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## On the semilocal convergence of efficient Chebyshev–Secant-type methods

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### ABSTRACT

We introduce a three-step Chebyshev–Secant-type method (CSTM) with high efficiency index for solving nonlinear equations in a Banach space setting. We provide a semilocal convergence analysis for (CSTM) using recurrence relations. Numerical examples validating our theoretical results are also provided in this study.

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## 1. Introduction

In this study we are concerned with the problem of approximating a locally unique solution  $x^*$  of an equation

$$F(x) = 0, \quad (1.1)$$

where  $F$  is a Fréchet-differentiable operator defined on a non-empty, open, convex subset  $\mathcal{D}$  of a Banach space  $\mathcal{X}$  with values in a Banach space  $\mathcal{Y}$ .

A large number of problems in applied mathematics and engineering are solved by finding the solutions of certain equations. For example, dynamic systems are mathematically modeled by difference or differential equations, and their solutions usually represent the states of the systems. For the sake of simplicity, we assume that a time-invariant system is driven by the equation  $\dot{x} = Q(x)$ , for some suitable operator  $Q$ , where  $x$  is the state. Then the equilibrium states are determined by solving Eq. (1.1). Similar equations are used in the case of discrete systems. The unknowns of engineering equations can be functions (difference, differential, and integral equations), vectors (systems of linear or nonlinear algebraic equations), or real or complex numbers (single algebraic equations with single unknowns). Except in special cases, the most commonly used solution methods are iterative. In fact, starting from one or several initial approximations a sequence is constructed that converges to a solution of the equation. Iteration methods are also applied for solving optimization

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problems. In such cases, the iteration sequences converge to an optimal solution of the problem at hand. Since all of these methods have the same recursive structure, they can be introduced and discussed in a general framework.

A classic iterative process for solving nonlinear equations is Chebyshev's method (see [1,2]):

$$\begin{cases} x_0 \in \mathcal{D}, \\ y_k = x_k - F'(x_k)^{-1} F(x_k), \\ x_{k+1} = y_k - \frac{1}{2} F'(x_k)^{-1} F''(x_k)(y_k - x_k)^2, \quad k \geq 0. \end{cases}$$

This one-point iterative process depends explicitly on the two first derivatives of  $F$  (namely,  $x_{n+1} = \psi(x_n, F(x_n), F'(x_n), F''(x_n))$ ). Ezquerro and Hernández introduced in [1] some modifications of Chebyshev's method that avoid the computation of the second derivative of  $F$  and reduce the number of evaluations of the first derivative of  $F$ . Actually, these authors have obtained a modification of the Chebyshev iterative process which only need to evaluate the first derivative of  $F$ , (namely,  $x_{n+1} = \bar{\psi}(x_n, F'(x_n))$ ), but with third-order of convergence. In this paper we recall this method as the Chebyshev–Newton-type method (CNTM) and it is written as follows:

$$\begin{cases} x_0 \in \mathcal{D}, \\ y_k = x_k - F'(x_k)^{-1} F(x_k), \\ z_k = x_k + a(y_k - x_k) \\ x_{k+1} = x_k - \frac{1}{a^2} F'(x_k)^{-1} ((a^2 + a - 1) F(x_k) + F(z_k)), \quad k \geq 0, \end{cases}$$

where  $F'(x)(x \in \mathcal{D})$  is the Fréchet-derivative of  $F$ . A semilocal convergence analysis was provided by Ezquerro and Hernández in [1].

The main aim of this paper is focused on constructing a family of iterative processes free of derivatives as the classic Secant method (SM) [3]. To obtain this new family we consider an approximation of the first derivative of  $F$  from a divided difference of first order, that is,  $F'(x_n) \approx [x_{n-1}, x_n, F]$ , where,  $[x, y; F]$  is a divided difference of order one for the operator  $F$  at the points  $x, y \in \mathcal{D}$ . Then, we introduce the Chebyshev–Secant-type method (CSTM)

$$\begin{cases} x_{-1}, x_0 \in \mathcal{D}, \\ y_k = x_k - A_k^{-1} F(x_k), \quad A_k = [x_{k-1}, x_k; F], \\ z_k = x_k + a(y_k - x_k), \\ x_{k+1} = x_k - A_k^{-1} (b F(x_k) + c F(z_k)), \quad k \geq 0, \end{cases}$$

where  $a, b, c$  are non-negative parameters to be chosen so that sequence  $\{x_k\}$  converges to  $x^*$ . Note that (CSTM) is reduced to (SM) if  $a = 0, b = c = 1/2$ , and  $y_k = x_{k+1}$ . Moreover, if  $x_{k-1} = x_k$ , and  $F$  is differentiable on  $\mathcal{D}$ , then,  $F'(x_k) = [x_k, x_k; F]$ , and (CSTM) reduces to Newton's method (NM).

Bosarge and Falb [4], Dennis [5], Potra [6], Argyros [7–11], Hernández et al. [12] and others [3,13,14], have provided sufficient convergence conditions for the (SM) based on Lipschitz-type conditions on divided difference operator (see, also relevant works in [15,4,16,5,17,18,6,19–21]).

