approximated Lagrange multipliers, in order to be able to hybridize the

* Corresponding author. Tel.: +33 0 327511927; fax: +33 0 327511900.
E-mail addresses: lahcen.boulaajine@inria.fr (L. Boulaajine), snicaise@univ-valenciennes.fr (S. Nicaise), Luc.Paquet@univ-valenciennes.fr (L. Paquet), rafilipo@wanadoo.mg (Rafilipojaona).
URL: http://www.univ-valenciennes.fr/macs/nicaise (S. Nicaise).

0377-0427/$ - see front matter
doi:10.1016/j.cam.2007.10.061

problem. We furthermore want to take into account the singular behaviour of the solution near the singular points of the domain by using refined meshes. Therefore, contrary to [7], we do not use quasi-uniform meshes but instead use locally refined meshes. As a consequence we need to modify the norm of the approximation space in order to obtain a uniform discrete inf-sup condition. In [23,24] the authors used a weighted mesh-dependent norm, we here prefer to use a standard L^2-norm (see below for the details). In comparison with the norm used in [7] and in [23,24], our norm is more simple from a practical point of view and allows us to consider $\mathbb{P}_1$-discontinuous approximations for the Lagrange multiplier.

Here as in [14,16] and contrary to [2], we split up the stress tensor into the strain tensor and the pressure, allowing us to avoid cancellation errors which could appear for nearly incompressible materials (i.e., μ quite smaller than λ), when trying to compute the strain tensor from the stress tensor. Moreover the hybridization of the method allows us to use reduction to solve a linear system involving only the Lagrange multiplier (u along the edges of the triangulation) and the pressure. Its resolution is then relatively cheap (see below).

The paper is organized as follows. In Section 2, we introduce the considered boundary value problem and give its new dual-mixed formulation. We then show the equivalence between this formulation and the standard one. We finally prove that the dual-mixed formulation has a unique solution by establishing an inf-sup condition and a coerciveness result uniform with respect to λ. In Section 3, we introduce the discrete dual-mixed formulation and show again a uniform discrete inf-sup condition and a coerciveness result uniform with respect to λ. Section 4 gives some regularity results of the solution of our elasticity system in terms of weighted Sobolev spaces. In Section 5, based on some interpolation error estimates in these weighted Sobolev spaces, we prove some optimal error estimates. Finally Section 6 is devoted to the hybridization of the problem and to numerical tests confirming our theoretical results.

2. The dual mixed variational formulation

Let Ω be a simply connected domain of $\mathbb{R}^2$ with polygonal boundary Γ such that the interior angle at each corner lies in $(0, 2\pi)$. Let Γ_D and Γ_N be disjoint open subsets of Γ such that $|\Gamma_D| \neq 0$ and $|\Gamma_N| \neq 0$ and $\Gamma = \Gamma_D \cup \Gamma_N$.

In the static theory of linear isotropic elasticity, the equation satisfied by the displacement field u is

$$-\mathrm{div}\sigma_s(u) = f \quad \text{in } \Omega, \tag{1}$$

where f represents the body force density, $\varepsilon(u) = \frac{1}{2}(\nabla u + (\nabla u)^{\mathrm{T}})$ is the strain tensor,

$$\sigma_s(u) = 2\mu\varepsilon(u) + \lambda \mathrm{tr}\,\varepsilon(u)\delta,$$

is the stress tensor, δ is the identity tensor, and finally μ, λ are the Lamé coefficients with $\mu \in [\mu_1, \mu_2]$ and $\lambda > 0$.

This balance equation is completed by boundary conditions to get the system:

$$\begin{cases} -\mathrm{div}\sigma_s(u) = f & \text{in } \Omega, \\ u = 0 & \text{on } \Gamma_D, \\ \sigma_s(u)n = g & \text{on } \Gamma_N, \end{cases} \tag{2}$$

where g corresponds to the surface force density and n is the unit outward normal vector to Γ.

In the sequel, we will use the following notations. For $\tau = (\tau_{ij}) \in (H(\mathrm{div}; \Omega))^2$; we denote by

$$\mathrm{div}\tau = \left(\frac{\partial \tau_{11}}{\partial x_1} + \frac{\partial \tau_{12}}{\partial x_2}, \frac{\partial \tau_{21}}{\partial x_1} + \frac{\partial \tau_{22}}{\partial x_2}\right),$$
$$as(\tau) = \tau_{21} - \tau_{12}.$$

For $v = (v_1, v_2) \in (H^1(\Omega))^2$, we recall that

$$\mathrm{curl}\ v = \frac{\partial v_2}{\partial x_1} - \frac{\partial v_1}{\partial x_2}.$$

As usual, we denote by $L^2(.)$ the Lebesgue space and by $H^s(.)$, $s \geq 0$, the standard Sobolev space. The usual norm and seminorm of $H^s(D)$ are denoted by $\|\cdot\|_{s,D}$ and $|\cdot|_{s,D}$. For brevity the $L^2(D)$-norm will be denoted by $\|\cdot\|_D$ and in the case $D = \Omega$, we will drop the index Ω. The inner product in $(L^2(\Omega))^2$ will be written $(\cdot,\cdot)$ and the duality pairing between $(H^{-\frac{1}{2}}(\Gamma))^2$ and $(H^{\frac{1}{2}}(\Gamma))^2$ will be denoted by $\langle\cdot,\cdot\rangle_\Gamma$. If $\sigma = (\sigma_{ij}), \tau = (\tau_{ij}) \in (L^2(\Omega))^{2\times 2}$, then we denote by

$$\sigma : \tau = \sum_{i,j} \sigma_{ij}\tau_{ij},$$

$$(\sigma, \tau) = \int_\Omega \sigma : \tau \mathrm{d}x.$$

We now introduce the Hilbert space

$$H^1_{0,\Gamma_D}(\Omega) = \{v \in H^1(\Omega); v_{|\Gamma_D} = 0\}.$$

We further recall that the Hilbert space $H^{-\frac{1}{2}}(\Gamma_N)$ is the dual space of $H^{\frac{1}{2}}_{00}(\Gamma_N)$ defined as follows:

$$H^{\frac{1}{2}}_{00}(\Gamma_N) = \{v_{|\Gamma_N}; v \in H^1_{0,\Gamma_D}(\Omega)\}.$$

The duality pairing will be denoted by $\langle\cdot,\cdot\rangle_{\Gamma_N}$.

In the sequel, the symbol $|\cdot|$ will denote either the Euclidean norm in $\mathbb{R}^2$, or the length of a line segment, or finally the area of a domain of $\mathbb{R}^2$. Finally the notation $a \lesssim b$ means here and below that there exists a positive constant C independent of a and b, of the mesh-size of the triangulation (see below) and of the Lamé parameter λ (but it may depend on μ_1, μ_2 and Ω), such that $a \le C\, b$. The notation $a \sim b$ means that $a \lesssim b$ and $b \lesssim a$ hold simultaneously.

Before going on, let us recall the following Green's formula (see [16])

Lemma 2.1. *Let $\tau \in (H(\mathrm{div};\Omega))^2$ and $v \in (H^1(\Omega))^2$, then*

$$(\varepsilon(v), \tau) = \langle \tau n, v\rangle_\Gamma - (\mathrm{div}\tau, v) - \frac{1}{2}(as(\tau), \mathrm{curl}\ v),$$

where $\tau n = (\tau_{11}n_1 + \tau_{12}n_2, \tau_{21}n_1 + \tau_{22}n_2)$.

The variational formulation of (2) is well known (see Section I.1.2 of [11]), and is summarized in the next lemma.

Lemma 2.2. *Let $f \in (L^2(\Omega))^2$ and $g \in (H^{-\frac{1}{2}}(\Gamma_N))^2$; then there exists a unique solution $u \in (H^1_{0,\Gamma_D}(\Omega))^2$ of*

$$\int_\Omega (2\mu\varepsilon(u) : \varepsilon(v) + \lambda \mathrm{tr}\,\varepsilon(u)\mathrm{tr}\,\varepsilon(v))\mathrm{d}x = \int_\Omega f v \mathrm{d}x + \langle g, v\rangle_{\Gamma_N}, \quad \forall v \in (H^1_{0,\Gamma_D}(\Omega))^2. \tag{3}$$

For the mixed formulation of problem (3), we introduce the additional unknowns

$$\sigma = 2\mu\varepsilon(u), \quad p = -\lambda \mathrm{div} u, \quad \omega = \frac{1}{2}\mathrm{curl}\ u, \quad \xi = -u_{|\Gamma_N}.$$

This last unknown is a Lagrange multiplier, which is introduced in order to impose the boundary condition on Γ_N (see below).

Let us further define the spaces

$$\Sigma = \{(\tau, q) \in (L^2(\Omega))^{2\times 2} \times L^2(\Omega) : \mathrm{div}(\tau - q\delta) \in (L^2(\Omega))^2\},$$
$$Q = (L^2(\Omega))^2 \times L^2(\Omega),$$
$$M = Q \times (H^{\frac{1}{2}}_{00}(\Gamma_N))^2.$$

For shortness we often write the pairs $(\sigma, p), (\tau, q) \in \Sigma$ by $\underline{\sigma} = (\sigma, p), \underline{\tau} = (\tau, q)$, and similarly the pairs $(u, \omega), (v, \theta) \in Q$ by $\underline{u} = (u, \omega), \underline{v} = (v, \theta)$. Clearly the space Σ is a Hilbert space equipped with the norm $\|\underline{\tau}\|_\Sigma := \|\tau\| + \|q\| + \|\mathrm{div}(\tau - q\delta)\|$.

With these notations, the mixed variational formulation of problem (3) is: Find $(\underline{\sigma}, (\underline{u}, \xi)) \in \Sigma \times M$ such that

$$\begin{cases} A(\underline{\sigma}, \underline{\tau}) + B(\underline{\tau}, (\underline{u}, \xi)) = 0 & \forall \underline{\tau} \in \Sigma, \\ B(\underline{\sigma}, (\underline{v}, \alpha)) = F(\underline{v}, \alpha) & \forall (\underline{v}, \alpha) \in M, \end{cases} \tag{4}$$

where the bilinear forms $A : \Sigma \times \Sigma \to \mathbb{R}$, $B : \Sigma \times M \to \mathbb{R}$ and the linear form $F : M \to \mathbb{R}$ are defined by

$$A(\underline{\sigma}, \underline{\tau}) = \frac{1}{2\mu}(\sigma, \tau) + \frac{1}{\lambda}(p, q),$$
$$B(\underline{\tau}, (\underline{v}, \alpha)) = (\mathrm{div}(\tau - q\delta), v) + (as(\tau), \theta) + \langle(\tau - q\delta)n, \alpha\rangle_{\Gamma_N},$$
$$F(\underline{v}, \alpha) = -\int_\Omega f v \mathrm{d}x + \langle g, \alpha\rangle_{\Gamma_N}.$$

Let us first show the equivalence between the standard and mixed formulations:

Proposition 2.3. *$u \in (H^1_{0,\Gamma_D}(\Omega))^2$ is solution of (3) if and only if $((\sigma, p), ((u, \omega), \xi)) \in \Sigma \times M$ is solution of (4), where $\sigma = 2\mu\varepsilon(u)$, $p = -\lambda \mathrm{div} u$, $\omega = \frac{1}{2}\mathrm{curl}\, u$, $\xi = -u_{|\Gamma_N}$.*

Proof. (i) (3) ⇒ (4). Let $u \in (H^1_{0,\Gamma_D}(\Omega))^2$ be the unique solution of (3). From the above considerations, we set

$$\sigma = 2\mu\varepsilon(u), \qquad p = -\lambda \mathrm{div} u, \omega = \frac{1}{2}\mathrm{curl}\, u, \qquad \xi = -u_{|\Gamma_N}.$$

Let us point out that p and ω belong to $L^2(\Omega)$ because $u \in (H^1(\Omega))^2$.

The proof of the first identity of (4) is similar to the corresponding one of part (i) of the proof of Theorem 3.2 of [16].

Let us now prove the second identity of (4): By (3) we may write

$$\int_\Omega (2\mu\varepsilon(u) : \varepsilon(v) + \lambda tr\varepsilon(u) tr\varepsilon(v))\mathrm{d}x = \int_\Omega f v \mathrm{d}x, \quad \forall v \in (\mathcal{D}(\Omega))^2.$$

Since $2\mu\varepsilon(u) + \lambda tr\varepsilon(u)\delta$ is symmetric, and $(\mathcal{D}(\Omega))^2$ is dense in $(L^2(\Omega))^2$, we can conclude that

$$(-\mathrm{div}(\sigma - p\delta), v) = (f, v), \quad \forall v \in (L^2(\Omega))^2. \tag{5}$$

In this identity, restricting ourselves to function $v \in (H^1_{0,\Gamma_D}(\Omega))^2$ and appealing to the Green's formula (from Lemma 2.1), we obtain

$$(\varepsilon(v), \sigma - p\delta) - \langle(\sigma - p\delta)n, v\rangle_{\Gamma_N} + \frac{1}{2}(as(\sigma - p\delta), \mathrm{curl}\, v) = (f, v), \quad \forall v \in (H^1_{0,\Gamma_D}(\Omega))^2.$$

Again by the definition of σ, p and by the property $as(\sigma - p\delta) = 0$, this identity is equivalent to

$$2\mu(\varepsilon(u), \varepsilon(v)) + \lambda(tr\varepsilon(u), tr\varepsilon(v)) - \langle(\sigma - p\delta)n, v\rangle_{\Gamma_N} = (f, v), \quad \forall v \in (H^1_{0,\Gamma_D}(\Omega))^2.$$

And by (3), we arrive at

$$(f, v) + \langle g, v\rangle_{\Gamma_N} - \langle(\sigma - p\delta)n, v\rangle_{\Gamma_N} = (f, v)$$

which yields

$$\langle(\sigma - p\delta)n, \alpha\rangle_{\Gamma_N} = \langle g, \alpha\rangle_{\Gamma_N} \quad \forall \alpha \in (H^{1/2}_{00}(\Gamma_N))^2. \tag{6}$$

As $as(\sigma) = 0$, using the identities (5) and (6), we may write

$$(\mathrm{div}(\sigma - p\delta), v) + (as(\sigma), \theta) + \langle(\sigma - p\delta)n, \alpha\rangle_{\Gamma_N} = -(f, v) + \langle g, \alpha\rangle_{\Gamma_N}, \quad \forall(\underline{v}, \alpha) \in M,$$

which is equivalent to the second identity of (4) by the definition of B.

