Semilocal generic formal fibers

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Abstract

Let T be a complete local ring and C a finite set of incomparable prime ideals of T. We find necessary and sufficient conditions for T to be the completion of an integral domain whose generic formal fiber is semilocal with maximal ideals the elements of C. In addition, if the characteristic of T is zero, we give necessary and sufficient conditions for T to be the completion of an excellent integral domain whose generic formal fiber is semilocal with maximal ideals the elements of C.
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1. Introduction

If A is a local integral domain with maximal ideal M, quotient field K, and M-adic completion $\widehat{A}$, then $\operatorname{Spec}(\widehat{A} \otimes_A K)$ is called the generic formal fiber of A. Note that there is a one-to-one correspondence between the elements of the generic formal fiber of A and the prime ideals in the inverse image of the ideal (0) under the map $\operatorname{Spec} \widehat{A} \to \operatorname{Spec} A$. In light of this correspondence, if $Q \in \operatorname{Spec} \widehat{A}$ and $Q \cap A = (0)$, we will say that Q is in the generic formal fiber of A. Furthermore, if the ring $\widehat{A} \otimes_A K$ is semilocal with maximal ideals $P_1 \otimes_A K, P_2 \otimes_A K, \ldots, P_n \otimes_A K$, then we will say that the generic formal fiber of A is semilocal with maximal ideals $P_1, P_2, \ldots, P_n$.

Because most of the standard integral domains we study have generic formal fibers that are far from semilocal, at first glance one might guess that, except for trivial examples, integral domains possessing a semilocal generic formal fiber do not exist. However, in [4], it was shown that such rings do exist and perhaps even more surprisingly, in [5], it was

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shown that these integral domains can be constructed to be excellent. In this paper, we show that these domains are more plentiful than one might suspect (both in the nonexcellent and excellent case).

In Section 3, we characterize which complete local rings are completions of integral domains possessing a semilocal generic formal fiber. Specifically, suppose T is a complete local ring with maximal ideal M, and $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. We show that there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold:

(1) $M \notin G$ and G contains all the associated prime ideals of T.
(2) If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$.
(3) If $Q \in G$, then $Q \cap$ prime subring of $T = (0)$.

It is easily seen that the above three conditions are necessary and so the bulk of the proof is dedicated to showing that the conditions are sufficient. It is worth pointing out that the three conditions in our theorem are relatively weak, and so in some sense most complete local rings can be realized as the completion of an integral domain whose generic formal fiber is semilocal where the maximal ideals can be prescribed.

In Section 4 we tackle the analogous version of the above problem where we require the additional condition that A be excellent. We are successful in characterizing the complete local rings of characteristic zero that are completions of excellent integral domains possessing a semilocal generic formal fiber. Specifically, let T be a complete local ring containing the integers and M the maximal ideal of T. Let $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. We show there exists an excellent local domain A with $\widehat{A} = T$ and such that A has generic formal fiber exactly the elements of G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold:

(1) $M \notin G$ and G contains all the associated prime ideals of T.
(2) If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$.
(3) If $Q \in G$, then $Q \cap$ prime subring of $T = (0)$.
(4) T is equidimensional.
(5) T_P is a regular local ring for all maximal elements $P \in G$.

Showing that the above five conditions are necessary, although maybe not immediately obvious, is relatively short. Our proof, then, will focus on proving that they are sufficient.

For both theorems, to show that the respective conditions are sufficient we construct the desired integral domain A. Our construction is based on the techniques used in [3] and is inspired by the construction of Heitmann in [1]. We start with the prime subring of T, localized at the appropriate prime ideal. We then successively adjoin elements of T to this ring in order to get our final result. Naturally, we must be careful which elements we choose to adjoin. For example, we must avoid the zero divisors of T, so that A will be an integral domain. We must also avoid nonzero elements of prime ideals that we wish to be in the generic formal fiber of A. We will adjoin enough elements of T to our domain A so that if I is a finitely generated ideal of A then $IT \cap A = I$. Furthermore, we will be

adjoining elements of T until we have obtained the property that for every ideal J of T such that $J \nsubseteq P$ for all $P \in G$, our ring contains a nonzero element of every coset in the ring T/J. Thus our ring will satisfy the property that if J is an ideal of T where $J \nsubseteq P$ for all $P \in G$, then the map $A \to T/J$ is onto. In particular, this means that $A \to T/M^2$ is onto. This fact, along with the condition that $IT \cap A = I$ for every finitely generated ideal I of A will force the completion of A to be T. Moreover, what is also interesting about the condition that $A \to T/J$ be onto is that it turns out if T contains the integers then it will force A to be excellent. By adjoining nonzero elements of each ideal J where $J \nsubseteq P$ for all $P \in G$ while avoiding nonzero elements of the prime ideals contained in G, we also ensure that the generic formal fiber of A is exactly the elements of G.

All rings in this paper are to be assumed commutative with unity. If we say a ring is local, we mean it is a Noetherian ring with one maximal ideal. The term quasi-local will be reserved for a ring with one maximal ideal that need not be Noetherian. We will use c to denote the cardinality of the real numbers. Finally, when we say that (T, M) is a local ring, we mean that T is a local ring with maximal ideal M.