Here, we provide a semilocal convergence analysis for (CSTM) using recurrence relations, as it was done in [1] for (CNTM). Three numerical examples are also provided. First, we consider a scalar equation where the main study of the paper is applied. Second, we discretize a nonlinear integral equation and approximate a numerical solution by a method of (CSTM) and its computational order of convergence. Thirdly, we do a comparative study of the methods of (CSTM), depending on the parameter  $c$ .

## 2. Semilocal convergence analysis of (CSTM)

We shall show the semilocal convergence of (CSTM) under the following conditions

( $\mathcal{C}_1$ )  $F : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$  is a Fréchet-differentiable operator, and there exists a divided difference denoted by  $[x, y; F]$  satisfying

$$[x, y; F](x - y) = F(x) - F(y) \quad \text{for all } x, y \in \mathcal{D};$$

( $\mathcal{C}_2$ ) There exist  $x_{-1}$  and  $x_0$  in  $\mathcal{D}$  such that

$$A_0^{-1} = [x_{-1}, x_0; F]^{-1} \in \mathcal{L}(\mathcal{Y}, \mathcal{X})$$

exists and

$$0 < \|A_0^{-1}\| \leq \beta;$$

( $\mathcal{C}_3$ ) There exists  $d > 0$  such that

$$\|x_0 - x_{-1}\| \leq d;$$

( $\mathcal{C}_4$ ) There exists  $\eta > 0$  such that

$$0 < \|A_0^{-1} F(x_0)\| \leq \eta;$$

(C<sub>5</sub>) There exists a constant  $M > 0$ , such that for all  $x, y, z$  in  $\mathcal{D}$

$$\| [x, y; F] - F'(z) \| \leq \frac{M}{2} (\|x - z\| + \|y - z\|);$$

(C<sub>6</sub>) For  $a \in [0, 1], b \in [0, 1]$  and  $c > 0$  given in (CSTM), we suppose

$$(1 - a)c = 1 - b.$$

Note that in view of (C<sub>5</sub>), the following assumption holds:

(C<sub>7</sub>) There exists  $M_0 > 0$  such that, for all  $z$  in  $\mathcal{D}$ ,

$$\| [x_{-1}, x_0; F] - F'(z) \| \leq \frac{M_0}{2} (\|x_{-1} - z\| + \|x_0 - z\|).$$

(C<sub>8</sub>)

$$\alpha = (1 + d + a c \gamma (a + d_0)) \gamma < 1,$$

where,

$$\gamma = \frac{\beta M \eta}{2}, \quad d_0 = \frac{d}{\eta};$$

(C<sub>9</sub>)

$$\bar{U}(x_0, r \eta) = \{x \in \mathcal{X} : \|x - x_0\| \leq r \eta\} \subseteq \mathcal{D},$$

for some  $r > 0$  to be precised later in Theorem 2.5.

Note that

$$\frac{M_0}{M} = \lambda \leq 1$$

holds in general, and  $\lambda$  can be arbitrarily large [9–11,15].

We note by (C) the set of conditions (C<sub>1</sub>)–(C<sub>9</sub>).

**Definition 2.1.** Let  $\gamma$  and  $d_0$  as defined in (C<sub>8</sub>). It is convenient to define for  $\mu_0 = w_0 = 1, q_{-1} = d_0$ , and  $n \geq 0$ , the following sequences

$$p_n = a c \gamma \mu_n (a w_n + q_{n-1}) w_n,$$

$$q_n = p_n + w_n,$$

$$\mu_{n+1} = \frac{\mu_n}{1 - \gamma \mu_n (q_{n-1} + q_n)},$$

$$c_n = \frac{M}{2} ((q_n + q_{n-1}) q_n + a c (a w_n + q_{n-1}) w_n),$$

and

$$w_{n+1} = \gamma \mu_{n+1} ((q_n + q_{n-1}) q_n + a c (a w_n + q_{n-1}) w_n).$$

Note that

$$w_{n+1} = \beta \eta \mu_{n+1} c_n.$$

Next, we give some Ostrowski-type approximations for (CSTM) that are needed later.

**Lemma 2.2.** Assume sequence  $\{x_k\}$  generated by (CSTM) is well-defined,  $(1 - a) c = 1 - b$  holds for  $a \in [0, 1], b \in [0, 1]$ , and  $c \geq 0$ .