(ii) (4) ⇒ (3). Let $(\underline{\sigma}, (\underline{u}, \xi)) \in \Sigma \times M$ be a solution of (4).

By the same arguments as those used in the proof of Theorem 3.2 of [16], it follows that $\sigma = 2\mu\varepsilon(u)$, $\omega = \frac{1}{2}\mathrm{curl}\, u$, and $u \in (H^1(\Omega))^2$.

Now in the first identity of (4), let us take $q = 0$ and $\tau \in (C^\infty(\bar{\Omega}))^{2\times 2}$. We obtain:

$$\frac{1}{2\mu}(\sigma, \tau) + (\mathrm{div}\tau, u) + (as(\tau), \omega) + \langle\tau n, \xi\rangle_{\Gamma_N} = 0, \quad \forall \tau \in (C^\infty(\bar{\Omega}))^{2\times 2}.$$

Knowing that $\omega = \frac{1}{2}\text{curl}\, u$ and that $\sigma = 2\mu\varepsilon(u)$, applying Green's formula in the above identity we obtain

$$\langle \tau n, u\rangle_{\Gamma} + \langle \tau n, \xi\rangle_{\Gamma_N} = 0, \quad \forall \tau \in (C^{\infty}(\bar{\Omega}))^{2\times 2},$$

or equivalently

$$\langle \tau n, \beta\rangle_{\Gamma} = 0,$$

where $\beta \in (H^{1/2}(\Gamma))^2$ is defined by

$$\beta = \begin{cases} u+\xi & \text{on } \Gamma_N, \\ u & \text{on } \Gamma_D. \end{cases}$$

Taking respectively τ in the forms

$$\begin{pmatrix} \phi & 0 \\ 0 & \phi \end{pmatrix}, \begin{pmatrix} 0 & -\psi \\ \psi & 0 \end{pmatrix},$$

with $(\phi, \psi \in C^{\infty}(\bar{\Omega}))$ (as in the proof of Theorem 3.2 of [16]), we deduce that $\beta \cdot n = 0$ on Γ, and $\beta \cdot t = 0$ on Γ and therefore $\beta = 0$ on Γ. This shows that $\xi = -u_{|\Gamma_N}$ and $u = 0$ on Γ_D.

In the second identity of (4), taking $v = 0$ and recalling that $as(\sigma) = 0$, we get

$$\langle (\sigma - p\delta)n, \alpha\rangle_{\Gamma_N} = \langle g, \alpha\rangle_{\Gamma_N}, \quad \forall \alpha \in (H^{1/2}_{00}(\Gamma_N))^2. \tag{7}$$

It remains to prove (3): For that purpose, in the second identity of (4), taking $v \in (H^1_{0,\Gamma_D}(\Omega))^2$, $\theta = 0$ and $\alpha = 0$, we get

$$(\text{div}(2\mu\varepsilon(u) + \lambda \text{tr}\, \varepsilon(u)\delta), v) = -(f, v).$$

Applying the Green's formula, we have:

$$(2\mu\varepsilon(u) + \lambda \text{tr}\, \varepsilon(u)\delta, \varepsilon(v)) - \langle (\sigma - p\delta)n, v\rangle_{\Gamma_N} = (f, v).$$

Thus, using identity (7), we obtain (3). ■

The previous proposition guarantees in particular the well-posedness of problem (4). But for further purposes, we need to check that the so-called inf-sup condition holds, as well as a uniform coerciveness result with respect to the Lamé coefficient λ.

We start with the inf-sup condition.

Lemma 2.4. *The bilinear form B satisfies the inf-sup condition, which means that*

$$\sup_{\underline{\tau} \in \Sigma, \underline{\tau} \neq 0} \frac{B(\underline{\tau}, (\underline{v}, \alpha))}{\|\underline{\tau}\|_{\Sigma}} \gtrsim \|(\underline{v}, \alpha)\|_M, \quad \forall (\underline{v}, \alpha) \in M. \tag{8}$$

Proof. Fix an arbitrary element $(\underline{v} = (v, \theta), \alpha) \in M$.

In the whole proof, we fix $\lambda^* > 0$ and $\mu^* > 0$ independently of λ, μ and of $(\underline{v}, \alpha)$.

(1) Let $w \in H^1(\Omega)^2$ be the unique solution of the problem

$$\begin{cases} \text{div}(2\mu^*\varepsilon(w) + \lambda^*\, \text{div} w\, \delta) = v & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_D, \\ (2\mu^*\varepsilon(w) + \lambda^*\, \text{div} w\, \delta)n = 0 & \text{on } \Gamma_N. \end{cases}$$

By Korn's inequality, w satisfies (the constant below depends on λ^*, μ^* and on Ω)

$$|w|_{1,\Omega} \lesssim \|v\|. \tag{9}$$

Setting $\tau^1 = 2\mu^*\varepsilon(w) + \lambda^* \text{div} w \delta$ and $\underline{\tau}^1 = (\tau^1, 0)$, we see that $\underline{\tau}^1$ belongs to Σ and is symmetric (i.e. $as(\tau^1) = 0$). Consequently

$$\begin{aligned} B(\underline{\tau}^1, (\underline{v}, \alpha)) &= (\text{div}\tau^1, v) + (as(\tau^1), \theta) + \langle \tau^1 n, \alpha\rangle_{\Gamma_N} \\ &= (\text{div}\tau^1, v) + \langle \tau^1 n, \alpha\rangle_{\Gamma_N}. \end{aligned}$$

And by the above problem solved by w, we get

$$B(\underline{\tau}^1, (\underline{v}, \alpha)) = \|v\|^2. \tag{10}$$

By the definition of $\underline{\tau}^1$, we further have $\|\underline{\tau}^1\|_\Sigma = \|\tau^1\| + \|\text{div}\tau^1\|$ and consequently (9) yields

$$\|\underline{\tau}^1\|_\Sigma \lesssim \|v\|. \tag{11}$$

(2) Let us fix $\Phi \in (H^{-\frac{1}{2}}(\Gamma_N))^2$ such that

$$\langle \Phi, \alpha\rangle_{\Gamma_N} = \|\alpha\|^2_{(H^{\frac{1}{2}}_{00}(\Gamma_N))^2}, \tag{12}$$

$$\|\Phi\|_{(H^{-\frac{1}{2}}(\Gamma_N))^2} \le \|\alpha\|_{(H^{\frac{1}{2}}_{00}(\Gamma_N))^2}. \tag{13}$$

As before let us consider te $w \in (H^1(\Omega))^2$ solution of

$$\begin{cases} \text{div}(2\mu^*\varepsilon(w) + \lambda^* \,\text{div} w\, \delta) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_D, \\ (2\mu^*\varepsilon(w) + \lambda^* \,\text{div} w\, \delta)n = \Phi & \text{on } \Gamma_N. \end{cases}$$

Setting $\tau^2 = 2\mu^*\varepsilon(w) + \lambda^* \,\text{div} w\, \delta$, we see that τ^2 is symmetric, i.e., $as(\tau^2) = 0$, $\text{div}\tau^2 = 0$ and $\tau^2 n = \Phi$ on Γ_N. Hence $\underline{\tau}^2 = (\tau^2, 0)$ fulfils

$$B(\underline{\tau}^2, (\underline{v}, \alpha)) = (\text{div}\tau^2, v) + (as(\tau^2), \theta) + \langle \tau^2 n, \alpha\rangle_{\Gamma_N} = \langle \Phi, \alpha\rangle_{\Gamma_N},$$

and by the above property of Φ:

$$B(\underline{\tau}^2, (\underline{v}, \alpha)) = \|\alpha\|^2_{(H^{\frac{1}{2}}_{00}(\Gamma_N))^2}. \tag{14}$$

Again Korn's inequality implies that

$$\|\underline{\tau}^2\|_\Sigma = \|\tau^2\| \lesssim \|\Phi\|_{(H^{-\frac{1}{2}}(\Gamma_N))^2},$$

and again by the above property of Φ:

$$\|\underline{\tau}^2\|_\Sigma \lesssim \|\alpha\|_{(H^{\frac{1}{2}}_{00}(\Gamma_N))^2}. \tag{15}$$

(3) Since $\mathcal{D}(\Omega)$ is dense in $L^2(\Omega)$, there exists a sequence $\theta_k \in \mathcal{D}(\Omega)$ such that

$$\theta_k \longrightarrow \theta \text{ in } L^2(\Omega), \text{ as } k \longrightarrow \infty.$$

Now, for each $k \in \mathbb{N}$, consider $w_k \in (H^1_{0,\Gamma_D}(\Omega))^2$, the unique solution of the problem:

$$\begin{cases} \text{div}(2\mu^*\varepsilon(w_k) + \lambda^* \,\text{div} w_k\, \delta) = \frac{1}{2}\text{curl}\, \theta_k & \text{in } \Omega, \\ w_k = 0 & \text{on } \Gamma_D, \\ (2\mu^*\varepsilon(w_k) + \lambda^* \,\text{div} w_k\, \delta)n = 0 & \text{on } \Gamma_N. \end{cases}$$

Its variational formulation is

$$\int_\Omega (2\mu^*\varepsilon(w_k) : \varepsilon(v) + \lambda^* \text{tr}\, \varepsilon(w_k)\text{tr}\, \varepsilon(v))\text{d}x = -\frac{1}{2}\int_\Omega \theta_k \text{curl}\, v \text{d}x, \quad \forall v \in (H^1_{0,\Gamma_D}(\Omega))^2.$$

Consequently by Korn's inequality we have

$$|w_k|_{1,\Omega} \lesssim \|\theta_k\|. \tag{16}$$

Then we set

$$\tau^3_k = 2\mu\varepsilon(w_k) + \lambda \text{div} w_k \delta + \frac{1}{2}\theta_k \chi,$$

where χ is the antisymmetric matrix defined by:

$$\chi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

With respect to the above problem, we remark that

$$\operatorname{div}\tau_k^3 = 0 \quad \text{in } \Omega,$$
$$as(\tau_k^3) = \theta_k \quad \text{in } \Omega,$$
$$\tau_k^3 n = 0 \quad \text{on } \Gamma_N.$$

Moreover, from (16), we clearly have

$$\|\tau_k^3 - \tau_l^3\| \lesssim \|\theta_k - \theta_l\|, \quad \forall k, l \in \mathbb{N}.$$

This means that the sequence $(\tau_k^3)_k$ is a Cauchy sequence in $H(\operatorname{div}, \Omega)^2$. Denote by τ^3 its limit. By the above properties of τ_k^3, τ^3 satisfies

$$\operatorname{div}\tau^3 = 0 \quad \text{in } \Omega,$$
$$as(\tau^3) = \theta \quad \text{in } \Omega,$$
$$\tau^3 n = 0 \quad \text{on } \Gamma_N,$$
$$\|\tau^3\| \lesssim \|\theta\|.$$

Setting $\underline{\tau}^3 = (\tau^3, 0)$, we then have

$$B(\underline{\tau}^3, (\underline{v}, \alpha)) = (\operatorname{div}\tau^3, v) + (as(\tau^3), \theta) + \langle \tau^3 n, \alpha\rangle_{\Gamma_N} = (\theta, \theta) = \|\theta\|^2, \tag{17}$$

as well as

$$\|\underline{\tau}^3\|_\Sigma = \|\tau^3\| \lesssim \|\theta\|. \tag{18}$$

(4) The three above points suggest that we set

$$\underline{\tau} = \underline{\tau}^1 + \underline{\tau}^2 + \underline{\tau}^3.$$

Indeed the bilinearity of B and the identities (10), (14) and (17) lead to

$$B(\underline{\tau}, (\underline{v}, \alpha)) = \|(\underline{v}, \alpha)\|_M^2,$$

while the estimates (11), (15) and (18) show that

$$\|\underline{\tau}\|_\Sigma \lesssim \|(\underline{v}, \alpha)\|_M.$$

Therefore, we may conclude that (8) holds. ■

Lemma 2.5. *The bilinear form A is uniformly coercive in λ on the kernel V of B in Σ defined by*

$$V = \left\{\underline{\tau} \in \Sigma : B(\underline{\tau}, (\underline{v}, \alpha)) = 0, \forall (\underline{v}, \alpha) \in M\right\}.$$

Proof. $\underline{\tau} = (\tau, q)$ belongs to V if and only if

$$(\operatorname{div}(\tau - q\delta), v) + (as(\tau), \theta) + \langle(\tau - q\delta)n, \alpha\rangle = 0, \quad \forall(\underline{v}, \alpha) \in M.$$

This identity implies that

$$\operatorname{div}(\tau - q\delta) = 0, \qquad as(\tau) = 0, \tag{19}$$

and

$$(\tau - q\delta)n = 0 \quad \text{on } \Gamma_N.$$

By Lemma 3.3 of [9], it follows that

$$\|\operatorname{tr}(\tau - q\delta)\| \lesssim \|(\tau - q\delta)^D\|,$$

where we recall that $\tau^D = \tau - \frac{1}{2}\mathrm{tr}\,(\tau)\delta$ denotes the deviatoric of τ. But in our case, we may write

$$(\tau - q\delta)^D = \tau - q\delta - \frac{1}{2}(\mathrm{tr}\,(\tau) - 2q)\delta = \tau^D.$$

Consequently the above estimate becomes

$$\|\mathrm{tr}\,\tau - 2q\| \lesssim \|\tau^D\| \lesssim \|\tau\|.$$

By the triangle inequality we get

$$\|q\| \le \frac{1}{\sqrt{2}}\|q\delta\| \le \|\tau - q\delta\| + \|\tau\|,$$

and by the above estimate there exists a positive constant C depending only on $|\Omega|$ such that

$$\|q\| \le C\|\tau\|. \tag{20}$$

Since $\mu \in [\mu_1, \mu_2]$, we may write

$$\begin{aligned} A(\underline{\tau}, \underline{\tau}) &= \frac{1}{2\mu}\|\tau\|^2 + \frac{1}{\lambda}\|q\|^2 \\ &\ge \frac{1}{2\mu}\|\tau\|^2 \ge \frac{1}{2\mu_2}\|\tau\|^2 \\ &\ge \frac{1}{4\mu_2}\|\tau\|^2 + \frac{1}{4\mu_2}\|\tau\|^2. \end{aligned}$$

By using the estimate (20), we conclude that

$$A(\underline{\tau}, \underline{\tau}) \ge \frac{1}{4\mu_2}\|\tau\|^2 + \frac{1}{4C^2\mu_2}\|q\|^2 \ge C(\mu_2)(\|\tau\|^2 + \|q\|^2). \tag{21}$$

Therefore the coerciveness of A in V holds uniformly in λ. ■

Theorem 2.6. *There exists a unique solution $(\underline{\sigma}, (\underline{u}, \xi)) \in \Sigma \times M$ of the mixed variational formulation (4) such that*

$$\|(\underline{\sigma}, (\underline{u}, \xi))\|_{\Sigma\times M} \lesssim \left(1 + \frac{1}{\lambda}\right)^2 (\|f\| + \|g\|_{(H^{-\frac{1}{2}}(\Gamma_N))^2}).$$

Proof. By the two previous lemmas, the inf-sup condition and the coerciviness are satisfied, so a straightforward application of Corollary I.4.1 of [19] yields the results. ■

3. The discrete problem

Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ made of triangles K of diameter h_K. As usual, the letter h will also denote $h = \max\{h_K, K \in \mathcal{T}_h\}$ (the meaning of h is indicated by the context). We further suppose that the points of $\overline{\Gamma_D} \cap \overline{\Gamma_N}$ are vertices of $\mathcal{T}_h$, for all $h > 0$.