2. The construction

We now begin the construction of our integral domain A. The following proposition is [2, Proposition 1]. It will be used to show that the ring A we construct has the desired completion.

Proposition 2.1. *If $(A, M \cap A)$ is a quasi-local subring of a complete local ring (T, M), the map $A \to T/M^2$ is onto and $IT \cap A = I$ for every finitely generated ideal I of A, then A is Noetherian and the natural homomorphism $\widehat{A} \to T$ is an isomorphism.*

Although Lemma 2.2 is well-known, we will use it repeatedly. So, we state it here without proof.

Lemma 2.2. *Let T be an integral domain and I a nonzero ideal of T. Then $|I| = |T|$.*

Lemma 2.3. *Let (T, M) be a complete local ring of dimension at least one. Let P be a nonmaximal prime ideal of T. Then $|T/P| = |T| \geqslant c$.*

Proof. Clearly, T/P is reduced. Furthermore, since T is complete and $\dim T \geqslant 1$, T/P is complete and $\dim(T/P) \geqslant 1$, as P is nonmaximal. Since T/P is reduced, complete and $\dim(T/P) \geqslant 1$, we have $|T/P| \geqslant c$. But clearly $|T/P| \leqslant |T|$, so $|T| \geqslant c$. Now, define a map $f : T \to \prod_{i=1}^{\infty} T/M^i$ by $f(t) = (t + M, t + M^2, t + M^3, \ldots)$. It is easy to see that f is injective and so $|T| = \sup\{c, |T/M|\}$. Now, $|T/P| \leqslant |T|$ and $|T/P| \geqslant \sup\{c, |T/M|\} = |T|$, so $|T/P| = |T|$ as desired. $\quad\square$

Armed with the previous two lemmas, we can now prove the following critical lemma. It will be used to adjoin elements to a specific subring of T so that the resulting ring maintains certain properties of the original subring.

Lemma 2.4. *Let* (T, M) *be a complete local ring such that* $\dim T \geqslant 1$, *C a finite set of nonmaximal prime ideals of T such that no ideal in C is contained in another ideal of C, and D a subset of T such that* $|D| < |T|$. *Let I be an ideal of T such that* $I \nsubseteq P$ *for all* $P \in C$. *Then* $I \nsubseteq \bigcup \{r + P \mid r \in D, \ P \in C\}$.

Proof. Let $C = \{P_1, P_2, \dots, P_n\}$. From the Prime Avoidance Theorem, we know that $I \nsubseteq \bigcup_{i=1}^{n} P_i$. Let $x \in I$, $x \notin \bigcup_{i=1}^{n} P_i$. Define a family of maps $f_i : \{P_i\} \times D \to T$. Let $(P_i, r) \in \{P_i\} \times D$. If $r + P_i \notin (x + P_i)(T/P_i)$, define $f_i(P_i, r) = 0$. Otherwise, it must be the case that $r + P_i = (x + P_i)(s_i + P_i)$ for some $s_i \in T$, so choose one such s_i and define $f_i(P_i, r) = s_i$. One should note that f_i is not unique. The element s_i can be any element of the coset $s_i + P$ and for our proof, it does not matter which one is chosen. Now let $S_i = \text{Image } f_i$. Note that we then have the inequality $|S_i| \leqslant |D| < |T| = |T/P_i|$.

First, suppose $n = 1$. Then $|S_1| \leqslant |D| < |T| = |T/P_1|$. So, there exists $t \in T$ such that $t + P_1 \neq s + P_1$ for all $s \in S_1$. Now, if $xt \in \bigcup \{r + P_1 \mid r \in D\}$, then $xt + P_1 = r + P_1$ for some $r \in D$. But then $r + P_1 \in (x + P_1)(T/P_1)$, so $r + P_1 = (x + P_1)(s + P_1)$ for some $s \in S_1$. So, we have

$$(x + P_1)(t + P_1) = r + P_1 = (x + P_1)(s + P_1),$$

which implies that $t + P_1 = s + P_1$, a contradiction. It follows that the lemma holds if $n = 1$.

If $n > 1$, we claim that

$$|T/P_i| = \left| \frac{P_i + \bigcap_{j=1, j \neq i}^{n} P_j}{P_i} \right|.$$

Notice that since T/P_i is an integral domain, this is true by Lemma 2.2 if we can simply show that $(P_i + \bigcap_{j=1, j \neq i}^{n} P_j)/P_i$ is not the zero ideal of T/P_i. Suppose that this were not true. Then it must be the case that $\bigcap_{j=1, j \neq i}^{n} P_j \subseteq P_i$. We know, however, that since no P_i is contained in any other ideal in C this cannot happen. Hence $(P_i + \bigcap_{j=1, j \neq i}^{n} P_j)/P_i$ is not the zero ideal of T/P_i, and it follows that

$$|D| < \left| \frac{P_i + \bigcap_{j=1, j \neq i}^{n} P_j}{P_i} \right|.$$

Thus there exists a $t_i \in \bigcap_{j=1, j \neq i}^{n} P_j$ such that $t_i + P_i \neq s_i + P_i$ for all $s_i \in S_i$ and for all $i = 1, \dots, n$. We claim that