Then, the following items hold for all  $k \geq 0$ :

$$F(z_k) = (1 - a)F(x_k) + a \int_0^1 (F'(x_k + a t (y_k - x_k)) - F'(x_k))(y_k - x_k) dt + a (F'(x_k) - A_k)(y_k - x_k), \tag{2.1}$$

$$x_{k+1} - y_k = -a c A_k^{-1} \left( \int_0^1 (F'(x_k + a t (y_k - x_k)) - F'(x_k))(y_k - x_k) dt + (F'(x_k) - A_k)(y_k - x_k) \right), \tag{2.2}$$

and

$$F(x_{k+1}) = \int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)) (x_{k+1} - x_k) dt + (F'(x_k) - A_k)(x_{k+1} - x_k) - a c \left( \int_0^1 (F'(x_k + a t(y_k - x_k)) - F'(x_k)) (y_k - x_k) dt + (F'(x_k) - A_k)(y_k - x_k) \right). \quad (2.3)$$

**Proof.** We have in turn using (CSTM)

$$\begin{aligned} a F(x_k) &= a A_k (x_k - y_k) \implies 0 = -a F(x_k) + A_k (x_k - z_k) \implies \\ F(z_k) &= F(x_k) - a F(x_k) + A_k (x_k - z_k) \\ &= (1 - a) F(x_k) + F(z_k) - F(x_k) + A_k (x_k - z_k) \\ &= (1 - a) F(x_k) + \int_0^1 (F'(x_k + t(z_k - x_k)) - A_k) (z_k - x_k) dt \\ &= (1 - a) F(x_k) + \int_0^1 (F'(x_k + t a (y_k - x_k)) - F'(x_k)) a (y_k - x_k) dt + a (F'(x_k) - A_k) (y_k - x_k), \end{aligned}$$

and (2.1) is proved.

By eliminating  $x_k$  from the first and the third approximations in (CSTM), we get:

$$\begin{aligned} x_{k+1} - y_k &= A_k^{-1} F(x_k) - A_k^{-1} b F(x_k) - A_k^{-1} c F(z_k) \\ &= A_k^{-1} ((1 - b) F(x_k) - c F(z_k)) \\ &= A_k^{-1} ((1 - a) F(x_k) - F(z_k)) c \\ &= -A_k^{-1} (-(1 - a) F(x_k) + F(z_k)) c \\ &= -A_k^{-1} c \left( a \int_0^1 (F'(x_k + a t (y_k - x_k)) - F'(x_k)) (y_k - x_k) dt + a (F'(x_k) - A_k) (y_k - x_k) \right), \end{aligned}$$

(by (2.1)), which proves (2.2).

Finally, we have:

$$\begin{aligned} F(x_{k+1}) &= F(x_{k+1}) - F(x_k) - A_k (y_k - x_k) \\ &= \int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)) (x_{k+1} - x_k) dt + F'(x_k) (x_{k+1} - x_k) - A_k (y_k - x_k) \\ &= \int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)) (x_{k+1} - x_k) dt \\ &\quad + (F'(x_k) - A_k) (x_{k+1} - x_k) + A_k (x_{k+1} - x_k) - A_k (y_k - x_k) \\ &= \int_0^1 (F'(x_k + t(x_{k+1} - x_k)) - F'(x_k)) (x_{k+1} - x_k) dt + (F'(x_k) - A_k) (x_{k+1} - x_k) + A_k (x_{k+1} - y_k), \end{aligned}$$

and (2.3) is proved by (2.2).

This completes the proof of Lemma 2.2.  $\square$

The following relates (CSTM) with scalar sequences introduced in Definition 2.1.

**Lemma 2.3.** Under the (C) conditions, we assume:

$$x_n \in \mathcal{D} \quad \text{and} \quad \gamma \mu_n (q_{n-1} + q_n) < 1 \quad (n \geq 0).$$

Then, the following items hold for all  $n \geq 0$ :

(I<sub>n</sub>)

$$\|A_n^{-1}\| \leq \mu_n \beta,$$

(II<sub>n</sub>)

$$\|y_n - x_n\| = \|A_n^{-1} F(x_n)\| \leq w_n \eta,$$

(III<sub>n</sub>)

$$\|x_{n+1} - y_n\| \leq p_n \eta,$$

(IV<sub>n</sub>)

$$\|x_{n+1} - x_n\| \leq q_n \eta.$$

**Proof.** We use induction.

We have  $\|y_0 - x_0\| \leq \eta$ , and  $\|z_0 - x_0\| \leq a \eta$ , so that  $x_0, z_0 \in \mathcal{D}$ .

Items **(I<sub>0</sub>)** and **(II<sub>0</sub>)** hold by **(C<sub>2</sub>)** and **(C<sub>4</sub>)**, respectively. To prove **(III<sub>0</sub>)**, we use Lemma 2.2 for  $n = 0$  to obtain by **(C<sub>2</sub>)**–**(C<sub>5</sub>)**

$$\begin{aligned} \|x_1 - y_0\| &\leq a c \|A_0^{-1}\| \frac{M}{2} (a \|y_0 - x_0\| + \|x_0 - x_{-1}\|) \|y_0 - x_0\| \\ &\leq \frac{a c \beta M}{2} (a \eta + d) \eta \\ &= a c \gamma (a + d_0) \eta = p_0 \eta. \end{aligned}$$