For $K \in \mathcal{T}_h$, let us denote by b_K the standard bubble function defined by $b_K(x) = \lambda_1(x)\lambda_2(x)\lambda_3(x)$ where $\lambda_i, i = 1, 2, 3$, are the barycentric coordinates on K associated with the vertices of K. The set of the edges of K will be denoted by $\mathcal{E}_K$.

Let us now set

$$\begin{aligned} \Sigma_h &= \{(\tau_h, q_h) \in \Sigma : q_{h|K} \in \mathbb{P}_1(K) \text{ and } \tau_{h|K} \in (\mathbb{P}_1(K))^{2\times 2} \oplus (\mathbb{R}\,\mathrm{curl}\, b_K)^2, \forall K \in \mathcal{T}_h\}, \\ L_h^2 &= \{v_h \in (L^2(\Omega))^2 : v_{h|K} \in (\mathbb{P}_0(K))^2, \forall K \in \mathcal{T}_h\}, \\ Q_h &= \{\theta_h \in L^2(\Omega) : \theta_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\}. \end{aligned}$$

Here, by $\tau_{h|K} \in (\mathbb{P}_1(K))^{2\times 2} \oplus (\mathbb{R}\text{curl } b_K)^2$ we mean that there exist polynomials $p_{11}, p_{12}, p_{21}, p_{22}$ of degree ≤ 1 and two real numbers a_1 and a_2 such that

$$\tau_{h|K} = \begin{pmatrix} p_{11} + a_1 \dfrac{\partial b_K}{\partial x_2} & p_{12} - a_1 \dfrac{\partial b_K}{\partial x_1} \\ p_{21} + a_2 \dfrac{\partial b_K}{\partial x_2} & p_{22} - a_2 \dfrac{\partial b_K}{\partial x_1} \end{pmatrix}.$$

Let $\{I_1, \dots, I_m\}$ be the partition of Γ_N induced by the triangulation $\mathcal{T}_h$, i.e., each $I_i = K \cap \bar{\Gamma}_N$ for some triangle K of $\mathcal{T}_h$ and $\bar{\Gamma}_N = \cup_{j=1}^m I_j$. Due to our previous hypotheses on the triangulation $\mathcal{T}_h$, each I_i is contained in one side of the polygonal line Γ.

Let us finally introduce the approximated space of each component of the Lagrange multiplier

$$L^2_{h,\Gamma_N} = \{\alpha_h \in L^2(\Gamma_N) : \alpha_{h|I_j} \in \mathbb{P}_1(I_j),\ j = 1, \dots, m\}.$$

The approximation space of M is then defined by

$$M_h = L^2_h \times Q_h \times (L^2_{h,\Gamma_N})^2.$$

Contrary to [7], we use discontinuous approximated Lagrange multipliers; hence the space M_h is equipped with the L^2-norm, namely

$$\|(\underline{v}_h, \alpha_h)\|_{\tilde{M}} := \|v_h\| + \|\theta_h\| + \|\alpha_h\|_{\Gamma_N},$$

with $\underline{v}_h = (v_h, \theta_h)$.

Another reason is that we want to use non quasi-uniform meshes, for which the applicability of the uniform inf-sup condition with the term $\|\alpha_h\|_{(H^{1/2}(\Gamma_N))^2}$ instead of $\|\alpha_h\|_{\Gamma_N}$ seems to be difficult to prove.

Accordingly the discrete problem associated with the (continuous) mixed problem (4) is: Find $\underline{\sigma}_h = (\sigma_h, p_h) \in \Sigma_h$, and $(\underline{u}_h = (u_h, \omega_h), \xi_h) \in M_h$ such that

$$\begin{cases} A(\underline{\sigma}_h, \underline{\tau}_h) + B(\underline{\tau}_h, (\underline{u}_h, \xi_h)) = 0 & \forall \underline{\tau}_h \in \Sigma_h, \\ B(\underline{\sigma}_h, (\underline{v}_h, \alpha_h)) = F(\underline{v}_h, \alpha_h) & \forall (\underline{v}_h, \alpha_h) \in M_h. \end{cases} \tag{22}$$

To get appropriate error estimates, we need to show that the uniform discrete inf-sup condition holds, as well as uniform coerciveness on the discrete kernel of B. For these purposes, we will use the BDM_1 interpolation operator I_h defined as follows (see [10,28,1]): for any $\delta \in (0, 1)$,

$$I_h : (H^\delta(\Omega))^{2\times 2} \cap (H(\text{div}; \Omega))^2 \to H_h : \tau \to I_h(\tau),$$

where $I_h(\tau) \in H_h$ is uniquely determined by the conditions

$$\int_{\partial K} I_h(\tau) n \cdot p_1 \mathrm{d}s = \int_{\partial K} \tau n \cdot p_1 \mathrm{d}s, \forall p_1 \in (\mathcal{R}_1(\partial K))^2, \forall K \in \mathcal{T}_h,$$

where

$$R_1(\partial K) = \{\psi \in L^2(\partial K) : \psi_{|E} \in \mathbb{P}_1(E), \forall E \in \mathcal{E}_K\}$$

and

$$H_h = \{\tau_h \in (H(\text{div}; \Omega))^2 : \tau_{h|K} \in BDM_1(K)^2 = (\mathbb{P}_1(K))^{2\times 2}, \forall K \in \mathcal{T}_h\}.$$

If P_h denotes the orthogonal projection from $(L^2(\Omega))^2$ to L^2_h, then we recall that the following diagram commutes [10]:

$$\begin{array}{ccccc} (H^\delta(\Omega))^{2\times 2} & \cap & (H(\text{div}; \Omega))^2 & \xrightarrow{\text{div}} & (L^2(\Omega))^2 \\ I_h & \downarrow & & & \downarrow \quad P_h \\ & H_h & & \xrightarrow{\text{div}} & L^2_h \end{array}$$

Consequently

$$\operatorname{div} I_h(\tau) = P_h(\operatorname{div}\tau), \quad \forall \tau \in (H^{\delta}(\Omega))^{2\times 2} \cap (H(\operatorname{div};\Omega))^2.$$

In addition, the following approximation property holds (see Theorems 3.2 and 3.3 of [1])

$$\|\tau - I_h(\tau)\| \lesssim h^{\delta}\|\tau\|_{(H^{\delta}(\Omega))^{2\times 2}} + h\|\operatorname{div}\tau\|, \quad \forall \tau \in (H^{\delta}(\Omega))^{2\times 2} \cap (H(\operatorname{div};\Omega))^2. \tag{23}$$

To prove the uniform discrete inf-sup condition, we start with three preliminary lemmas. As in the continuous case, in the proof of these three lemmas, we fix $\lambda^* > 0$ and $\mu^* > 0$ independently of λ and μ.

Lemma 3.1. *Let $v_h \in L_h^2$. Then there exists $\tau_h^1 \in H_h$ such that $\operatorname{div}\tau_h^1 = v_h$ in Ω, $\tau_h^1 n = 0$ on Γ_N and*

$$\|\tau_h^1\|_{(H(\operatorname{div};\Omega))^2} \lesssim \|v_h\|. \tag{24}$$

Proof. Let $w \in (H^1(\Omega))^2$ be the unique solution of the problem

$$\begin{cases} \operatorname{div}(2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta) = v_h & \text{in } \Omega,\\ w = 0 & \text{on } \Gamma_D,\\ (2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta)n = 0 & \text{on } \Gamma_N. \end{cases}$$

By the elliptic regularity of the Lamé system (see for instance [22]), there exists $\delta_0 \in (0,1)$ such that $w \in (H^{1+\delta}(\Omega))^2$, for all $\delta \in (0,\delta_0)$ and satisfies

$$\|w\|_{(H^{1+\delta}(\Omega))^2} \lesssim \|v_h\|. \tag{25}$$

We now fix $\delta \in (0,\delta_0)$, $\delta \neq \frac{1}{2}$ and set $\tau^1 = 2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta$. As $\operatorname{div}\tau^1 = v_h \in (L^2(\Omega))^2$, we deduce that $\tau^1 \in (H^{\delta}(\Omega))^{2\times 2} \cap (H(\operatorname{div};\Omega))^2$. Therefore, we may set $\tau_h^1 = I_h\tau^1$, which then belongs to H_h.

By the above commuting diagram we further have

$$\operatorname{div}(\tau_h^1) = P_h(\operatorname{div}\tau^1) = P_h(v_h) = v_h = \operatorname{div}\tau^1.$$

By the triangular inequality we have

$$\|I_h(\tau^1)\|_{(H(\operatorname{div};\Omega))^2} \le \|I_h(\tau^1) - \tau^1\|_{(H(\operatorname{div};\Omega))^2} + \|\tau^1\|_{(H(\operatorname{div};\Omega))^2}. \tag{26}$$

Since $\operatorname{div}(I_h(\tau^1) - \tau^1) = 0$, we can write

$$\|I_h(\tau^1) - \tau^1\|_{(H(\operatorname{div};\Omega))^2} = \|I_h(\tau^1) - \tau^1\| \lesssim h^{\delta}\|\tau^1\|_{(H^{\delta}(\Omega))^{2\times 2}} + h\|\operatorname{div}\tau^1\|,$$

owing to (23). This estimate in (26) yields

$$\|I_h(\tau^1)\|_{(H(\operatorname{div};\Omega))^2} \lesssim h^{\delta}\|\tau^1\|_{(H^{\delta}(\Omega))^{2\times 2}} + \|\tau^1\|_{(H(\operatorname{div};\Omega))^2}. \tag{27}$$

But, owing to (25), we have

$$\|\tau^1\|_{(H^{\delta}(\Omega))^{2\times 2}} \lesssim \|w\|_{(H^{1+\delta}(\Omega))^2} \lesssim \|v_h\|. \tag{28}$$

On the other hand, due to Korn's inequality, we have

$$\|\tau^1\|_{(H(\operatorname{div};\Omega))^2} \lesssim \|w\|_{H^1(\Omega)^2} + \|v_h\| \lesssim \|v_h\|. \tag{29}$$

Therefore using (27)–(29), we get (24).

Besides, by the definition of I_h

$$\int_E I_h(\tau^1)n \cdot p_1\,\mathrm{d}s = \int_E \tau^1 n \cdot p_1\,\mathrm{d}s, \quad \forall p_1 \in (\mathbb{P}_1(E))^2, \forall E \in \mathcal{E}_K.$$

Taking $E \subset \Gamma_N$ and recalling that $\tau^1 n = 0$ in Γ_N, we deduce that

$$\tau_h^1 n = I_h(\tau^1)n = 0 \quad \text{on } \Gamma_N. \quad \blacksquare$$

Lemma 3.2. *Let $\alpha_h \in (L^2_{h,\Gamma_N})^2$. Then there exists $\tau_h^2 \in H_h$ such that $\operatorname{div}\tau_h^2 = 0$ in Ω, $\tau_h^2 n = \alpha_h$ on Γ_N and*

$$\|\tau_h^2\|_{(H(\mathrm{div};\Omega))^2} \lesssim \|\alpha_h\|_{\Gamma_N}. \tag{30}$$

Proof. Consider $w \in (H^1(\Omega))^2$ the unique solution of

$$\begin{cases} \operatorname{div}(2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta) = 0 & \text{in } \Omega, \\ w = 0 & \text{on } \Gamma_D, \\ (2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta)n = \alpha_h & \text{on } \Gamma_N. \end{cases}$$

As usual, we set $\tau^2 = 2\mu^*\varepsilon(w) + \lambda^* \operatorname{div} w\,\delta$. Since $\alpha_h \in (L^2_{h,\Gamma_N})^2 \subset (L^2(\Gamma_N))^2$, by the elliptic regularity of the Lamé system [22], there exists $\delta_0 \in (0, 1)$ such that that $w \in (H^{1+\delta}(\Omega))^2$, for all $\delta \in (0, \delta_0)$ and satisfies

$$\|w\|_{(H^{1+\delta}(\Omega))^2} \lesssim \|\alpha_h\|_{\Gamma_N}.$$

By the definition of τ^2, we get

$$\|\tau^2\|_{(H^\delta(\Omega))^{2\times 2}} \lesssim \|w\|_{(H^{1+\delta}(\Omega))^2} \lesssim \|\alpha_h\|_{\Gamma_N}. \tag{31}$$

As $\operatorname{div}\tau^2 = 0$, we conclude that

$$\|\tau^2\|_{(H(\mathrm{div};\Omega))^2} = \|\tau^2\| \lesssim \|w\|_{(H^{1+\delta}(\Omega))^2} \lesssim \|\alpha_h\|_{\Gamma_N}. \tag{32}$$

We can now take $\tau_h^2 = I_h(\tau^2)$. Then from the commuting diagram we have

$$\operatorname{div}\tau_h^2 = \operatorname{div} I_h(\tau^2) = P_h(\operatorname{div}\tau^2) = 0. \tag{33}$$

Furthermore the definitions of I_h yield

$$\tau_h^2 n = \alpha_h \quad \text{on } \Gamma_N.$$

Therefore using (23), (32) and (33), we have:

$$\begin{aligned} \|\tau_h^2\|_{(H(\mathrm{div};\Omega))^2} &= \|\tau_h^2\| \le \|\tau_h^2 - \tau^2\| + \|\tau^2\| \\ &\lesssim h^\delta \|\tau^2\|_{(H^\delta(\Omega))^{2\times 2}} + \|\tau^2\| \lesssim (1 + h^\delta)\|\alpha_h\|_{\Gamma_N}. \quad \blacksquare \end{aligned}$$

Contrary to the continuous case, since $as(\tau_h^1+\tau_h^2)$ is not necessarily equal to zero (recall that $as(\tau^1) = as(\tau^2) = 0$, but this is not automatically the case for their interpolant), the construction of τ_h^3 is not independent of τ_h^1 and τ_h^2.