$$x \sum_{j=1}^{n} t_j \notin \bigcup \{r + P_i \mid r \in D, \ P_i \in C\}.$$

To see this, suppose that

$$x \sum_{j=1}^{n} t_j \in \bigcup \{r + P_i \mid r \in D, \ P_i \in C\}.$$

Then $x \sum_{j=1}^{n} t_j + P_i = r + P_i$ for some $P_i \in C$, $r \in D$. But this means that $x t_i + P_i = r + P_i$, implying that $(x + P_i)(t_i + P_i) = r + P_i$ and thus $r + P_i \in (x + P_i)(T/P_i)$. But then

$$(x + P_i)(t_i + P_i) = r + P_i = (x + P_i)(s_i + P_i)$$

for some $s_i \in S_i$. Thus $t_i + P_i = s_i + P_i$ for some $s_i \in S_i$, a contradiction. $\square$

Definition. Let (T, M) be a complete local ring and C a set of prime ideals of T. Suppose that $(R, R \cap M)$ is a quasi-local subring of T such that $|R| < |T|$ and $R \cap P = (0)$ for every $P \in C$. Then we call R a small C-avoiding subring of T and will denote it by *SCA*-subring.

SCA-subrings will be essential in our proof. If R is an *SCA*-subring of T, then note that if we choose our set C such that the associated prime ideals of T are contained in prime ideals in C, then the condition $R \cap P = (0)$ for all $P \in C$ implies that $R \cap Q = (0)$ for every $Q \in \text{Ass}\, T$, and thus R contains no zero divisors of T—certainly a condition that any domain we might wish to construct must enjoy. Furthermore, this condition will ensure that the prime ideals of C are in the generic formal fiber of our final domain A. It is worth noting too that the condition $|R| < |T|$ implies that $|R| < |T/P|$ for all nonmaximal prime ideals P of T from Lemma 2.3. This cardinality condition will allow us to adjoin an element to R so that the resulting ring will not contain zero divisors of T or nonzero elements of the prime ideals in C.

Recall that one property that we would like our constructed ring, call it A, to possess, is that if J is an ideal of T with $J \nsubseteq P$ for all $P \in C$, then the map $A \to T/J$ is onto. Lemma 2.5 allows us to adjoin an element of a coset of T/J, which eventually will force our ring A to satisfy this property. The proof of Lemma 2.5 closely parallels the proof of [6, Lemma 3] and [3, Lemma 3].

Lemma 2.5. *Let (T, M) be a complete local ring of dimension at least one. Let C be a finite set of nonmaximal prime ideals of T such that no ideal in C is contained in any other ideal in C. Let J be an ideal of T such that $J \nsubseteq P$ for all $P \in C$. Let R be an SCA-subring of T and $u + J \in T/J$. Then there exists an infinite SCA-subring S of T such that $R \subseteq S \subseteq T$ and $u + J$ is in the image of the map $S \to T/J$. Moreover, if $u \in J$, then $S \cap J \neq (0)$.*

Proof. Let $P \in C$. Let $D_{(P)}$ be a full set of coset representatives of the cosets $t + P$ that make $(u + t) + P$ algebraic over R. Note that as $|R| < |T|$ and $|T| \geqslant c$, we have $|D_{(P)}| < |T|$. Let $D = \bigcup_{P \in C} D_{(P)}$, and note that $|D| < |T|$. Now use Lemma 2.4 with $I = J$ to find an $x \in J$ such that $x \notin \bigcup \{r + P \mid r \in D, \ P \in C\}$. We claim that $S = R[u + x]_{(R[u+x] \cap M)}$ is the desired *SCA*-subring. It is easy to see that $|S| < |T|$. Now suppose that $f \in R[u + x] \cap P$ for some $P \in C$. Then

$$f = r_n (u + x)^n + \cdots + r_1 (u + x) + r_0 \in P,$$

where $r_i \in R$. But we chose x such that $(u + x) + P$ is transcendental over R. Therefore $r_i \in R \cap P = (0)$ for every $i = 1, 2, \ldots, n$ and it follows that $f = 0$. So $S \cap P = (0)$ and we

have that S is an *SCA*-subring. Note further that if $u \in J$, then $u + x \in J$. Since $(u + x) + P$ is transcendental over R, it must be the case that $u + x \neq 0$. It follows that $S \cap J \neq (0)$. $\square$

The following lemma will help us ensure that $IT \cap A = I$ for every finitely generated ideal I of A. Recall that this is a necessary condition in order to be able to use Proposition 2.1. The proof of Lemma 2.6 resembles that of [5, Lemma 6], as well as that of [3, Lemma 4].