Moreover,

$$\|x_1 - x_0\| \leq \|x_1 - y_0\| + \|y_0 - x_0\| \leq p_0 \eta + \eta = (1 + p_0) \eta = q_0 \eta,$$

which implies **(IV<sub>0</sub>)**. Note also that  $z_1 \in \mathcal{D}$ . Following an inductive argument, assume  $x_k \in \mathcal{D}$ , and  $\gamma \mu_k (q_{k-1} + q_k) < 1$ . Then, we have

$$\begin{aligned} \|A_k^{-1}\| \|A_{k+1} - A_k\| &\leq \|A_k^{-1}\| \|[x_k, x_{k+1}; F] - [x_{k-1}, x_k; F]\| \\ &\leq \|A_k^{-1}\| \frac{M}{2} (\|x_k - x_{k-1}\| + \|x_{k+1} - x_k\|) \\ &\leq \frac{\beta M}{2} \mu_k (q_{k-1} + q_k) \eta = \gamma \mu_k (q_{k-1} + q_k) < 1. \end{aligned}$$

It follows from the Banach lemma on invertible operators [7,11] that  $A_{k+1}^{-1}$  exists, and

$$\begin{aligned} \|A_{k+1}^{-1}\| &\leq \frac{\|A_k^{-1}\|}{1 - \|A_k^{-1}\| \frac{M}{2} (\|x_k - x_{k-1}\| + \|x_{k+1} - x_k\|)} \\ &\leq \frac{\beta \mu_k}{1 - \gamma \mu_k (q_{k-1} + q_k)} = \mu_{k+1} \beta, \end{aligned}$$

which shows **(I<sub>k+1</sub>)**. Using Lemma 2.2, **(C<sub>5</sub>)**, and the induction hypotheses, we get

$$\begin{aligned} \|F(x_{k+1})\| &\leq \frac{M}{2} \|x_{k+1} - x_k\|^2 + \frac{M}{2} \|x_{k+1} - x_k\| \|x_k - x_{k-1}\| \\ &\quad + a c \left( \frac{a M}{2} \|y_k - x_k\|^2 + \frac{M}{2} \|x_k - x_{k-1}\| \|y_k - x_k\| \right) \\ &\leq \frac{M}{2} q_k^2 \eta^2 + \frac{M}{2} q_k \eta q_{k-1} \eta + a c \left( \frac{a M}{2} w_k^2 \eta^2 + \frac{M}{2} q_{k-1} \eta w_k \eta \right) \\ &= c_k \eta^2. \end{aligned} \tag{2.4}$$

Then, we get

$$\|y_{k+1} - x_{k+1}\| \leq \|A_{k+1}^{-1}\| \|F(x_{k+1})\| \leq \mu_{k+1} \beta c_k \eta^2 = w_{k+1} \eta.$$

Moreover, by Lemma 2.2, we have

$$\begin{aligned} \|x_{k+2} - y_{k+1}\| &\leq a c \|A_{k+1}^{-1}\| \frac{M}{2} (a \|y_{k+1} - x_{k+1}\| + \|x_{k+1} - x_k\|) \|y_{k+1} - x_{k+1}\| \\ &\leq a c \mu_{k+1} \frac{\beta M}{2} (a w_{k+1} + q_k) w_{k+1} \eta^2 = p_{k+1} \eta, \end{aligned}$$

and consequently,

$$\|x_{k+2} - x_{k+1}\| \leq \|x_{k+2} - y_{k+1}\| + \|y_{k+1} - x_{k+1}\| \leq (p_{k+1} + w_{k+1}) \eta = q_{k+1} \eta.$$

This completes the proof of Lemma 2.3. □

We shall establish the convergence of sequence  $\{x_n\}$  generated by (CSTM). This can be achieved by showing that  $\{q_n\}$  is a Cauchy sequence, if the following conditions hold for  $n \geq 0$ :

- (A<sub>1</sub>)  $x_n \in \mathcal{D}$ , and
- (A<sub>2</sub>)  $\gamma \mu_n (q_{n-1} + q_n) < 1$ .

In the next result, we show the Cauchy property for sequence  $\{q_n\}$ .

**Lemma 2.4.** Assume  $(\mathcal{C}_8)$ . Note that  $\alpha \in [0, 1)$  implies  $\gamma (q_{-1} + q_0) < 1$ . Then, the scalar sequence:

- (a)  $\{\mu_n\}$  is increasing.
- (b)  $\{q_n\}$  is decreasing and  $\lim_{n \rightarrow \infty} q_n = 0$ .

**Proof.** (a) We show using induction that all scalar sequences involved are positive. By Definition 2.1, and  $(\mathcal{C}_8)$ , we have for  $j = 0$ :  $\mu_j, p_j, q_j, w_j, c_j$ , and  $1 - \gamma \mu_j (q_{j-1} + q_j)$  are positive. Assume  $\mu_k, p_k, q_k, w_k, c_k$ , and  $1 - \gamma \mu_k (q_{k-1} + q_k)$  are positive for all  $k \leq n$ . Since  $c_k > 0$ , it follows from the definition of the scalar sequences that  $w_{k+1}, \mu_{k+1}, p_{k+1}, d_{k+1}$  have the same sign. Assume the common sign to be negative. Then

$$\begin{aligned} q_{k-1} + q_k + q_{k+1} < q_{k-1} + q_k &\implies 1 - \gamma \mu_k (q_{k-1} + q_k + q_{k+1}) > 1 - \gamma \mu_k (q_{k-1} + q_k) \\ &\implies \frac{1 - \gamma \mu_k (q_{k-1} + q_k + q_{k+1})}{1 - \gamma \mu_k (q_{k-1} + q_k)} > 1. \end{aligned}$$