Lemma 3.3. *Let $((v_h, \theta_h), \alpha_h) \in M_h$. Then there exists $\tau_h^3 \in H_h$ such that $\operatorname{div}\tau_h^3 = 0$ in Ω, $\tau_h^3 n = 0$ on Γ_N, $as(\tau_h^3) = \theta_h - as(\tau_h^1 + \tau_h^2)$ and*

$$\|\tau_h^3\|_{(H(\mathrm{div};\Omega))^2} \lesssim \|v_h\| + \|\theta_h\| + \|\alpha_h\|_{\Gamma_N}. \tag{34}$$

Proof. Let us set $\gamma_h = \theta_h - as(\tau_h^1 + \tau_h^2)$, where τ_h^1 and τ_h^2 have been previously determined. Let us further set

$$X_h = \{v_h \in H^1(\Omega) : v_{h|K} \in \mathbb{P}_2(K) \oplus \mathbb{R} b_K\}.$$

Let us fix a nonempty open connected subset Γ_0 of Γ_D such that Γ_0 is included into one edge of Ω. Moreover by an eventual change of variables, we may suppose that the outward normal vector along Γ_0 is the vector $(0, -1)$. Fix further another nonempty open connected subset Γ_{00} such that $\bar{\Gamma}_{00} \subset \Gamma_0$. Fix a smooth function η defined on $\bar{\Omega}$ such that $0 \le \eta \le 1$ and such that $\eta = 1$ on an open neighborhood of $\overline{\Gamma \setminus \Gamma_0}$ and $\eta = 0$ on an open neighborhood of Γ_{00}.

Let us consider $\eta_h \in X_h$ as the $\mathbb{P}_1$ Lagrange interpolant of η, which then fulfils $0 \le \eta_h \le 1$ and $\eta_h = 1$ on $\Gamma \setminus \Gamma_0$ (note that $\Gamma_N \subset \Gamma \setminus \Gamma_0$ and consequently $\eta_h = 1$ on Γ_N).

We now fix the vector

$$c = \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbb{R}^2$$

such that

$$\int_{\Omega}(\gamma_h - c\cdot\nabla\eta_h)\mathrm{d}x = 0. \tag{35}$$

Indeed this condition is equivalent to

$$\int_{\Omega}\gamma_h \mathrm{d}x = \int_{\Gamma}\eta_h c\cdot n\,\mathrm{d}s. \tag{36}$$

Using the properties of η_h, we have

$$\int_{\Gamma}\eta_h c\cdot n\,\mathrm{d}s = \int_{\Gamma\setminus\Gamma_0}\eta_h c\cdot n\mathrm{d}s + \int_{\Gamma_0}\eta_h c\cdot n\mathrm{d}s$$
$$= \int_{\Gamma\setminus\Gamma_0} c\cdot n\mathrm{d}s + \int_{\Gamma_0}\eta_h c\cdot n\mathrm{d}s.$$

Moreover the property $\int_{\Omega}\operatorname{div} c\,\mathrm{d}x = 0$ and the Green's formula yield

$$\int_{\Gamma} c\cdot n\mathrm{d}s = 0,$$

and therefore

$$\int_{\Gamma\setminus\Gamma_0} c\cdot n\mathrm{d}s = -\int_{\Gamma_0} c\cdot n\mathrm{d}s.$$

These identities in (36) lead to the condition

$$\int_{\Gamma_0}(\eta_h - 1)c\cdot n\mathrm{d}s = \int_{\Omega}\gamma_h\,\mathrm{d}x.$$

By the choice of the normal vector along Γ_0, we get the condition

$$e = \frac{\int_{\Omega}\gamma_h \mathrm{d}x}{\int_{\Gamma_0}(1-\eta_h)\,\mathrm{d}s}. \tag{37}$$

Now as (35) means that $\gamma_h - c\cdot\nabla\eta_h$ is of zero mean on Ω, by Corollary I.2.4 of [19], there exists $r \in (H_0^1(\Omega))^2$ such that

$$\operatorname{div} r = \gamma_h - c\cdot\nabla\eta_h,$$
$$\|r\|_{1,\Omega} \lesssim \|\gamma_h - c\cdot\nabla\eta_h\|.$$

By the definition of e, this estimate becomes

$$\|r\|_{1,\Omega} \lesssim \|\gamma_h\|.$$

Using the definition of γ_h and the estimates (24) and (30), we finally obtain

$$\|r\|_{1,\Omega} \lesssim \|\theta_h\| + \|v_h\| + \|\alpha_h\|_{\Gamma_N}. \tag{38}$$

We now look for $\omega_h \in (X_h)^2$ such that

$$\begin{cases}(\operatorname{div}\omega_h, q_h) = (\gamma_h, q_h), & \forall q_h \in Q_h,\\ \omega_h = c & \text{on } \Gamma_N,\end{cases} \tag{39}$$

with the above vector c.

For that purpose, using the fact that the discretization of the Stokes problem by the pair $((X_h)^2 \cap H_0^1(\Omega)^2, Q_h \cap L_0^2(\Omega))$ is stable (Section II.2.2 of [19]) and making use of Fortin's lemma (Lemma II.1.1 of [19]), there exists $r_h \in (X_h)^2 \cap H_0^1(\Omega)^2$ such that (with the vector function r above)

$$\int_{\Omega}\operatorname{div}(r - r_h)q_h \mathrm{d}x = 0, \quad \forall q_h \in Q_h,$$
$$\|r_h\|_{1,\Omega} \lesssim \|r\|_{1,\Omega}.$$

The vector valued function ω_h defined by

$$\omega_h = r_h + \eta_h c$$

belongs to $(X_h)^2$ and satisfies, owing to the previous inequalities

$$\|\omega_h\|_{1,\Omega} \lesssim \|r_h\|_{1,\Omega} + |c| \lesssim \|\theta_h\| + \|v_h\| + \|\alpha_h\|_{\Gamma_N}. \tag{40}$$

Moreover for any $q_h \in Q_h$, one has

$$\int_\Omega \mathrm{div}\omega_h q_h \mathrm{d}x = \int_\Omega (\mathrm{div} r_h + c \cdot \nabla \eta_h) q_h \mathrm{d}x = \int_\Omega \gamma_h q_h \mathrm{d}x.$$

Besides

$$\omega_h = r_h + \eta_h c = 0 + c = c \quad \text{ on } \Gamma_N,$$

which shows that ω_h satisfies (39).

Setting

$$\tau_h^3 = \begin{pmatrix} \mathrm{curl}\ \omega_{h1} \\ \mathrm{curl}\ \omega_{h2} \end{pmatrix},$$

by (39), we remark that it fulfils (34) due to (40) as well as

$$as(\tau_h^3) = \mathrm{div}\,\omega_h = \gamma_h,$$
$$\tau_h^3 n = 0 \quad \text{ on } \Gamma_N. \quad \blacksquare$$

Theorem 3.4. *There exists a $\beta_3 > 0$ independent of h such that*

$$\sup_{\underline{\tau}_h \in \Sigma_h, \underline{\tau}_h \neq 0} \frac{B(\underline{\tau}_h, (\underline{v}_h, \alpha_h))}{\|\underline{\tau}_h\|_\Sigma} \geq \beta_3 \|(\underline{v}_h, \alpha_h)\|_{\tilde{M}}, \quad \forall (\underline{v}_h, \alpha_h) \in M_h.$$

Proof. With the notations from the three previous lemmas, we set

$$\tau_h^* = \tau_h^1 + \tau_h^2 + \tau_h^3,$$

and $\underline{\tau}_h^* = (\tau_h^*, 0) \in \Sigma_h$, which satisfies

$$B(\underline{\tau}_h^*, (\underline{v}_h, \alpha_h)) = \|v_h\|^2 + \|\theta_h\|^2 + \|\alpha_h\|^2_{\Gamma_N}. \tag{41}$$

Indeed by the above properties of $\tau_h^1, \tau_h^2, \tau_h^3$ stated in the three previous lemmas, we have

$$\mathrm{div}\underline{\tau}_h^* = v_h,$$
$$as(\tau_h^*) = as(\tau_h^1 + \tau_h^2) + as(\tau_h^3) = \theta_h,$$
$$\tau_h^* n = (\tau_h^1 + \tau_h^2 + \tau_h^3) n = \tau_h^3 n = \alpha_h \quad \text{ on } \Gamma_N.$$

By the definition of B we obtain (41).

Finally the estimates (24), (30) and (34) lead to

$$\begin{aligned} \|\underline{\tau}_h^*\|_\Sigma &= \|\tau_h^*\|_{(H(\mathrm{div};\Omega))^2} \leq \|\tau_h^1\|_{(H(\mathrm{div};\Omega))^2} + \|\tau_h^2\|_{(H(\mathrm{div};\Omega))^2} + \|\tau_h^3\|_{(H(\mathrm{div};\Omega))^2} \\ &\lesssim \|\theta_h\| + \|v_h\| + \|\alpha_h\|_{\Gamma_N}. \end{aligned} \tag{42}$$

The inf-sup condition follows from the identity (41) and the estimate (42). $\blacksquare$

Lemma 3.5. *The bilinear form A is coercive uniformly with respect to λ on*

$$V_h = \left\{ \underline{\tau}_h \in \Sigma_h : B(\underline{\tau}_h, (\underline{v}_h, \alpha_h)) = 0, \forall (\underline{v}_h, \alpha_h) \in M_h \right\}.$$

In other words

$$A(\underline{\tau}_h, \underline{\tau}_h) \gtrsim \|\tau_h\|^2 + \|q_h\|^2, \quad \forall \underline{\tau}_h = (\tau_h, q_h) \in V_h.$$

Proof. As in Theorem 2.6, taking $(\underline{v}_h, \alpha_h) \in M_h$ in the form $((v_h, 0), 0)$ with arbitrary $v_h \in L^2_h$, we see that $\underline{\tau}_h \in V_h$ satisfies

$$(\operatorname{div}(\tau_h - q_h\delta), v_h) = 0, \quad \forall v_h \in L^2_h.$$

As $\operatorname{div}(\tau_h - q_h\delta)$ is piecewise constant, we deduce that $\tau_h - q_h\delta$ satisfies $\operatorname{div}(\tau_h - q_h\delta) = 0$. The rest of the proof now follows the proof of the continuous case (cf. Theorem 2.6). ■

This lemma and Theorem 3.4 guarantee the existence and uniqueness of a solution to problem (22).

4. Some regularity results

Let us decompose $\Gamma = \cup_{j=1}^{n_e} \overline{\Gamma}_j$, where each Γ_j is an open segment. Denote furthermore by S_j the common vertex between Γ_j and Γ_{j+1} (modulo n_e) and by ω_j the interior opening of Ω at S_j. We will distinguish three kinds of vertices, namely the set $\mathcal{S}_{DD}$ of Dirichlet–Dirichlet vertices, in the sense that S_j belongs to $\mathcal{S}_{DD}$ if and only if Γ_j and Γ_{j+1} are included into Γ_D; similarly S_j belongs to the Neumann–Neumann set $\mathcal{S}_{NN}$ if and only if Γ_j and Γ_{j+1} are included into Γ_N; and finally S_j belongs to the Dirichlet–Neumann set $\mathcal{S}_{DN}$ if and only if either Γ_j is included in Γ_D and Γ_{j+1} is included into Γ_N, or the converse. Later on, we will denote by (r_j, θ_j) the polar coordinates centred at the vertex S_j.

It is well known (see [21] or [20,12,22]) that the weak solution of problem (2) presents vertex singularities. To describe them, we need to introduce the following notations: with each vertex S_j, we associate the following characteristic equation:

$$\begin{cases} \sin^2(\alpha\omega_j) = \left(\dfrac{\lambda+\mu}{\lambda+3\mu}\right)^2 \alpha^2 \sin^2\omega_j & \text{if } S_j \in \mathcal{S}_{DD}, \\ \sin^2(\alpha\omega_j) = \alpha^2 \sin^2\omega_j & \text{if } S_j \in \mathcal{S}_{NN}, \\ \sin^2(\alpha\omega_j) = \dfrac{(\lambda+2\mu)^2 - (\lambda+\mu)^2\alpha^2\sin^2\omega_j}{(\lambda+\mu)(\lambda+3\mu)} & \text{if } S_j \in \mathcal{S}_{DN}. \end{cases} \tag{43}$$

Denote by Λ_j the set of complex roots of this equation. We denote by $\nu(\alpha)$ the multiplicity of $\alpha \in \Lambda_j$; it is well known that it is either 1 or 2.