Lemma 2.6. *Let (T, M) be a complete local ring of dimension at least one. Let C be a finite set of nonmaximal prime ideals of T such that if $Q \in \operatorname{Ass} T$ then $Q \subseteq P$ for some $P \in C$ and no ideal in C is contained in any other ideal in C, and let R be an SCA-subring of T. Suppose that I is a finitely generated ideal of R and $c \in IT \cap R$. Then there exists an SCA-subring S of T such that $R \subseteq S \subseteq T$ and $c \in IS$.*

Proof. We will induct on the number of generators of I. Suppose $I = aR$. Now if $a = 0$, then $c = 0$ and thus $S = R$ is the desired *SCA*-subring of T. Thus consider the case where $a \neq 0$. In this case, $c = au$ for some $u \in T$. We claim that $S = R[u]_{(R[u] \cap M)}$ is the desired *SCA*-subring. To see this, first note that $|S| < |T|$. Now let $f \in R[u] \cap P$ where $P \in C$. Then $f = r_n u^n + \cdots + r_1 u + r_0 \in P$. Multiplying through by a^n, we get

$$a^n f = r_n (au)^n + \cdots + r_1 a^{n-1}(au) + r_0 a^n$$

and it follows that

$$a^n f = r_n c^n + \cdots + r_1 a^{n-1} c + r_0 a^n \in P \cap R = (0).$$

Now, $a \in R$, $R \cap P = (0)$ for every $P \in C$, and all associated prime ideals of T are contained in an element of C. It follows that a is not a zero divisor in T. It must be the case then that $f = 0$, giving us that S is an *SCA*-subring of T. Thus we have proven the base case, when I is principal.

Now let I be an ideal of R that is generated by $m > 1$ elements, and suppose that the lemma holds true for all ideals of R generated by $m - 1$ elements. Let $I = (y_1, \ldots, y_m)R$. Then $c = y_1 t_1 + y_2 t_2 + \cdots + y_m t_m$ for some $t_1, t_2, \ldots, t_m \in T$. By adding 0, note that we then have the equality

$$c = y_1 t_1 + y_1 y_2 t - y_1 y_2 t + y_2 t_2 + \cdots + y_m t_m$$
$$= y_1(t_1 + y_2 t) + y_2(t_2 - y_1 t) + y_3 t_3 + \cdots + y_m t_m$$

for any $t \in T$. Let $x_1 = t_1 + y_2 t$ and $x_2 = t_2 - y_1 t$ where we will choose the element t later. Now let $P \in C$. If $(t_1 + y_2 t) + P = (t_1 + y_2 t') + P$, then it must be the case that $y_2(t - t') \in P$. But $y_2 \in R$, $R \cap P = (0)$ and $y_2 \neq 0$, so we have $t - t' \in P$. Thus $t + P = t' + P$. The contrapositive of this result indicates that if $t + P \neq t' + P$, then $(t_1 + y_2 t) + P \neq (t_1 + y_2 t') + P$. Let $D_{(P)}$ be a full set of coset representatives of the cosets $t + P$ that make $x_1 + P$ algebraic over R. Let $D = \bigcup_{P \in C} D_{(P)}$. Note that $|D| < |T|$. Now we can use Lemma 2.4 with $I = T$ to find an element $t \in T$ such that $x_1 + P$ is transcendental over R for every $P \in C$. It can be easily shown (as in the proof of

Lemma 2.5) that $R' = R[x_1]_{(R[x_1] \cap M)}$ is an *SCA*-subring of T. Now let $J = (y_2, \ldots, y_m)R'$ and $c^* = c - y_1 x_1$. It is then the case that $c^* \in JT \cap R'$, so we can use our induction assumption to draw the conclusion that there exists an *SCA*-subring S of T such that $R' \subseteq S \subseteq T$ and $c^* \in JS$. Thus $c^* = y_2 s_2 + \cdots + y_m s_m$ for some $s_1, s_2, \ldots, s_m \in S$. It follows that $c = y_1 x_1 + y_2 s_2 + \cdots + y_m s_m \in IS$, and thus S is the desired *SCA*-subring. $\quad\square$

Definition. Let Ω be a well-ordered set and $\alpha \in \Omega$. We define $\gamma(\alpha) = \sup\{\beta \in \Omega \mid \beta < \alpha\}$.

Lemma 2.7 allows us to put many of our desired conditions together. We note here that the proof of Lemma 2.7 is based on the proof of [5, Lemma 12].

Lemma 2.7. *Let (T, M) be a complete local ring of dimension at least one. Let J be an ideal of T with $J \not\subseteq P$ for all $P \in C$, where C is a finite set of nonmaximal ideals of T such that if $Q \in \mathrm{Ass}\, T$ then $Q \subseteq P$ for some $P \in C$ and no ideal in C is contained in any other ideal in C, and let $u + J \in T/J$. Suppose R is an SCA-subring. Then there exists an SCA-subring S of T such that*

(1) *$R \subseteq S \subseteq T$.*
(2) *If $u \in J$, then $S \cap J \neq (0)$.*
(3) *$u + J$ is in the image of the map $S \to T/J$.*
(4) *For every finitely generated ideal I of S, we have $IT \cap S = I$.*

Proof. We first apply Lemma 2.5 to find an infinite *SCA*-subring R' of T such that $R \subseteq R' \subseteq T$, $u + J$ is in the image of the map $R' \to T/J$, and if $u \in J$ then $R' \cap J \neq (0)$. We will construct the desired S such that $R' \subseteq S \subseteq T$ which will ensure that the first three conditions of the lemma hold true. Now let

$$\Omega = \{(I, c) \mid I \text{ is a finitely generated ideal of } R' \text{ and } c \in IT \cap R'\}.$$