But it follows from the definition of sequence  $\{\mu_k\}$  that

$$\begin{aligned} 1 - \gamma \mu_{k+1} (q_k + q_{k+1}) &= \frac{1 - \gamma \mu_k (q_{k-1} + 2q_k + q_{k+1})}{1 - \gamma \mu_k (q_{k-1} + q_k)} \\ \implies 1 - \gamma \mu_{k+1} q_{k+1} &= \frac{1 - \gamma \mu_k (q_{k-1} + 2q_k + q_{k+1})}{1 - \gamma \mu_k (q_{k-1} + q_k)} + \gamma \mu_{k+1} q_k \\ &= \frac{1 - \gamma \mu_k (q_{k-1} + q_k + q_{k+1})}{1 - \gamma \mu_k (q_{k-1} + q_k)} > 1, \end{aligned}$$

which is a contradiction, since we get  $\gamma \mu_{k+1} q_{k+1} < 0$ , but  $\mu_{k+1} q_{k+1}$  have the same sign, and  $\gamma > 0$ . The induction is then completed.

By the definition of sequence  $\{\mu_n\}$  and  $\mu_0 = 1$ , we have

$$\begin{aligned} 1 - \gamma \mu_k (q_{k-1} + q_k) &= \frac{\mu_k}{\mu_{k+1}} \\ \implies q_{k-1} + q_k &= \frac{1}{\gamma} \left( \frac{1}{\mu_k} - \frac{1}{\mu_{k+1}} \right) \\ \implies \sum_{i=0}^{k-1} (q_{i-1} + q_i) &= \frac{1}{\gamma} \left( \frac{1}{\mu_0} - \frac{1}{\mu_k} \right) = \frac{1}{\gamma} \left( 1 - \frac{1}{\mu_k} \right) \\ \implies \mu_k &= \frac{1}{1 - \gamma \sum_{i=0}^{k-1} (q_{i-1} + q_i)}. \end{aligned}$$

But  $1 - \gamma \sum_{i=0}^{k-1} (q_{i-1} + q_i)$  decreases. Therefore, sequence  $\{\mu_k\}$  increases, and consequently  $\mu_k \geq \mu_0 = 1$ .

- (b) We have that sequence  $\mu_k > 1$  is increasing, so that  $0 \leq \frac{1}{\mu_k} \leq 1$ . Since  $\left\{ \frac{1}{\mu_k} \right\}$  is monotonic on a compact set, it converges to  $\frac{1}{\mu_\infty}$ . Then, we have

$$\lim_{k \rightarrow \infty} (q_{k-1} + q_k) = \frac{1}{\gamma} \lim_{k \rightarrow \infty} \left( \frac{1}{\mu_k} - \frac{1}{\mu_{k+1}} \right) = \frac{1}{\gamma} \left( \frac{1}{\mu_\infty} - \frac{1}{\mu_\infty} \right) = 0.$$

This completes the proof of Lemma 2.4.  $\square$

We can show the main semilocal convergence theorem for (CSTM).

**Theorem 2.5.** Let  $F : \mathcal{D} \subseteq \mathcal{X} \rightarrow \mathcal{Y}$  be a Fréchet-differentiable operator defined on a non-empty open, convex domain  $\mathcal{D}$  of a Banach space  $\mathcal{X}$ , with values in a Banach space  $\mathcal{Y}$ . Assume that the  $(\mathcal{C})$  conditions hold. Then, sequence  $\{x_n\}$  ( $n \geq -1$ ), generated by (CSTM), is well-defined, remains in  $\bar{U}(x_0, r \eta)$  for all  $n \geq 0$ , and converges to a solution  $x^* \in \bar{U}(x_0, r \eta)$  of equation  $F(x) = 0$ , where,

$$r = \sum_{n=0}^{\infty} q_n. \tag{2.5}$$

Moreover, the following estimate holds

$$\|x_n - x^*\| \leq \sum_{k=n+1}^{\infty} q_k \eta < r \eta.$$

Furthermore,  $x^*$  is the unique solution of  $F(x) = 0$  in  $U(x_0, r_0) \cap \mathcal{D}$ , provided that  $r_0 \geq r \eta$ , where,

$$r_0 = \frac{2}{\beta M_0} - d - r \eta. \tag{2.6}$$

**Proof.** According to Lemmas 2.3 and 2.4, sequence  $\{x_n\}$  is of Cauchy ( $\{q_n\}$  is of Cauchy) in a Banach space  $\mathcal{X}$ , and it converges to some  $x^* \in \bar{U}(x_0, r \eta)$  (since,  $\bar{U}(x_0, r \eta)$  is a closed set). The sequence  $\{\mu_n\}$  is bounded above. Indeed, we have

$$\mu_n = \frac{1}{1 - \gamma \sum_{i=0}^{n-1} q_i} \leq \frac{1}{1 - \gamma \sum_{i=0}^{\infty} q_i},$$

and  $\lim_{n \rightarrow \infty} q_n = 0$ , which imply  $\lim_{n \rightarrow \infty} c_n = 0$ . By letting  $n \rightarrow \infty$  in (2.4), we get  $F(x^*) = 0$ . We also have

$$\|x_{n+1} - x_0\| \leq \sum_{i=0}^n \|x_{i+1} - x_i\| \leq \sum_{i=0}^n q_i \eta < r \eta, \tag{2.7}$$

which imply  $x_n \in \bar{U}(x_0, r \eta)$ . Consequently, we obtain  $x^* \in \bar{U}(x_0, r \eta)$ .