The next result was shown in [21]:

Theorem 4.1. *Assume that the characteristic equation (43) has no root on the vertical line $\Re\alpha = 1$ (except $\alpha = 1$ in the NN-case), that $f \in (L^2(\Omega))^2$ and that $g \in (H^{\frac{1}{2}}(\Gamma_N))^2$. Then the weak solution u of problem (2) admits the following decomposition*

$$u = u_R + \sum_{j=1}^{n_e} \sum_{\alpha\in\Lambda_j:\Re\alpha\in]0,1[} r_j^\alpha \sum_{k=0}^{\nu(\alpha)-1} c_{j,\alpha,k} (\ln r_j)^k \varphi_{j,\alpha,k}(\theta_j), \tag{44}$$

where u_R belonging to $(H^2(\Omega))^2$ is the regular part of u, $c_{j,\alpha,k} \in \mathbb{C}$ is a so-called coefficient of singularity and $\varphi_{j,\alpha,k}$ is a smooth function (explicitly known, cf. [21]). Moreover the following estimate holds

$$\|u_R\|_{2,\Omega} + \sum_{j=1}^{n_e} \sum_{\alpha\in\Lambda_j:\Re\alpha\in]0,1[} \sum_{k=0}^{\nu(\alpha)-1} |c_{j,\alpha,k}| \lesssim \|f\| + \|g\|_{(H^{\frac{1}{2}}(\Gamma_N))^2}. \tag{45}$$

The above decomposition allows us to show that u belongs to appropriated weighted Sobolev spaces that we next define.

Definition 4.2. For any scalar function $\phi \in C^0(\overline{\Omega})$ such that $\phi(x) > 0\ \forall x \in \overline{\Omega} \setminus \{S_1, \ldots, S_{n_e}\}$, and any $m, k \in \mathbb{N}$, we define

$$H^{m,k}_\phi(\Omega) = \{v \in H^m(\Omega) : \phi D^\beta v \in L^2(\Omega), \forall \beta \in \mathbb{N}^2 : m < |\beta| \le m+k\}.$$

$H^{m,k}_{\phi}(\Omega)$ is a Hilbert space with the norm

$$\|v\|_{m,k;\phi,\Omega} = \left(\|v\|^2_{m,\Omega} + \sum_{m<|\beta|\le m+k} \|\phi D^{\beta} v\|^2\right)^{\frac{1}{2}}.$$

We also define the semi-norm:

$$|v|_{m,k;\phi,\Omega} = \left(\sum_{|\beta|=m+k} \|\phi D^{\beta} v\|^2\right)^{\frac{1}{2}}.$$

For all $j \in \{1, 2, \dots, n_e\}$, we now fix a non-negative real number $\alpha_j < 1$ such that

$$\alpha_j > 1 - \Re\alpha, \quad \forall \alpha \in \Lambda_j : \Re\alpha \in]0, 1[.$$

Corollary 4.3. *Let the assumptions of Theorem 4.1 be satisfied. Let us fix $\phi \in C^0(\overline{\Omega})$ to be as in Definition 4.2 and such that $\phi = r_j^{\alpha_j}$ in a neighborhood of the vertex S_j for every $j = 1, 2, \dots, n_e$. Then $u \in (H^{1,1}_{\phi}(\Omega))^2$ and consequently $\sigma = 2\mu\varepsilon(\mu) \in (H^{0,1}_{\phi}(\Omega))^{2\times 2}$, $p = -\lambda \mathrm{div} u \in H^{0,1}_{\phi}(\Omega)$ and $\omega = \frac{1}{2}\mathrm{curl}\ u \in H^{0,1}_{\phi}(\Omega)$. Moreover one has*

$$\|u\|_{1,1;\phi,\Omega} \lesssim \|f\| + \|g\|_{(H^{\frac{1}{2}}(\Gamma_N))^2}. \tag{46}$$

Proof. It suffices to check that each singular function $r_j^{\alpha}(\ln r_j)^k \varphi_{j,\alpha,k}(\theta_j)$ belongs to $(H^{1,1}_{\phi}(\Omega))^2$, and to use the estimate (45). ■

For further purposes, we need to give a meaning to the traces of functions in $H^{0,1}_{\phi}(\Omega)$; namely we show the

Lemma 4.4. *Let ϕ be a function like in Corollary 4.3. If $w \in H^{0,1}_{\phi}(\Omega)$, then for all triangles $K \in \mathcal{T}_h$, it holds that*

$$w_{|E} \in L^1(E), \quad \forall E \in \mathcal{E}_K.$$

Proof. By Lemma 2.5 of [13], there exists a $p > 1$ such that the next continuous embedding holds

$$H^{0,1}_{\phi}(\Omega) \hookrightarrow W^{1,p}(\Omega).$$

A standard trace theorem yields

$$w_{|E} \in W^{1-1/p,p}(E),$$

and the conclusion follows. ■

5. Error estimates

In this section, we take advantage of the previous results and some interpolation error estimates to obtain convergence results.

We first introduce a kind of Fortin operator (compare with Proposition 4.4 of [16]):

Proposition 5.1. *Let ϕ be a function like in Corollary 4.3. Then there exists an operator*

$$\begin{array}{llll} \Pi_h : & \Sigma \cap [(H^{0,1}_{\phi}(\Omega))^{2\times 2} \times H^{0,1}_{\phi}(\Omega)] & \longrightarrow & \Sigma_h \\ & \underline{\tau} = (\tau, q) & \longrightarrow & \Pi_h \tau = (\tau_h, q_h) \end{array}$$

such that

$$B(\underline{\tau} - \Pi_h \underline{\tau}, (\underline{v}_h, \alpha_h)) = 0, \quad \forall (\underline{v}_h, \alpha_h) \in M_h. \tag{47}$$

Proof. Let us fix $(\tau, q) \in \Sigma \cap (H_\phi^{0,1}(\Omega))^{2\times 2} \times H_\phi^{0,1}(\Omega)$. We first take $q_h = P_h^1 q$, where P_h^1 is the L^2-orthogonal projection from $L^2(\Omega)$ onto Q_h. We secondly define τ_h^* such that for all $K \in \mathcal{T}_h : \tau_{h|K}^* \in (P_1(K))^{2\times 2}$ and is uniquely determined by the condition:

$$\int_{\partial K} [(\tau_{h|K}^* - q_h\delta) - (\tau - q\delta)]n \cdot p_1 \, \mathrm{d}s = 0, \quad \forall p_1 \in (\mathcal{R}_1(\partial K))^2. \tag{48}$$

We can observe that the term on the left-hand side of (48) is meaningful, since $(\tau - q\delta)_{|K} \in (H_\phi^{0,1}(K))^{2\times 2}$, and by Lemma 4.4, $(\tau - q\delta)_{|E} \in (L^1(E))^2$ for all edges E of K.

Let us set $\gamma = as(\tau - \tau_h^*)$. Since γ is not necessarily of zero mean (contrary to the one in the proof of Proposition 4.4 of [16]), as in Theorem 3.4, we fix the vector

$$c = \begin{pmatrix} 0 \\ e \end{pmatrix} \in \mathbb{R}^2$$

such that

$$\int_\Omega (\gamma - c \cdot \nabla\eta_h) \mathrm{d}x = 0, \tag{49}$$

or equivalently

$$e = \frac{\int_\Omega \gamma \, \mathrm{d}x}{\int_{\Gamma_0} (1 - \eta_h) \, \mathrm{d}s}.$$

Now as $\gamma - c \cdot \nabla\eta_h$ is of zero mean in Ω, by Corollary I.2.4 of [19], there exists $r \in (H_0^1(\Omega))^2$ such that

$$\mathrm{div} r = \gamma - c \cdot \nabla\eta_h,$$

with the estimate

$$\|r\|_{1,\Omega} \lesssim \|\gamma\| \leq \|\tau - \tau_h^*\|,$$

using the above expression of e.

As before, using the fact that the discretization of the Stokes problem by the pair $((X_h)^2 \cap H_0^1(\Omega)^2, Q_h)$ is stable and making use of Fortin's lemma, there exists $r_h \in (X_h)^2 \cap H_0^1(\Omega)^2$ such that (with the vector function r above)

$$\int_\Omega \mathrm{div}(r - r_h) q_h \mathrm{d}x = 0, \quad \forall q_h \in Q_h,$$
$$\|r_h\|_{1,\Omega} \lesssim \|r\|_{1,\Omega}.$$

The function ω_h defined by

$$\omega_h = r_h + \eta_h c$$

belongs to $(X_h)^2$ and satisfies, owing to the properties of r:

$$\|\omega_h\|_{1,\Omega} \lesssim \|r_h\|_{1,\Omega} + |c| \lesssim \|\gamma\| \lesssim \|\tau - \tau_h^*\|. \tag{50}$$

Moreover for any $q_h \in Q_h$, one has

$$\int_\Omega \mathrm{div}\omega_h q_h \mathrm{d}x = \int_\Omega (\mathrm{div} r_h + c \cdot \nabla\eta_h) q_h \mathrm{d}x = \int_\Omega \gamma q_h \mathrm{d}x. \tag{51}$$

Besides, we clearly have

$$\omega_h = r_h + \eta_h c = 0 + c = c \quad \text{on } \Gamma_N. \tag{52}$$

We finally set $\Pi_h(\tau, q) = (\tau_h, q_h)$, where

$$\tau_h = \tau_h^* + \begin{bmatrix} \mathrm{curl}\ \omega_{h1} \\ \mathrm{curl}\ \omega_{h2} \end{bmatrix}.$$

Clearly $\Pi_h(\tau, q)$ belongs to Σ_h and satisfies

$$\begin{aligned}
&\operatorname{div}(\tau_h - q_h\delta) = \operatorname{div}(\tau_h^* - q_h\delta),\\
&as(\tau_h) = as(\tau_h^*) + \operatorname{div}\omega_h,\\
&\tau_h n = \tau_h^* n \quad \text{on } \Gamma_N.
\end{aligned}$$

From these properties we get

$$\begin{aligned}
B(\underline{\tau} - \Pi_h\underline{\tau}, (\underline{v}_h, \alpha_h)) = {}& (\operatorname{div}(\tau - \tau_h^* - (q - q_h)\delta), v_h) + (as(\tau - \tau_h^*) - \operatorname{div}\omega_h, \theta_h)\\
&+ \langle(\tau - \tau_h^* - (q - q_h)\delta)n, \alpha_h\rangle_{\Gamma_N}.
\end{aligned}$$

But Green's formula and the definition of τ_h^* yield

$$\begin{aligned}
(\operatorname{div}(\tau - \tau_h^* - (q - q_h)\delta), v_h) &= \sum_{K\in\mathcal{T}_h} \int_K \operatorname{div}(\tau - \tau_h^* - (q - q_h)\delta)\cdot v_h \,\mathrm{d}x\\
&= \sum_{K\in\mathcal{T}_h} \int_{\partial K} (\tau - \tau_h^* - (q - q_h)\delta)n\cdot v_h \,\mathrm{d}x = 0.
\end{aligned}$$

On the other hand, by (51), we have

$$(as(\tau - \tau_h^*) - \operatorname{div}\omega_h, \theta_h) = (\gamma - \operatorname{div}\omega_h, \theta_h) = 0.$$

Finally we may write

$$\begin{aligned}
\langle(\tau - q\delta - \tau_h^* + q_h\delta)n, \alpha_h\rangle_{\Gamma_N} &= \int_{\Gamma_N} (\tau - q\delta - \tau_h^* + q_h\delta)n\alpha_h \mathrm{d}s\\
&= \sum_{K\in\mathcal{T}_h:\partial K\cap\Gamma_N\neq\emptyset}\ \sum_{E\in\mathcal{E}_K\cap\bar{\Gamma}_N} \int_E ((\tau - \tau_h^*) - (q - q_h)\delta)n\cdot\alpha_h \,\mathrm{d}s = 0,
\end{aligned}$$

this last identity coming from (48) and the fact that α_h is in $(\mathbb{P}_1(E))^2$, for any $E \in \mathcal{E}_K \cap \bar{\Gamma}_N$.

The above identities lead to the conclusion. ■

Corollary 5.2. *Under the assumptions of the previous proposition, we have*

$$\|\underline{\tau} - \Pi_h\underline{\tau}\| \lesssim \|(\tau - q\delta) - (\tau_h^* - q_h\delta)\| + \|q - q_h\|, \tag{53}$$

where τ_h^* is defined by (48) and $q_h = P_h^1 q$.

Proof. By the construction of Π_h and (50), we may write, keeping the notations from the proof of Proposition 5.1,

$$\begin{aligned}
\|\underline{\tau} - \Pi_h\underline{\tau}\| &\le \|\tau - \tau_h\| + \|q - q_h\|\\
&\le \|\tau - \tau_h^*\| + \|\omega_h\|_{1,\Omega} + \|q - q_h\|\\
&\lesssim \|\tau - \tau_h^*\| + \|q - q_h\|,
\end{aligned}$$

by the estimate (50). The conclusion follows from the triangular inequality. ■

We now need to define local weighted Sobolev spaces:

Definition 5.3. Let K be an arbitrary triangle in the plane and A a vertex of K. For $m = 0$ or 1 and $\beta \in [0, 1[$, we will denote

$$H_A^{m,1;\beta}(K) = \{\psi \in H^m(K);\ |x - A|^\beta D^\alpha\psi \in L^2(K) \quad \forall\alpha \in \mathbb{N}^2 : |\alpha| = m + 1\},$$

equipped with the norm

$$\|\psi\|_{m,1;\beta,K} = (\|\psi\|_{m,K}^2 + |\psi|_{m,1;\beta,K}^2)^{\frac{1}{2}}$$

and semi-norm

$$|\psi|_{m,1;\beta,K} = \left(\sum_{|\alpha|=m+1} \| |x-A|^{\beta} D^{\alpha}\psi \|_K^2 \right)^{\frac{1}{2}}.$$

By Lemma 4.4, the trace of an element of $H_A^{0,1;\beta}(K)$ with $\beta \in [0,1[$ is well-defined and is in $L^1(\partial K)$. Thus, given $v \in [H_A^{0,1;\beta}(K)]^2$, its Brezzi–Douglas–Marini interpolant $\rho_K v \in BDM_1(K) = (\mathbb{P}_1(K))^2$ [10, p. 125] is well defined by the relations:

$$\int_{\partial K} \rho_K v \cdot n p_1 \mathrm{d}s = \int_{\partial K} v \cdot n p_1 \, \mathrm{d}s, \quad \forall p_1 \in \mathcal{R}_1(\partial K).$$

Using the so-called Piola transformation and Bramble–Hilbert arguments, Farhloul and Paquet have shown in Proposition 4.12 from [16] the next result:

Lemma 5.4. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. For any $\beta \in [0,1[$, and every $K \in \mathcal{T}_h$, it holds that*

$$\|v - \rho_K v\|_K \lesssim h_K^{1-\beta} |v|_{0,1;\beta,K}, \quad \forall v \in (H_A^{0,1;\beta}(K))^2.$$

Direct consequences of this lemma are the next global interpolation error estimates under appropriate refinement conditions on the regular family of triangulations $(\mathcal{T}_h)_{h>0}$ (see Theorem 4.13 and its Corollary in [16]):

Theorem 5.5. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$. We suppose that $(\mathcal{T}_h)_{h>0}$ satisfies the two following refinement rules:*

1. *If K is a triangle of $\mathcal{T}_h$ admitting S_j as a vertex, then*

$$h_K \lesssim h^{\frac{1}{1-\alpha_j}}, \tag{54}$$

where α_j has been defined in Section 4.