Letting $I = R'$, we can see that $|\Omega| \geqslant |R'|$. But then since R' is infinite, the number of finitely generated ideals of R' is $|R'|$, and therefore $|R'| \geqslant |\Omega|$, giving us the equality $|R'| = |\Omega|$. Moreover, as R' is an *SCA*-subring of T, we have $|\Omega| = |R'| < |T|$. Well order Ω so that it does not have a maximal element and let 0 denote its first element. We will now inductively define a family of *SCA*-subrings of T, one for each element of Ω. Let $R_0 = R'$ and let $\alpha \in \Omega$. Assume that R_β has been defined for all $\beta < \alpha$. If $\gamma(\alpha) < \alpha$ and $\gamma(\alpha) = (I, c)$, then define R_α to be the *SCA*-subring obtained from Lemma 2.6. In this manner, R_α will have the properties that $R_{\gamma(\alpha)} \subseteq R_\alpha \subseteq T$, and $c \in IR_\alpha$. If $\gamma(\alpha) = \alpha$, define $R_\alpha = \bigcup_{\beta < \alpha} R_\beta$. Note that in both cases, R_α is an *SCA*-subring of T. Now let $R_1 = \bigcup_{\alpha \in \Omega} R_\alpha$. We know that $|\Omega| < |T|$ and $|R_\alpha| < |T|$ for every $\alpha \in \Omega$, and thus $|R_1| < |T|$ as well. Moreover, as $R_\alpha \cap P = (0)$ for every $P \in C$ and every $\alpha \in \Omega$, we have $R_1 \cap P = (0)$ for every $P \in C$. It follows that R_1 is an *SCA*-subring. Furthermore, notice that if I is a finitely generated ideal of R_0 and $c \in IT \cap R_0$, then $(I, c) = \gamma(\alpha)$ for some $\alpha \in \Omega$ with $\gamma(\alpha) < \alpha$. It follows from the construction that $c \in IR_\alpha \subseteq IR_1$. Thus $IT \cap R_0 \subseteq IR_1$ for every I a finitely generated ideal of R_0.

Following this same pattern, build an *SCA*-subring R_2 of T such that $R_1 \subseteq R_2 \subseteq T$ and $IT \cap R_1 \subseteq IR_2$ for every finitely generated ideal I of R_1. Continue to form a chain $R_0 \subseteq R_1 \subseteq R_2 \subseteq \cdots$ of *SCA*-subrings of T such that $IT \cap R_n \subseteq IR_{n+1}$ for every finitely generated ideal I of R_n.

We now claim that $S = \bigcup_{i=1}^{\infty} R_i$ is the desired *SCA*-subring. To see this, first note that S is indeed an *SCA*-subring, and that $R \subseteq S \subseteq T$. Now set $I = (y_1, y_2, \ldots, y_k)S$ and let $c \in IT \cap S$. Then there exists an $N \in \mathbb{N}$ such that $c, y_1, \ldots, y_k \in R_N$. Thus $c \in (y_1, \ldots, y_k)T \cap R_N \subseteq (y_1, \ldots, y_k)R_{N+1} \subseteq IS$. From this it follows that $IT \cap S = I$, so the fourth condition of the lemma holds. $\square$

We now construct a domain A that has the desired completion, as well as other interesting properties.

Lemma 2.8. *Let (T, M) be a complete local ring of dimension at least one, and G a set of nonmaximal prime ideals of T where G contains the associated prime ideals of T and such that the set of maximal elements of G, call it C, is finite. Moreover, suppose that if $q \in \operatorname{Spec} T$ with $q \subseteq P$ for some $P \in G$ then $q \in G$. Also suppose that for each prime ideal $P \in G$, P contains no nonzero integers of T. Then there exists a local domain A such that*

(1) *$\widehat{A} = T$.*
(2) *If p is a nonzero prime ideal of A, then $T \otimes_A k(p) \cong k(p)$ where $k(p) = A_p/pA_p$.*
(3) *The generic formal fiber of A is exactly the elements of G (and so has maximal ideals the elements of C).*
(4) *If I is a nonzero ideal of A, then A/I is complete.*

We note here that although the second and fourth conditions of this lemma may not seem relevant, they will prove useful later when, under certain circumstances, we show that A can be forced to be excellent.

Proof. The proof is quite similar to [3, Lemma 8]. Define

$$\Omega = \{u + J \in T/J \mid J \text{ is an ideal of } T \text{ with } J \nsubseteq P \text{ for every } P \in G\}.$$

We claim that $|\Omega| \leqslant |T|$. Since T is infinite and Noetherian, $|\{J$ is an ideal of T with $J \nsubseteq P$ for all $P \in G\}| \leqslant |T|$. Now, if J is an ideal of T, then $|T/J| \leqslant |T|$. It follows that $|\Omega| \leqslant |T|$.

Well order Ω so that each element has fewer than $|\Omega|$ predecessors. Let 0 denote the first element of Ω. Define R_0' to be the prime subring of T and let R_0 simply denote R_0' localized at $R_0' \cap M$. Note that R_0 is an *SCA*-subring.