Finally, we shall show the uniqueness of the solution  $x^*$  in  $U(x_0, r_0)$ . Let  $y^*$  be a solution of equation  $F(x) = 0$  in  $U(x_0, r_0)$ . Define linear operator

$$\mathcal{L} = \int_0^1 F'(x^*_t) dt, \quad \text{where } x^*_t = x^* + t(y^* - x^*).$$

We shall show  $\mathcal{L}^{-1}$  exists. Using  $(\mathcal{C}_2)$  and  $(\mathcal{C}_7)$ , we get

$$\begin{aligned} \|A_0^{-1} \| \|A_0 - \mathcal{L}\| &\leq \frac{\beta M_0}{2} \int_0^1 (\|x_{-1} - x^*_t\| + \|x_0 - x^*_t\|) dt \\ &\leq \frac{\beta M_0}{2} \int_0^1 (\|x_0 - x_{-1}\| + 2 \|x_0 - x^*_t\|) dt \\ &\leq \frac{\beta M_0}{2} (d + \|x_0 - x^*\| + \|y^* - x^*\|) \\ &\leq \frac{\beta M_0}{2} (d + r \eta + r_0) = 1. \end{aligned} \tag{2.8}$$

It follows from (2.8), and the Banach lemma on invertible operators, that  $\mathcal{L}$  is invertible.

Finally, in view of the equality

$$0 = F(y^*) - F(x^*) = \mathcal{L}(y^* - x^*),$$

we obtain

$$x^* = y^*.$$

This completes the proof of Theorem 2.5.  $\square$

**Remark 2.6.** (a) It follows from the proof of Lemma 2.4 that

$$\mu_k = \frac{1}{1 - \gamma \sum_{i=0}^{k-1} (q_{i-1} + q_i)},$$

so that

$$\sum_{i=0}^{k-1} (q_{i-1} + q_i) = \frac{1}{\gamma} \left( 1 - \frac{1}{\mu_k} \right). \tag{2.9}$$

By (2.9), the following relation between  $\mu_\infty$  and  $r$  holds:

$$r = 0.5 \left( q_{-1} + \frac{1}{\gamma} \left( 1 - \frac{1}{\mu_\infty} \right) \right).$$

Set

$$\bar{r}_n = 0.5 \left( q_{-1} + \frac{1}{\gamma} \left( 1 - \frac{1}{\mu_n} \right) \right), \quad \bar{r} = 0.5 \left( q_{-1} + \frac{1}{\gamma} \right) \quad \text{and} \quad \bar{r}_0 = \frac{2}{\beta M} - d - \bar{r} \eta.$$

Then, we have

$$\bar{r} > r \quad \text{and} \quad \bar{r}_0 < r_0.$$

In view of the proof of Theorem 2.5,  $\bar{r}$  can replace  $r$ . However, this approach is less accurate but it avoids the computation of  $\mu_\infty$ .

(b) Condition  $(\mathcal{C}_5)$  implies that for  $x = y$ ,

$$\|F'(x_0)^{-1} (F'(x) - F'(z))\| \leq M \|x - z\| \quad \text{for all } x, z \in \mathcal{D}.$$

Then the conclusions of [1, Theorem 4.4] can be obtained from Theorem 2.5 for

$$b = \frac{a^2 + a - 1}{a^2}, \quad c = \frac{1}{a^2}.$$

Theorem 2.5 provides a larger uniqueness ball if  $M_0 < M$ . To obtain the uniqueness ball of [1, Theorem 4.4], simply set  $M = M_0$ .

### 3. Numerical examples

To illustrate the theoretical results introduced previously, we present some numerical examples. In these examples we show some situations where the results provided in the paper can be applied. In addition, the application of the methods introduced in (CSTM) for equations defined in functional spaces is also shown.