2. *if K is a triangle of $\mathcal{T}_h$ admitting no S_j $(j = 1, \ldots, n_e)$ as a vertex, then*

$$h_K \lesssim h \inf_{x \in K} \phi(x), \tag{55}$$

where ϕ is a function like in Corollary 4.3.

Then for every vector field $v \in (H_\phi^{0,1}(\Omega))^2$, it Holds that

$$\|v - \rho_h v\| \lesssim h |v|_{0,1;\phi,\Omega}, \tag{56}$$

where $\rho_h v$ denotes the BDM_1 interpolant of v, i.e., for all $K \in \mathcal{T}_h$, $(\rho_h v)_{|K} = \rho_K v$.

Similarly for every $q \in H_\phi^{0,1}(\Omega)$, it holds that

$$\|q - P_h^1 q\| \lesssim h |q|_{0,1;\phi,\Omega}, \tag{57}$$

where we recall that P_h^1 denotes the L^2-orthogonal projection on Q_h.

Corollary 5.6. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (54) and (55). Then for every $\underline{\tau} = (\tau, q) \in (H_\phi^{0,1}(\Omega))^{2\times 2} \times H_\phi^{0,1}(\Omega)$*

$$\|\underline{\tau} - \Pi_h \underline{\tau}\| \lesssim h(|\tau|_{0,1;\phi,\Omega} + |q|_{0,1;\phi,\Omega}). \tag{58}$$

Proof. We simply notice that the definition of τ_h^* in Proposition 5.1 means that each line of $\tau_h^* - q_h\delta$ is the BDM_1-interpolant of the corresponding line of $\tau - q\delta$. By the estimate (56) we then obtain

$$\|\tau - q\delta - (\tau_h^* - q_h\delta)\| \lesssim h|\tau - q\delta|_{0,1;\phi,\Omega} \lesssim h(|\tau|_{0,1;\phi,\Omega} + |q|_{0,1;\phi,\Omega}).$$

This estimate and (57) in (53) lead to the conclusion. ■

Remark 5.7. Regular families of meshes satisfying the refinement conditions (54) and (55) are easily built; see for instance [26].

We end up with a local interpolation error estimate on the partition of Γ_N.

Lemma 5.8. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (54) and (55). For $v \in H^{1,1}_{\phi}(\Omega) \cap H^1_{0,\Gamma_D}(\Omega)$, denote by $L_h v$ its $\mathbb{P}_1$-Lagrange interpolant, in the sense that $L_h v$ is the unique continuous function in L^2_{h,Γ_N} such that $L_h v(x) = v(x)$, for all nodal points $x \in \bar{\Gamma}_N$ (which is meaningful). Then for all triangle $K \in \mathcal{T}_h$ having an edge E included into $\bar{\Gamma}_N$, it holds that*

$$\|v - L_h v\|_E \lesssim h_K^{1/2} h |v|_{1,1;\phi,K}. \tag{59}$$

In particular, we clearly have

$$\|v - L_h v\|_{\Gamma_N} \lesssim h|v|_{1,1;\phi,\Omega}. \tag{60}$$

Proof. By Corollary 2.6 of [13], for any $\beta \in [0, 1[$, the space $H^{1,1;\beta}_A(K)$ is continuously embedded into $C(K)$ and compactly embedded into $H^1(K)$. The first property implies that $L_h v$ is meaningful. Standard scaling arguments and the two embeddings further lead to the estimate

$$\|v - L_h v\|_E \lesssim h_K^{\frac{3}{2}-\beta} |v|_{1,1;\beta,K}.$$

For a triangle K having some S_j as a vertex, we apply this estimate with $\beta = \alpha_j$ and by the first refinement rule (54), we arrive at the estimate (59). For a triangle K having no vertex S_j as vertex, we apply the above estimate with $\beta = 0$ and by the second refinement rule (55), we still arrive at the estimate (59). ■

Theorem 5.9. *Let $(\mathcal{T}_h)_{h>0}$ be a regular family of triangulations of $\overline{\Omega}$ satisfying the refinement conditions (54) and (55). Let $((\sigma, p), ((u, \omega), \xi))$ be the unique solution of problem (4) and let $((\sigma_h, p_h), ((u_h, \omega_h), \xi_h))$ be the unique solution of problem (22). We suppose that $f \in (L^2(\Omega))^2$, $g \in (H^{\frac{1}{2}}(\Gamma_N))^2$ and that the characteristic equation (43) (cf. Theorem 4.1) has no root on the vertical line $\Re(\alpha) = 1$ for each $j = 1, 2, \ldots, n_e$ (except $\alpha = 1$ if $S_j \in \mathcal{S}_{NN}$). Then the following error estimates hold:*

$$\|\underline{\sigma} - \underline{\sigma}_h\| \lesssim \left(1 + \frac{1}{\lambda}\right) h\, (\|f\| + \|g\|_{(H^{\frac{1}{2}}(\Gamma_N))^2}), \tag{61}$$

$$\|u - u_h\| + \|\omega - \omega_h\| + \|\xi - \xi_h\|_{\Gamma_N} \lesssim \left(1 + \frac{1}{\lambda}\right)^2 h\, (\|f\| + \|g\|_{(H^{\frac{1}{2}}(\Gamma_N))^2}). \tag{62}$$

Proof. From (4) and (22) we have

$$A(\underline{\sigma} - \underline{\sigma}_h, \underline{\tau}_h) + B(\underline{\tau}_h, (\underline{u} - \underline{u}_h, \xi - \xi_h)) = 0, \quad \forall \underline{\tau}_h \in \Sigma_h, \tag{63}$$

$$B(\underline{\sigma} - \underline{\sigma}_h, (\underline{v}_h, \alpha_h)) = 0, \quad \forall (\underline{v}_h, \alpha_h) \in M_h. \tag{64}$$

We recall that, in these relations,

$$\underline{\sigma} = (\sigma, p), \qquad \underline{u} = (u, \omega), \qquad \underline{\sigma}_h = (\sigma_h, p_h), \qquad \underline{u}_h = (u_h, \omega_h).$$

Let us set $\Pi_h \underline{\sigma} = (\sigma_h^*, p_h^*)$. Taking $\underline{\tau}_h = \Pi_h \underline{\sigma} - \underline{\sigma}_h$ in (63), we have

$$A(\underline{\sigma} - \underline{\sigma}_h, \Pi_h \underline{\sigma} - \underline{\sigma}_h) + (\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - u_h) + (as(\sigma_h^* - \sigma_h), \omega - \omega_h)$$
$$+ \langle(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)n, \xi - \xi_h\rangle_{\Gamma_N} = 0.$$

Introducing $(P_h^0 u, P_h^1 \omega, L_h \xi)$ in this last equation, where P_h^0 is the standard L^2-orthogonal projection on L_h^2, we get

$$A(\underline{\sigma} - \underline{\sigma}_h, \Pi_h \underline{\sigma} - \underline{\sigma}_h) + (\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - P_h^0 u)$$
$$+ (\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), P_h^0 u - u_h) + (as(\sigma_h^* - \sigma_h), \omega - P_h^1 \omega)$$

$$+ (as(\sigma_h^* - \sigma_h), P_h^1\omega - \omega_h) + \langle(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}$$
$$+ \langle(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)n, L_h\xi - \xi_h\rangle_{\Gamma_N} = 0. \tag{65}$$

Since $\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)$ is a constant vector on each triangle K, and from the definition of $P_h^0 u$, we deduce that

$$(\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), u - P_h^0 u) = 0.$$

From (47) and (64), we have

$$(\mathrm{div}(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta), P_h^0 u - u_h) + (as((\sigma_h^* - \sigma_h), P_h^1\omega - \omega_h))$$
$$+ \langle(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)n, L_h\xi - \xi_h\rangle_{\Gamma_N} = 0.$$

By these two identities, (65) becomes

$$A(\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) + (as(\sigma_h^* - \sigma_h), \omega - P_h^1\omega) + \langle(\sigma_h^* - \sigma_h - (p_h^* - p_h)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N} = 0,$$

which yields

$$A(\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) = (as(\sigma_h - \sigma_h^*), \omega - P_h^1\omega) + \langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}. \tag{66}$$

Since

$$\begin{aligned} A(\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) &= A(\underline{\sigma} - \underline{\sigma}_h - (\Pi_h\underline{\sigma} - \underline{\sigma}_h), \Pi_h\underline{\sigma} - \underline{\sigma}_h) + A(\Pi_h\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) \\ &= A(\underline{\sigma} - \Pi_h\underline{\sigma}, \Pi_h\underline{\sigma} - \underline{\sigma}_h) + A(\Pi_h\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h), \end{aligned}$$

by the identity (66) we obtain

$$\begin{aligned} A(\Pi_h\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h) &= A(\Pi_h\underline{\sigma} - \underline{\sigma}, \Pi_h\underline{\sigma} - \underline{\sigma}_h) + (as(\sigma_h - \sigma_h^*), \omega - P_h^1\omega) \\ &\quad + \langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N} \\ &= \frac{1}{2\mu}(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h) + \frac{1}{\lambda}(p_h^* - p, p_h^* - p_h) + (as(\sigma_h - \sigma_h^*), \omega - P_h^1\omega) \\ &\quad + \langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}. \end{aligned} \tag{67}$$

Owing to (47) and (64), we see that $\Pi_h\underline{\sigma} - \underline{\sigma}_h$ belongs to V_h and by Lemma 3.5, we get

$$\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|^2 \lesssim A(\Pi_h\underline{\sigma} - \underline{\sigma}_h, \Pi_h\underline{\sigma} - \underline{\sigma}_h). \tag{68}$$

We will now estimate the four terms of the right-hand side of (67). From the Cauchy–Schwarz's inequality, we have

$$|(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h)| \le \|\sigma_h^* - \sigma\|\,\|\sigma_h^* - \sigma_h\| \le \|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\,\|\Pi_h\underline{\sigma} - \underline{\sigma}\|,$$
$$|(p_h^* - p, p_h^* - p_h)| \le \|p_h^* - p\|\,\|p_h^* - p_h\| \le \|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\,\|\Pi_h\underline{\sigma} - \underline{\sigma}\|,$$
$$|(as(\sigma_h - \sigma_h^*), \omega - P_h^1\omega)| \le \|\sigma_h - \sigma_h^*\|\|\omega - P_h^1\omega\| \le \|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\|\omega - P_h^1\omega\|.$$

Using the interpolation error estimates (57) and (58), we obtain

$$|(\sigma_h^* - \sigma, \sigma_h^* - \sigma_h)| \lesssim h\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\,(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}), \tag{69}$$
$$|(p_h^* - p, p_h^* - p_h)| \lesssim h\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\,(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}), \tag{70}$$
$$|(as(\sigma_h - \sigma_h^*), \omega - P_h^1\omega)| \lesssim h\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|\,|u|_{1,1;\phi,\Omega}. \tag{71}$$

Similarly, we estimate

$$|\langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}| \le \sum_{\substack{K\in\mathcal{T}_h\\ \partial K\cap\bar{\Gamma}_N\neq\emptyset}} \sum_{E\subset\partial K\cap\bar{\Gamma}_N} \|(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n\|_E\|\xi - L_h\xi\|_E.$$

Now using a standard inverse inequality and the estimate (59), we get

$$|\langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}| \lesssim h \sum_{\substack{K\in\mathcal{T}_h\\ \partial K\cap\bar{\Gamma}_N\neq\emptyset}} \sum_{E\in\mathcal{E}_K\cap\bar{\Gamma}_N} \|\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta\|_K |u|_{1,1;\phi,K}.$$

By the discrete Cauchy–Schwarz's inequality, we obtain

$$|\langle(\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta)n, \xi - L_h\xi\rangle_{\Gamma_N}| \lesssim h|u|_{1,1;\phi,\Omega}\|\sigma_h - \sigma_h^* - (p_h - p_h^*)\delta\| \\ \lesssim h|u|_{1,1;\phi,\Omega}\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\|. \quad (72)$$

Then the identity (67) and the estimates (69)–(72) in the estimate (68) lead to

$$\|\Pi_h\underline{\sigma} - \underline{\sigma}_h\| \lesssim \left(1 + \frac{1}{\lambda}\right) h\,(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}).$$

Therefore, (61) follows from this estimate, (58) and the triangle inequality.