Now recursively define a family of *SCA*-subrings as follows, starting with R_0. Let $\lambda \in \Omega$ and assume that R_β has already been defined for all $\beta < \lambda$. Then $\gamma(\lambda) = u + J$ for some ideal J of T with $J \nsubseteq P$ for all $P \in G$ and thus all $P \in C$. If $\gamma(\lambda) < \lambda$, use Lemma 2.7 to obtain an *SCA*-subring R_λ such that $R_{\gamma(\lambda)} \subseteq R_\lambda \subseteq T$, $u + J \in \operatorname{Image}(R_\lambda \to T/J)$ and for every finitely generated ideal I of R_λ the property $IT \cap R_\lambda = I$ holds. Moreover, this

gives us that $R_\lambda \cap J \neq (0)$. If $\gamma(\lambda) = \lambda$, define $R_\lambda = \bigcup_{\beta < \lambda} R_\beta$. Then we have R_λ is an *SCA*-subring for all $\lambda \in \Omega$. We claim that $A = \bigcup_{\lambda \in \Omega} R_\lambda$ is the desired domain.

We will first show that the generic formal fiber ring of A has the desired properties. As each R_λ is an *SCA*-subring, we have $R_\lambda \cap P = (0)$ for each $P \in C$ and thus each $P \in G$. Therefore $A \cap P = (0)$ for each $P \in G$ as well. Moreover, if J is an ideal of T with $J \nsubseteq P$ for all $P \in G$, then $0 + J \in \Omega$. Therefore, $\gamma(\lambda) = 0 + J$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. By construction, $R_\lambda \cap J \neq (0)$. It follows that $J \cap A \neq (0)$. Hence the generic formal fiber of A is exactly the elements of G, and has maximal ideals the elements of C.

Now we show that the completion of A is T. To do this, we will use Proposition 2.1. Note that as each prime ideal $P \in G$ is nonmaximal in T, we have that M^2 is not contained in any $P \in G$. Thus by the construction, the map $A \to T/M^2$ is surjective. Now let I be a finitely generated ideal of A with $I = (y_1, \ldots, y_k)$. Let $c \in IT \cap A$. Then $\{c, y_1, \ldots, y_k\} \subseteq R_\lambda$ for some $\lambda \in \Omega$ with $\gamma(\lambda) < \lambda$. Again by the construction, $(y_1, \ldots, y_k)T \cap R_\lambda = (y_1, \ldots, y_k)R_\lambda$. As $c \in (y_1, \ldots, y_k)T \cap R_\lambda$, we have that $c \in (y_1, \ldots, y_k)R_\lambda \subseteq I$. Hence $IT \cap A = I$ as desired, and it follows that A is Noetherian and its completion is T.

To show the fourth condition is fairly simple. Suppose that I is a nonzero ideal of A, and let $J = IT$. If $J \subseteq P$ for some $P \in G$, then $I \subseteq J \cap A \subseteq P \cap A = (0)$, a contradiction. Thus $J \nsubseteq P$ for every $P \in G$. It follows by construction that the map $A \to T/J$ is surjective. Now since $A \cap J = A \cap IT = I$, the map $A/I \to T/J$ is an isomorphism, making A/I complete.

Finally, we prove the second condition. Let p be a nonzero prime ideal of A. Then A/p is complete, so we have

$$T \otimes_A k(p) \cong (T/pT)_{\overline{A-p}} \cong (A/p)_{\overline{A-p}} \cong A_p/pA_p = k(p),$$

as desired. $\square$

3. The main theorem and corollaries

For the next proof, we will in fact only need two of the previous lemma's four results, namely the first one and the third one. We are finally ready to arrive at our main theorem.

Theorem 3.1. *Let (T, M) be a complete local ring, and $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. Then there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold*:

(1) *$M \notin G$ and G contains all the associated prime ideals of T.*
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$.*
(3) *If $Q \in G$, then $Q \cap$ prime subring of $T = (0)$.*

Proof. First, the forward direction. Suppose that T is not a field, and that there exists a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly G. Suppose that $M \in G$. But then from our assumptions $M \cap A = (0)$, which implies that the maximal

ideal of A is (0) and thus the maximal ideal of T is zero as well, which implies that T is a field, a contradiction. Thus $M \notin G$. Moreover, if G does not contain all of the associated prime ideals of T, then, as the generic formal fiber of A is exactly G, there must be an associated prime ideal P of T such that $P \cap A \neq (0)$, a contradiction. Therefore G contains all of the associated prime ideals of T.

That the second requirement holds true is clear, as if one ideal Q is in the generic formal fiber of A, then $Q \cap A = (0)$ and thus if $P \subseteq Q$ then $P \cap A = (0)$ and P is also in the generic formal fiber of A and thus is contained in G, as desired.

In order to see that the intersection of each $P \in G$ with the prime subring of T is (0), note that $P \cap A = (0)$ for each $P \in G$. Since T is the completion of an integral domain A, the unity element of T must be in A. Hence the prime subring of T is also in A, and P contains no nonzero integers of T for all $P \in G$.

On the other hand, suppose that T is a field. Then the only prime ideal of T is (0), and consequently $G = \{(0)\}$. Thus, since as a field T is a completion of itself, $A = T$ and the generic formal fiber of A is $\{(0)\} = G$ as desired, so we are done.