**Example 3.1.** Let  $\mathcal{X} = \mathcal{Y} = \mathbb{R}^2$  be equipped with the max-norm. Choose:

$$x_{-1} = (0.999, 0.999)^T, \quad x_0 = (1, 1)^T, \quad \mathcal{D} = U(x_0, 1 - \kappa), \quad \kappa \in \left[0, \frac{1}{2}\right).$$

Define function  $F$  on  $U_0$  by

$$F(x) = (\theta_1^3 - \kappa, \theta_2^3 - \kappa)^T, \quad x = (\theta_1, \theta_2)^T. \tag{3.1}$$

The Fréchet-derivative of operator  $F$  is given by

$$F'(x) = \begin{bmatrix} 3\theta_1^2 & 0 \\ 0 & 3\theta_2^2 \end{bmatrix}, \tag{3.2}$$

and the divided difference of  $F$  is defined by

$$[y, x; F] = \int_0^1 F'(x + t(y - x)) dt.$$

By the  $(\mathcal{C})$  conditions, Definition 2.1, and Remark 2.6(a), we have:

$$M = 6\beta(2 - \kappa), \quad M_0 = 3\beta(3 - \kappa), \quad \eta = (1 - \kappa)\beta.$$

Let  $\kappa = 0.49$ . Then, using Maple 13, we get for  $a = b = 0.5$ , and  $c = 1$ :

$$\begin{aligned} \beta &= 0.333666889, & M &= 3.023022014, \\ M_0 &= 2.512511674, & q_{-1} = d &= 0.001, & \eta &= 0.170170113, \\ \gamma &= 0.08582379485, & d_0 &= 0.005876472562, & \alpha &= 0.08777269180, \end{aligned}$$

$p_0 = 0.02170811930$		$q_0 = p_0 + w_0 = 1.02170811930$
$\mu_1 = \frac{\mu_0}{1 - \gamma \mu_0 (q_{-1} + q_0)} = 1.096218005$		
$\bar{r}_1 = 0.5118540590$		$\bar{r}_1 \eta = 0.08710226306$
$c_0 = 1.958025248$	$w_1 = \beta \eta \mu_1 c_0 = 0.1218741551$	
$p_1 = 0.006206866728$	$q_1 = p_1 + w_1 = 0.1280810218$	
$\mu_2 = 1.229183711$	$\bar{r}_2 = 1.086748630$	$\bar{r}_2 \eta = 0.1849321372$
$c_1 = 0.3223137037$	$w_2 = \beta \eta \mu_2 c_1 = 0.02249530917$	
$p_2 = 0.001215093015$	$q_2 = p_2 + w_2 = 0.02371040218$	
$\mu_3 = 1.249186897$	$\bar{r}_3 = 1.162644344$	$\bar{r}_3 \eta = 0.1978473194$
$c_2 = 0.007808702470$	$w_3 = \beta \eta \mu_3 c_2 = 0.0005538634354$	
$p_3 = 7.121800661 \times 10^{-7}$	$q_3 = p_3 + w_3 = 0.0005545756155$	
$\mu_4 = 1.252445067$	$\bar{r}_4 = 1.174776832$	$\bar{r}_4 \eta = 0.1999119063$
$c_3 = 0.00003038079472$	$w_4 = \beta \eta \mu_4 c_3 = 0.000002160499729$	
$p_4 = 6.452032164 \times 10^{-11}$	$q_4 = p_4 + w_4 = 0.000002160564249$	
$\mu_5 = 1.252520022$	$\bar{r}_5 = 1.175055202$	$\bar{r}_5 \eta = 0.1999592765$



$c_4 = 2.725422817 \times 10^{-9}$	$w_5 = \beta \eta \mu_5 c_4 = 1.938273079 \times 10^{-10}$	
$p_5 = 2.250940548 \times 10^{-17}$	$q_5 = p_5 + w_5 = 1.938273304 \times 10^{-10}$	
$\mu_6 = 1.252520313$	$\bar{r}_6 = 1.175056282$	$\bar{r}_6 \eta = 0.1999594603$
$c_5 = 9.495486543 \times 10^{-16}$	$w_6 = \beta \eta \mu_6 c_5 = 6.753025660 \times 10^{-17}$	
$p_6 = 7.035192511 \times 10^{-28}$	$q_6 = p_6 + w_6 = 6.753025660 \times 10^{-17}$	
$\mu_7 = 1.252520313$	$\bar{r}_7 = 1.175056282$	$\bar{r}_7 \eta = 0.1999594603$

We can stop the process, since  $\bar{r}_7 = \bar{r}_6$ . Then, we set  $r \simeq \bar{r}_7 = 1.175056282$ . Consequently

$$\bar{r}_0 = 2.184701893$$

and

$$\mathcal{D}_0 = U(x_0, 2.184701893) \cap \mathcal{D} = \mathcal{D}.$$

The hypotheses of Theorem 2.5 are satisfied. Hence, equation  $F(x) = 0$  has a solution

$$x^* = (\sqrt[3]{0.49}, \sqrt[3]{0.49})^T = (0.788373516, 0.788373516)^T,$$

which is unique in  $\mathcal{D}_0$  and can be obtained as the limit of  $\{x_k\}$  starting at  $x_0$ .

**Example 3.2.** In this example we present an application of the previous analysis to the Chandrasekhar equation [16]:

$$x(s) = 1 + \frac{s}{4} x(s) \int_0^1 \frac{x(t)}{s+t} dt, \quad s \in [0, 1]. \tag{3.3}$$

We determine where a solution is located, along with its region of uniqueness. Later, the solution is approximated by an iterative method of (CSTM).