To prove (62), first observe that the uniform discrete inf-sup condition (see Theorem 3.4) yields

$$\|P_h^0 u - u_h\| + \|P_h^1\omega - \omega_h\| + \|L_h\xi - \xi_h\|_{\Gamma_N} \lesssim \sup_{\underline{\tau}_h\in\Sigma_h\setminus\{0\}} \frac{B(\underline{\tau}_h, ((P_h^0 u, P_h^1\omega), L_h\xi) - (\underline{u}_h, \xi_h))}{\|\underline{\tau}_h\|_\Sigma}. \quad (73)$$

Now owing to the Eq. (63) and the properties of P_h^0, we have

$$B(\underline{\tau}_h, ((P_h^0 u, P_h^1\omega), L_h\xi) - (\underline{u}_h, \xi_h)) = A(\underline{\sigma}_h - \underline{\sigma}, \underline{\tau}_h) + (as(\tau_h), P_h^1\omega - \omega) \\ + \langle(\tau_h - q_h\delta)n, L_h\xi - \xi\rangle_{\Gamma_N}, \quad \forall\underline{\tau}_h \in \Sigma_h. \quad (74)$$

Applying the Cauchy–Schwarz inequality and the estimate (57) and (61) (rather its proof), we may write

$$|A(\underline{\sigma}_h - \underline{\sigma}, \underline{\tau}_h)| \lesssim \left(1 + \frac{1}{\lambda}\right)^2 h(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega})\|\underline{\tau}_h\|_\Sigma, \quad (75)$$

$$|(as(\tau_h), P_h^1\omega - \omega)| \lesssim h\,|u|_{1,1;\phi,\Omega}\,\|\underline{\tau}_h\|_\Sigma. \quad (76)$$

Moreover by the arguments used to obtain (72), we have

$$\left|\langle(\tau_h - q_h\delta)n, L_h\xi - \xi\rangle_{\Gamma_N}\right| \lesssim h|u|_{1,1;\phi,\Omega}\|\underline{\tau}_h\|_\Sigma. \quad (77)$$

The estimates (75)–(77) and the identity (74) lead to

$$|B(\underline{\tau}_h, ((P_h^0 u, P_h^1 w), L_h\xi) - (\underline{u}_h, \xi_h))| \lesssim \left(1 + \frac{1}{\lambda}\right)^2 h\,(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega})\,\|\underline{\tau}_h\|_\Sigma.$$

This estimate and (73) show that

$$\|P_h^0 u - u_h\| + \|P_h^1\omega - \omega_h\| + \|L_h\xi - \xi_h\|_{\Gamma_N} \lesssim \left(1 + \frac{1}{\lambda}\right)^2 h\,(|u|_{1,1;\phi,\Omega} + |p|_{0,1;\phi,\Omega}). \quad (78)$$

Moreover by standard scaling arguments, it holds that

$$\|u - P_h^0 u\| \lesssim h|u|_{1,\Omega}. \quad (79)$$

Therefore, (62) follows from the estimates (57), (60), (79), (78) and the triangle inequality. ■

Remark 5.10. It follows from the second equation of (22) that $\operatorname{div}(\sigma_h - p_h\delta) = -P_h f$, and therefore

$$\operatorname{div}(\sigma - p\delta) - \operatorname{div}(\sigma_h - p_h\delta) = -(f - P_h f).$$

Hence if we suppose that f belongs to $H^1(\Omega)^2$, then the error between $\operatorname{div}(\sigma - p\delta)$ and $\operatorname{div}(\sigma_h - p_h\delta)$ is of order h, i.e.,

$$\|\operatorname{div}(\sigma - p\delta) - \operatorname{div}(\sigma_h - p_h\delta)\| \lesssim h.$$

6. Numerical experiments

For the implementation of the mixed problem (22), a so-called "hybrid formulation" [10,17,25] should be used. In this hybrid form, the continuity of the normal trace $(\sigma_h - p_h\delta)n$ across the inter-element edges of the triangulation is relaxed by using a Lagrange multiplier λ_h. The Lagrange multiplier λ_h is an approximation of the trace of the displacement field on the edges of the triangulation. This technique enables us to eliminate the approximations of σ, u, and ω at the element level, and leads to a linear system that involves only the Lagrange multiplier λ_h and p_h as degrees of freedom.

We first introduce the enlarged space $\widetilde{\Sigma}_h$ (with respect to Σ_h) by suppressing the requirement for its elements to have continuous normal components at the interfaces of the triangulation $\mathcal{T}_h$

$$\widetilde{\Sigma}_h := \{\tau_h \in (L^2(\Omega))^{2\times 2} : \tau_{h|K} \in (\mathbb{P}_1(K))^{2\times 2} \oplus (\mathbb{R}\ \mathrm{curl}\ b_K)^2, \forall K \in \mathcal{T}_h\}$$
$$\times \{q_h \in L^2(\Omega) : q_{h|K} \in \mathbb{P}_1(K), \forall K \in \mathcal{T}_h\},$$

and the space of Lagrangian multipliers:

$$\Lambda_h := \{\mu_h \in \cup_{e\in\mathcal{E}_h} (L^2(e))^2 : \mu_{h|e} \in [\mathbb{P}_1(e)]^2, \forall e \in \mathcal{E}_h \text{ and } \mu_{h|e} = 0, \forall e \subset \Gamma_D\},$$

where $\mathcal{E}_h$ is the set of all edges of the triangulation $\mathcal{T}_h$.

The hybrid formulation of the discrete problem (22) is the following one: Find $(\widetilde{\sigma}_h, \widetilde{p}_h, \lambda_h) \in \widetilde{\Sigma}_h \times \Lambda_h$ and $(\widetilde{u}_h, \widetilde{\omega}_h) \in L^2_h \times Q_h$ such that

$$\begin{cases} \dfrac{1}{2\mu}(\widetilde{\sigma}_h, \tau_h) + \dfrac{1}{\lambda}(\widetilde{p}_h, q_h) + \displaystyle\sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}(\tau_h - q_h\delta)\cdot \widetilde{u}_h\, \mathrm{d}x \\ +(as\ (\tau_h), \widetilde{\omega}_h) - \displaystyle\sum_{K\in\mathcal{T}_h} \int_{\partial K} \lambda_h(\tau_h - q_h\delta)n_K\, \mathrm{d}s = 0, \quad \forall(\tau_h, q_h) \in \widetilde{\Sigma}_h, \\ \displaystyle\sum_{K\in\mathcal{T}_h} \int_K \mathrm{div}(\widetilde{\sigma}_h - \widetilde{p}_h\delta)\cdot v_h\, \mathrm{d}x + (as\ (\widetilde{\sigma}_h), \theta_h) + (f, v_h) = 0, \quad \forall(v_h, \theta_h) \in L^2_h \times Q_h, \\ \displaystyle\sum_{K\in\mathcal{T}_h} \int_{\partial K} \mu_h\cdot(\widetilde{\sigma}_h - \widetilde{p}_h\delta)n_K\, \mathrm{d}s = \int_{\partial K\cap\Gamma_N} \mu_h\cdot g\, \mathrm{d}s, \quad \forall\mu_h \in \Lambda_h. \end{cases} \tag{80}$$

It is easily proved that $\widetilde{\sigma}_h = \sigma_h$, $\widetilde{p}_h = p_h$, $\widetilde{u}_h = u_h$, $\lambda_{h|\Gamma_N} = -\xi_h$ and $\widetilde{\omega}_h = \omega_h$ where $((\sigma_h, p_h), ((u_h, \omega_h), \xi_h))$ is the solution of the non-hybridized mixed formulation (22). Taking advantage of the fact that $\widetilde{\Sigma}_h$ is a product space, we can decouple the first equation of the system (80) and obtain

$$\begin{cases} \dfrac{1}{2\mu}\displaystyle\int_K \sigma_K : \tau_K\, \mathrm{d}x + |K|\mathrm{div}\ \tau_K\cdot u_K - \int_{\partial K} \lambda_{\partial K}\cdot\tau_K n_K\, \mathrm{d}s \\ +\displaystyle\int_K as\ (\tau_K)\omega_K\, \mathrm{d}x = 0, \quad \forall\tau_K \in [\mathbb{P}_1(T)]^{2\times 2} \oplus [\mathbb{R}\ \mathrm{curl}\ b_T]^2, \\ \dfrac{1}{\lambda}\displaystyle\int_K p_K q_K\, \mathrm{d}x - |K|\nabla q_K\cdot u_K + \int_{\partial K} (\lambda_{\partial K}\cdot n_K)q_K\, \mathrm{d}s = 0, \quad \forall q_K \in \mathbb{P}_1(K), \\ \displaystyle\int_K as\ (\sigma_K)\theta_K\, \mathrm{d}x = 0, \quad \forall\theta_K \in \mathbb{P}_1(K), \\ |K|\mathrm{div}\ \sigma_K\cdot v_K - |K|\nabla p_K\cdot v_K = -\displaystyle\int_K f\cdot v_K\, \mathrm{d}x, \quad \forall v_K \in \mathbb{P}_0(K)^2, \\ \displaystyle\int_a ((\sigma_{K_1^a} - p_{K_1^a}\delta)n_{K_1^a} + (\sigma_{K_2^a} - p_{K_2^a}\delta)n_{K_2^a})\cdot\mu_a\, \mathrm{d}s = \int_a \mu_a\cdot g\, \mathrm{d}s, \\ \quad \forall\mu_a \in [\mathbb{P}_1(a)]^2 \text{ if } a \in \mathcal{E}_h \setminus \Gamma_D, \end{cases}$$

where $\sigma_K = \sigma_{h|K}$, $p_K = p_{h|K}$, $\lambda_{\partial K|e} = \lambda_{h|e}$ for all $e \in \mathcal{E}_h \cap \partial K$.

On each element, we consider the corresponding linear and bilinear forms:

- $A_K(\sigma_K, \tau_K) := \frac{1}{2\mu}\int_K \sigma_K : \tau_K \, dx$,
- $B_K(p_K, v_K) := |K|\nabla p_K \cdot v_K$,
- $C_K(\sigma_K, v_K) := |K| \operatorname{div} \sigma_K \cdot v_K$,
- $H_K(\sigma_K, \theta_K) := \int_K as\,(\sigma_K)\theta_K \, dx$,
- $P_K(p_K, q_K) := \frac{1}{\lambda}\int_K p_K q_K \, dx$,
- $F_K(v_K) := \int_K f_K \cdot v_K \, dx$.

The corresponding linear and bilinear forms on the collection of internal edges and edges contained in Γ_N are of the form

- $E_{K,a}(\sigma_K, \mu_a) := \int_a (\sigma_K n_K) \cdot \mu_a \, ds$,
- $G_{K,a}(p_K, \mu_a) := \int_a p_K n_K \cdot \mu_a \, ds$,
- $T_e(\mu_e) := \int_e g \cdot \mu_e \, ds$.

With these notations, the system (80) takes the following local form:

$$\begin{cases} A^K\sigma^K + (C^K)^\top u^K - (E^{K,a})^\top\lambda^a - (E^{K,b})^\top\lambda^b - (E^{K,c})^\top\lambda^c + (H^K)^\top\omega^K = 0, \\ P^K p^K - (B^K)^\top u^K + (G^{K,a})^\top\lambda^a + (G^{K,b})^\top\lambda^b + (G^{K,c})^\top\lambda^c = 0, \\ H^K\sigma^K = 0, \\ E^{K_1,e}\sigma^{K_1} + E^{K_2,e}\sigma^{K_2} - G^{K_1,e}p^{K_1} - G^{K_2,e}p^{K_2} = T^e, \\ C^K\sigma^K - B^K p^K = -F^K, \end{cases}$$

where A^K, B^K, C^K, $E^{K,e}, G^{K,e}, H^K$, and P^K denote local stiffness matrices corresponding to the previously defined bilinear forms explicitly computed using appropriate basis functions, see the appendix of [8]. σ^K, p^K, u^K, ω^K and λ^e denote vectors of the components of σ_K, p_K, u_K, ω_K and λ_e written in these basis functions. Still denoting by σ, p, u, ω and λ the vectors of the degrees of freedom of the unknowns σ, p, u, ω and λ, the global algebraic system generated by this last system has the following form:

$$\begin{cases} A\sigma + C^\top u - E^\top\lambda + H^\top\omega = 0, \\ Pp - B^\top u + G^\top\lambda = 0, \\ H\sigma = 0, \\ E\sigma - Gp = T, \\ C\sigma - Bp = -F. \end{cases} \tag{81}$$

In the system (81), we start by eliminating σ, then u and finally ω. These eliminations are made element by element. After this procedure, we obtain the following system:

$$\begin{cases} \mathbf{\Lambda}\lambda + \mathbf{B}^\top p = \mathbf{F_1}, \\ \mathbf{B}\lambda - \mathbf{C}p = \mathbf{F_2}, \end{cases} \tag{82}$$

where the matrices Λ, B and C depend on the previous ones, Λ being invertible.

The system of algebraic equations (82) is then solved by the use of the following extension of the Augmented Lagrangian algorithm (see [30]):

$$\begin{cases} \mathbf{\Lambda}\lambda_m + \mathbf{B}^\top p_m = \mathbf{F_1}, \\ \mathbf{B}\lambda_m - (\mathbf{C} + \epsilon\mathbf{I})p_m = \mathbf{F_2} - \epsilon p_{m-1}, \end{cases}$$

where $\epsilon > 0$ is a fixed parameter and $\mathbf{I}$ is simply the identity matrix. The convergence of this scheme is $O(\epsilon^m)$, for $m = 1, 2, \ldots$. The positive parameter ϵ does not have to be chosen too small, so that the condition number of the system is not too large and a few iterations can reduce the error due to penalization. The implementation issue is as follows:

1. Start with any p_0 and fix a tolerance Tol > 0.
2. p_{m-1} being given we calculate λ_m by

$$(\mathbf{\Lambda} + \mathbf{B}^\top(\mathbf{C} + \epsilon\mathbf{I})^{-1}\mathbf{B})\lambda_m = \mathbf{F_1} + \mathbf{B}^\top(\mathbf{C} + \epsilon\mathbf{I})^{-1}(\mathbf{F_2} - \epsilon p_{m-1}),$$

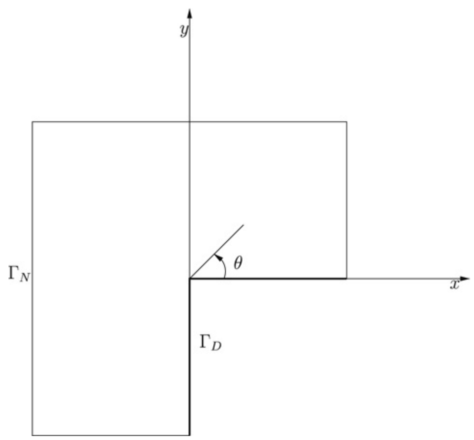

Fig. 1. The L-shaped domain.