Now we prove the backwards direction. If T is a field, then $A = T$ works. So suppose that T is not a field and that all the above conditions hold. The first condition gives us that $\dim T \geqslant 1$. Now, use Lemma 2.8 to construct the desired domain A. $\quad\square$

Example 1. Let $T = \mathbb{C}[\![x, y, z]\!]$, $C = \{(x, y), (z)\}$, and G be those prime ideals P of T such that $P \subseteq (x, y)$ or $P \subseteq (z)$. Is there a local domain A such that $\widehat{A} = T$ and the generic formal fiber of A is exactly the elements of G and has maximal ideals the elements of C?

Clearly T is local, with maximal ideal $M = (x, y, z)$, and C is finite. Thus we may use Theorem 3.1. Certainly $M \notin G$. Moreover, T is a domain, so it has no zero divisors, and hence no associated prime ideals other than (0), which is in G. The way in which we defined G makes it evident that the second condition of the theorem holds. The third condition is also easy to see since the prime subring of T is $\mathbb{Z}$ and all integers are units. Thus T satisfies the three conditions of Theorem 3.1, and hence there exists a domain A such that $\widehat{A} = \mathbb{C}[\![x, y, z]\!]$ and the generic formal fiber of A is exactly the elements of G with maximal ideals (x, y) and (z) as desired.

We now state the local version of Theorem 3.1 in the following corollary.

Corollary 3.2. *Let (T, M) be a complete local ring and P a prime ideal of T. Then there exists a local integral domain A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal P if and only if either T is a field and $P = (0)$ or the following two conditions hold*:

(1) *$P \neq M$ and P contains all the associated prime ideals of T.*
(2) *$P \cap$ prime subring of $T = (0)$.*

Proof. That the forward direction holds true is obvious from Theorem 3.1, letting $G = \{Q \in \operatorname{Spec} T \mid Q \subseteq P\}$.

To see the backwards direction in the case where T is not a field (if T is a field, then $A = T$ works), P is a nonmaximal ideal of T containing all the associated prime

ideals of T, and $P \cap$ prime subring of $T = (0)$. Now simply let the set $G = \{Q \in \operatorname{Spec} T \mid Q \subseteq P\}$. It follows from Theorem 3.1 that there exists a domain A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal P. $\square$

Example 2. Let $T = \mathbb{C}[\![x, y, z]\!]/(xy)$. Does there exist a local integral domain A such that $\widehat{A} = T$ and the generic formal fiber of A is local with maximal ideal (x, y)?

Let $P = (x, y)$. It is easy to see that P satisfies the two conditions of Corollary 3.2. So, there exists a local domain A such that $\widehat{A} = \mathbb{C}[\![x, y, z]\!]/(xy)$ and the generic formal fiber of A is local with maximal ideal (x, y).

4. The excellent case

We now consider under what conditions the ring A can be made excellent. Using the same building blocks as the previous theorem, we are able to come up with a characterization in the characteristic zero case of those complete local rings that are the completion of a local excellent domain possessing a specific generic formal fiber.

Theorem 4.1. *Let (T, M) be a complete local ring containing the integers. Let $G \subseteq \operatorname{Spec} T$ such that G is nonempty and the number of maximal elements of G is finite. Then there exists an excellent local domain A with $\widehat{A} = T$ and such that A has generic formal fiber exactly G if and only if T is a field and $G = \{(0)\}$ or the following conditions hold*:

(1) *$M \notin G$ and G contains all the associated prime ideals of T.*
(2) *If $Q \in G$ and $P \in \operatorname{Spec} T$ with $P \subseteq Q$, then $P \in G$.*
(3) *If $Q \in G$, then $Q \cap$ prime subring of $T = (0)$.*
(4) *T is equidimensional.*
(5) *T_P is a regular local ring for all maximal elements $P \in G$.*

Proof. Assume that T is the completion of an excellent domain A having generic formal fiber exactly G with maximal ideals the maximal elements of G. If $\dim T = 0$ then T is a field and $G = \{(0)\}$. Thus consider the case where $\dim T \geqslant 1$.

As A is excellent, it is universally catenary. Hence, A is formally catenary and it follows that $A/(0) \cong A$ is formally equidimensional. Thus the completion, T, is equidimensional.

The first three conditions can be shown by the exact same arguments as the three conditions in Theorem 3.1.

To see that the fifth condition holds, note that the maximal ideals of $T \otimes_A k(0)$ are the maximal elements of G. Let P be one of these maximal elements. Then $T \otimes_A k(0)$ localized at P is isomorphic to T_P. Since A is excellent, $T \otimes_A k(0)$ is regular, implying that T_P is a regular local ring for every maximal element of G as desired.

Conversely, first suppose that T is a field and $G = \{(0)\}$. Then $A = T$ works. So, suppose that T is not a field and that all of the five conditions hold true for some complete local ring T and some nonempty set G of prime ideals of T such that the number of maximal elements of G is finite. We want to show that there exists an excellent domain A possessing generic formal fiber exactly G. Note that conditions (1) and (5) imply that

T is reduced by the following argument. Suppose that these two conditions are true, but that T is not reduced. Then there exists a nonzero $x \in T$ such that $x^n = 0$ in T. Now consider the ideal $(0 : x)$ of T. Now, $(0 : x) \subseteq Q_1 \cup Q_2 \cup \cdots \cup Q_n$, where the Q_i are the associated prime ideals of T. But then by the Prime Avoidance Theorem, $(0 : x) \subseteq Q_i$ for some i. Moreover, as G contains the associated prime ideals of T, $Q_i \subseteq P$ for some P a maximal element of G. Consider the regular local ring T_P, and note that regular local rings are domains. Now $x^n/1 = 0/1$ in T_P. Thus, as T_P is a domain, $x/1 = 0/1$ in T_P. But then there exists an element $s \notin P$ such that $sx = 0$. This implies that $s \in (0 : x)$, however, which indicates that $s \in P$, a contradiction. Therefore, T must be reduced as desired.

Now if $\dim T = 0$ then T is a field and we are in the first case. Suppose, on the other hand, that $\dim T \geqslant 1$. Then use Lemma 2.8 to construct the domain A. We claim that A is excellent with generic formal fiber exactly G. From the construction of A, A has the desired generic formal fiber. To see that A is excellent, suppose that p is a nonzero prime ideal of A. Then from Lemma 2.8 we have $T \otimes_A k(p) \cong k(p)$. Now let L be a finite field extension of $k(p)$. Then

$$T \otimes_A L \cong T \otimes_A k(p) \otimes_{k(p)} L \cong k(p) \otimes_{k(p)} L \cong L.$$

Thus the fiber over p is geometrically regular. Now T_P is regular by assumption for every maximal element P of G. It follows that $T \otimes_A k(\!(0)\!)$ is regular. Now since T contains the integers, so does A. It follows that $k(\!(0)\!)$ is a field of characteristic zero, and hence that $T \otimes_A L$ is regular for every finite field extension L of $k(\!(0)\!)$. Thus all of the formal fibers of A are geometrically regular. Since A is formally equidimensional, it is universally catenary, and thus A is excellent. Hence A is the desired domain. $\quad\square$

Notice that this proof fails if the characteristic of T is $p > 0$, as the A we construct may not have a geometrically regular generic formal fiber. It is worth noting, though, that all the other fibers are geometrically regular and so the only obstruction to A being excellent is that the generic formal fiber may not be geometrically regular.

We now state the local version of Theorem 4.1. Arguably more elegant than the previous theorem, this more specific theorem has fewer conditions, and thus may prove to be more practical.

Corollary 4.2. *Let (T, M) be a complete local ring containing the integers and P a prime ideal of T. Then T is the completion of a local excellent domain A possessing a local generic formal fiber with maximal ideal P if and only if T is a field and $P = (0)$ or the following three conditions hold*:

(1) *P contains the associated prime ideals of T.*
(2) *P is a nonmaximal prime ideal of T such that P contains no nonzero integers of T.*
(3) *T_P is a regular local ring.*

Proof. The forward direction of this proof follows immediately from Theorem 4.1.
   Conversely, if T is a field and $P = (0)$, then $A = T$ works.

So suppose P is a nonmaximal prime ideal of T containing all the associated prime ideals of T such that $P \cap$ prime subring of $T = (0)$, and T_P is a regular local ring. It interesting to note that these conditions alone imply that T is an integral domain by the following argument. Suppose that $xy = 0$ in T for some $x, y \in T$, $x, y \neq 0$. But then both x and y are zerodivisors of T, so $x, y \in P$. But now since T_P is a domain, then either $x/1 = 0/1$ or $y/1 = 0/1$. WLOG, assume that $x/1 = 0/1$. This implies that there is some $s \notin P$ such that $sx = 0$ in T. But then s is a zero divisor or T, and hence $s \in P$, a contradiction. Therefore, T contains no nonzero zero divisors and T is an integral domain. Observe that T being an integral domain implies that T is both reduced and equidimensional.

Now if $\dim T = 0$, then since T is reduced, M must be an associated prime ideal of T, and moreover it is the only associated prime ideal of T. Hence T is a field and we are in the first case.

If $\dim T \geqslant 1$, let $G = \{Q \in \operatorname{Spec} T \mid Q \subseteq P\}$, and use Lemma 2.8 to construct the domain A. It is trivial to verify that the five conditions of Theorem 4.1 hold. Theorem 4.1 tells us that A is then excellent with generic formal fiber exactly G. But the maximal ideal of G is P by definition, so A is excellent with local generic formal with maximal ideal P as desired. $\quad\square$

Example 3. Consider the complete local ring $T = \mathbb{C}[\![x, y, z]\!]/(xy)$ and $G = \{Q \in \operatorname{Spec} T \mid Q \subseteq (x)$ or $Q \subseteq (y, z)\}$ a set of prime ideals of T with maximal elements (x) and (y, z). It is not difficult to check that T and G satisfy the conditions of Theorem 4.1 and so T is the completion of an excellent local domain A with generic formal fiber exactly G.

Note here that T in the above example is not a domain, and thus we should not be able to find a prime ideal P of T such that there exists an excellent domain A that completes to T with local generic formal fiber with maximal ideal P. (Recall that in the proof of Corollary 4.2, we showed that such a T is necessarily an integral domain.) Indeed, we can observe that this is true by seeing that any ideal we might choose will either not contain $\operatorname{Ass} T = \{(x), (y)\}$, or if it does then T_P will not be a regular local ring.

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