3. Calculate p_m by

$$p_m = (\mathbf{C} + \epsilon \mathbf{I})^{-1}(\epsilon p_{m-1} + \mathbf{B}\lambda_m - \mathbf{F_2}),$$

4. if $\|p_m - p_{m-1}\|/\|p_m\| < Tol$, stop. Else, $p_{m-1} \leftarrow p_m$, and we come back to step 2.

We now present some numerical results on a test problem in the L-shaped domain $\Omega =]-1, 1[^2 \backslash ([0, 1[\times]-1, 0])$ as shown in Fig. 1, where Γ_D is the part of the boundary included into the x-axis or the y-axis. Using polar coordinates (r, θ), $0 \le \theta \le \omega := \frac{3\pi}{2}$, centered at the reentrant corner (see Fig. 1), we take as exact solution, the singular function [20]

$$u(r, \theta) = r^{\alpha} \begin{pmatrix} \phi_{1,\alpha}(\theta) \\ \phi_{2,\alpha}(\theta) \end{pmatrix},$$

where

$$\phi_{1,\alpha}(\theta) = C_1(\rho + \tau)\{\cos(\alpha - 2)\theta - \cos(\alpha\theta)\} + C_2\{(\rho + \tau)\sin(\alpha - 2)\theta + (\rho - 3\tau)\sin(\alpha\theta)\},$$
$$\phi_{2,\alpha}(\theta) = C_1\{-(\rho + \tau)\sin(\alpha - 2)\theta + (3\rho - \tau)\sin(\alpha\theta)\} + C_2(\rho + \tau)\{\cos(\alpha - 2)\theta - \cos(\alpha\theta)\}.$$

The parameters are

$$C_1 = (\rho + \tau)\sin(\alpha - 2)\omega - (3\tau - \rho)\sin(\alpha\omega),$$
$$C_2 = (\rho + \tau)\{\cos(\alpha\omega) - \cos(\alpha - 2)\omega\},$$
$$\rho = \frac{\lambda + \mu}{\mu}(\alpha - 1) - 2, \qquad \tau = \frac{\lambda + \mu}{\mu}(\alpha + 1) + 2,$$

where α is the smallest strictly positive solution of the transcendental equation (43)(i) for $\omega = \frac{3\pi}{2}$. The right-hand side f and the surface force density g are fixed accordingly.

With the aim of corroborating the robustness of our mixed method, we fix the Lamé coefficient $\mu = 1000$ and take increasing values of the Lamé coefficient λ as shown below:

λ	α
1.E + 008	0.544485935531526
1.E + 010	0.544483758810418
1.E + 012	0.544483737042583
1.E + 014	0.544483736824905

We use two kinds of meshes. The first one (uniform) is obtained by dividing the intervals [0, 1] and [−1, 0] into n subintervals of length $\frac{1}{n}$, and then each square is divided into triangles (see Fig. 2 where we have chosen $n = 10$). The second kind of mesh (refined) is obtained from the first one by refinement near (0, 0) according to Raugel's

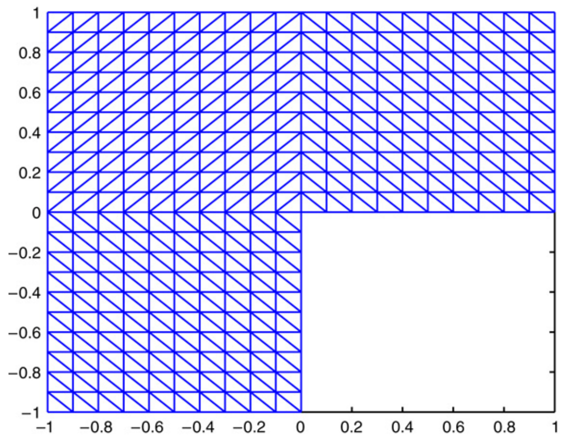

Fig. 2. The uniform mesh for $n = 10$.

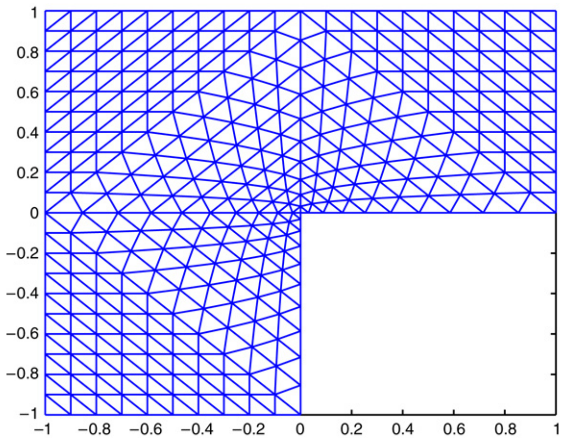

Fig. 3. The refined mesh for $n = 10$ and $\beta = 1.8$.

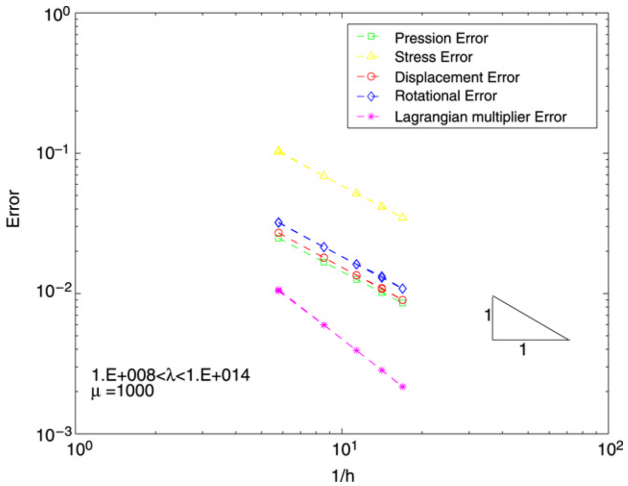

Fig. 4. Error for uniform meshes.

procedure [26]. Namely, Ω is divided into six big triangles; on the three which do not contain (0, 0), a uniform mesh is used; each big triangle containing (0, 0) is divided according to the ratios $(\frac{i}{n})^{\beta}$, $1 \leq i \leq n$, where $\beta \geq \frac{1}{(1-\alpha)}$

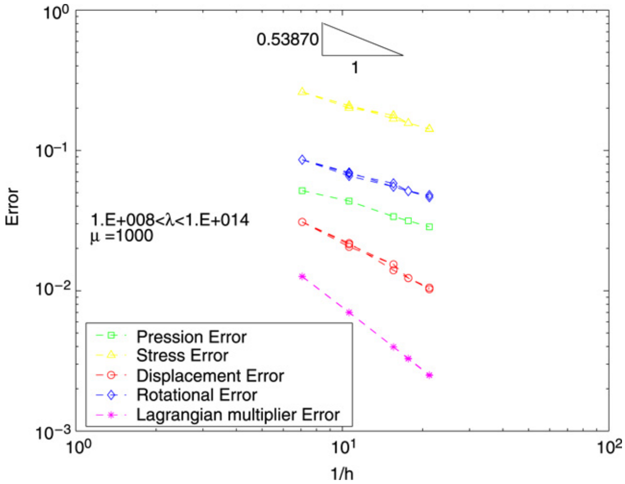

Fig. 5. Error for refined meshes.

along the sides which end at (0, 0), and finally each of these strips is divided uniformly (see Fig. 3 where we have chosen $n = 10$ and $\beta = 1.8$). We then represent the variations of the errors $\|p_h - p\|_{0,\Omega}$, $\|\sigma_h - \sigma\|_{0,\Omega}$, $\|u_h - u\|_{0,\Omega}$, $\|\omega_h - \omega\|_{0,\Omega}$ and $\|\xi - \xi_h\|_{\Gamma_N}$, with respect to the mesh size h for the four values of λ, in Figs. 4 and 5. A double logarithmic scale was used such that the slope of the curves yields the order of convergence $O(h)$ for refined meshes (see Fig. 5) according to our theoretical results, and $O(h^{\frac{2}{3}})$ for uniform meshes (see Fig. 4) due to the singular behaviuor of the solution. In these figures, since the curves are nearly confounded, we can deduce a strong stability with respect to the variations of the Lamé coefficient λ as expected. We further note a slightly better convergence of the error on the Lagrange multiplier (probably due to the use of the L^2-norm instead of the H^s-norm, with $s < 1/2$).

References

[1] A. Agouzal, J.-M. Thomas, An extension theorem for equilibrium finite element spaces, Japan J. Ind. Appl. Math. 13 (1996) 257–266.

[2] D.N. Arnold, F. Brezzi, J. Douglas, Jr., PEERS: A new mixed finite element for plane elasticity, Japan J. Appl. Math. 1 (1984) 347–367.

[3] D.N. Arnold, R.S. Falk, R. Winther, Finite element exterior calculus, homological techniques, and applications, Acta Numer. 15 (2006) 1–155.

[4] D.N. Arnold, R.S. Falk, R. Winther, Mixed finite element methods for linear elasticity with weakly imposed symmetry, Math. Comp. 76 (260) (2007) 1699–1723.

[5] D.N. Arnold, R. Winther, Mixed finite elements for elasticity, Numer. Math. 92 (2002) 401–419.

[6] I. Babuska, The finite element methods with Lagrangian multipliers, Numer. Mathods 20 (1973) 179–192.

[7] I. Babuska, G.N. Gatica, On the mixed finite element method with Lagrange multipliers, Numer. Methods Partial Differential Equations 19 (2003) 192–210.

[8] L. Boulaajine, Méthode des éléments finis mixte duale pour les problèmes de l'élasticité et de l'élastodynamique: analyse d'erreur a priori et a posteriori, Ph.D. thesis, Univ. of Valenciennes, France, July 2006.

[9] L. Boulaajine, M. Farhoul, L. Paquet, A posteriori error estimation for the dual mixed finite element method of the elasticity problem in a polygonal domain, Numer. Methods Partial Different Equations 21 (2005) 938–960.

[10] F. Brezzi, M. Fortin, Mixed and Hybrid Finite Element Methods, Springer-Verlag, New York, 1991.

[11] P.G. Ciarlet, The finite element method for elliptic problems, in: Studies in Math. and its Appl., vol. 4, North-Holland, 1978.

[12] M. Dauge, Elliptic boundary value problems in corner domains. Smoothness and asymptotics of solutions, in: L. N. in Math., vol. 1341, Springer-Verlag, 1988.

[13] H. El Bouzid, S. Nicaise, Nonconforming finite element methods and singularities in polygonal domains, Adv. Math. Sci. Appl. 7 (1997) 935–962.

[14] M. Farhoul, M. Fortin, Dual hybrid methods for the elasticity and the Stokes problems: A unified approach, Numer. Math. 76 (1997) 419–440.

[15] M. Farhoul, S. Nicaise, L. Paquet, A posteriori error estimation for the dual mixed finite element method of the Stokes problem, Numer. Funct. Anal. Optim. 27 (2006) 831–846.

[16] M. Farhoul, L. Paquet, Refined Mixed finite element method for the elasticity problem in a polygonal domain, Numer. Methods Partial Differential Equations 18 (2002) 323–339.

[17] B.F. Fraeijs de Veubeke, Displacement and equilibrium models in finite element method, in: O.C. Zienkiewicz, G.S. Holister (Eds.), Stress Analysis, Wiley, New York, 1965, pp. 145–195 (Chapter 9).

[18] G.N. Gatica, A. Marquez, S. Meddahi, A new dual-mixed finite element method for the plane linear elasticity problem with pure traction boundary conditions, Preprint 2006-23, Departemento de ing. Mat., Univ. de Concepcion, Chile, 2006.

[19] V. Giraut, P.A. Raviart, Finite Element Methods for Navier–Stokes Equations, Theory and Algorithms, Springer-Verlag, 1986.
[20] P. Grisvard, Boundary value problems in plane polygons. Instructions for use, EDF Bull. Direction Études Rech. Sér. C Math. Inform. (1986) 21–59 (in French).
[21] P. Grisvard, Singularité en Elasticité, Arch. for Rational Mech. Anal. 107 (1989) 157–180.
[22] P. Grisvard, Singularities in boundary value problems, in: RMA, vol. 22, Masson, Paris, 1992.
[23] J. Pitkäranta, Local stability conditions for the Babuska method of Lagrange multipliers, Math. Comput. 35 (1980) 1113–1129.
[24] J. Pitkäranta, The finite element method with Lagrange multipliers for domains with corners, Math. Comput. 37 (1981) 13–30.
[25] A. Quarteroni, A. Valli, Numerical Approximation of Partial Differential Equations, SCM 23, Springer-Verlag, 1994.
[26] G. Raugel, Résolution numérique par une méthode d'éléments finis du problème de Dirichlet pour le Laplacien dans un polygone, C. R. Acad. Sci. Paris, Série A 286 (1978) 791–794.
[27] P.A. Raviart, J.M. Thomas, A mixed finite element method for second order elliptic problems, in: I. Galligani, E. Magenes (Eds.), Mathematical aspects of Finite Element Methods, in: Lecture Notes in Mathematics, vol. 606, Springer-Verlag, 1977, pp. 292–315.
[28] J.E. Roberts, J.M. Thomas, Mixed and Hybrid Methods, in: J.L. Lions, P.G. Ciarlet (Eds.), Handbook of Numerical Analysis, Vol. II, Finite Element Methods (Part 1), North-Holland, Amsterdam, 1991, pp. 523–639.
[29] R. Stenberg, A family of mixed finite elements for the elasticity problem, Numer. Math. 53 (1988) 513–538.
[30] D. Yang, Iterative schemes for mixed finite element methods with applications to elasticity and compressible flow problems, Numer. Math 83 (2002) 177–200.
[31] S.-Y. Yi, A new nonconforming mixed finite element method for linear elasticity, Math. Models Methods Appl. Sci. 16 (2006) 979–999.