Note that solving (3.3) is equivalent to solve  $F(x) = 0$ , where  $F : C[0, 1] \rightarrow C[0, 1]$  and

$$[F(x)](s) = x(s) - 1 - \frac{s}{4} x(s) \int_0^1 \frac{x(t)}{s+t} dt, \quad s \in [0, 1]. \tag{3.4}$$

To obtain the existence of a unique solution of  $F(x) = 0$ , where  $F$  is given in (3.4), we need to evaluate  $d, \beta, \eta, M$  from operator (3.4) and the starting points  $x_{-1}$  and  $x_0$ . In addition, from (3.4), we have

$$\begin{aligned} [F'(x)y](s) &= y(s) - \frac{s}{4} x(s) \int_0^1 \frac{y(t)}{s+t} dt - \frac{s}{4} y(s) \int_0^1 \frac{x(t)}{s+t} dt, \quad s \in [0, 1], \\ [x, y; F]z(s) &= \int_0^1 F'(y + \tau(x - y))z(s) \tau \\ &= z(s) - \frac{1}{8} \int_0^1 \frac{s}{s+t} (3x(s) - y(s))z(t) + z(s)(3x(t) - y(t)) dt. \end{aligned}$$

On the other hand, from (3.3), we infer that  $x(0) = 1$ , so that reasonable choices of initial approximations seem to be  $x_{-1}(s) = 0.99$  and  $x_0(s) = 1$ , for all  $s \in [0, 1]$ , and  $d = \|x_0 - x_{-1}\| = 0.01$ . In consequence,

$$\begin{aligned} \|I - A_0\| &= \frac{1}{8} \max_{s \in [0,1]} \left| \int_0^1 \frac{s}{s+t} (3x(s) - y(s))z(t) + z(s)(3x(t) - y(t)) dt \right| \\ &\leq \frac{\ln 2}{4} \|3x_0 - x_{-1}\| \leq \frac{201}{400} \ln 2 < 1. \end{aligned}$$

Hence, by the Banach lemma, there exists  $A_0^{-1}$  and

$$\|A_0^{-1}\| \leq \frac{1}{1 - \|I - A_0\|} \leq \frac{400}{400 - 201 \ln 2} = 1.534463572 = \beta.$$

Moreover,

$$\|A_0^{-1}F(x_0)\| \leq \frac{100 \ln 2}{400 - 201 \ln 2} = 0.2659022747 = \eta.$$

Furthermore,

$$\|A_0^{-1}([x, y; F] - F'(z))\| \leq \frac{\ln 2}{4} \beta (\|y - z\| + \|x - z\|) \quad \text{and} \quad M = \frac{\ln 2}{2} \beta = 0.5318045495.$$

If we now choose  $a = b = 1/2$ ,  $c = 1$ , and using Maple 13, then

$$\gamma = 0.1084927426, \quad q_{-1} = d = 0.01, \quad d_0 = 0.03760780163, \quad \alpha = 0.1127416734 < 1,$$

$p_0 = 0.02916327242$	$q_0 = 1.029163272$	$\mu_1 = 1.127067473$
$\bar{r}_1 = 0.5245816365$	$\bar{r}_1 \eta = 0.1394874504$	
$c_0 = 0.3521792331$	$w_1 = 0.1619542298$	$p_1 = 0.0009008346109$
$q_1 = 0.1628550644$	$\mu_2 = 1.319377855$	$\bar{r}_2 = 1.120590804$
$\bar{r}_2 \eta = 0.2979676438$		
$c_1 = 0.07552214604$	$w_2 = 0.04065576601$	$p_2 = 0.0005330243797$
$q_2 = 0.04118879039$	$\mu_3 = 1.359072886$	$\bar{r}_3 = 1.222612732$
$\bar{r}_3 \eta = 0.3250955065$		
$c_2 = 0.003224873715$	$w_3 = 0.001788274139$	$p_3 = 0.000005548217528$
$q_3 = 0.001793822357$	$\mu_4 = 1.367741311$	$\bar{r}_4 = 1.244104037$
$\bar{r}_4 \eta = 0.3308100934$		
$c_3 = 0.00003050725442$	$w_4 = 0.00001702494524$	$p_4 = 2.276647831 \times 10^{-9}$
$q_4 = 0.00001702722189$	$\mu_5 = 1.368108938$	$\bar{r}_5 = 1.245009462$
$\bar{r}_5 \eta = 0.3310508480$		
$c_4 = 1.227832032 \times 10^{-8}$	$w_5 = 6.853908064 \times 10^{-9}$	$p_5 = 8.662850931 \times 10^{-15}$
$q_5 = 6.853916727 \times 10^{-9}$	$\mu_6 = 1.368112397$	$\bar{r}_6 = 1.245017978$
$\bar{r}_6 \eta = 0.3310531124$		
$c_5 = 4.656304816 \times 10^{-14}$	$w_6 = 2.5992126 \times 10^{-14}$	$p_6 = 1.322129312 \times 10^{-23}$
$q_6 = 2.599212677 \times 10^{-14}$	$\mu_7 = 1.368112398$	$\bar{r}_7 = 1.245017980$
$\bar{r}_7 \eta = 0.3310531129$		
$c_6 = 7.105511134 \times 10^{-23}$	$w_7 = 3.966392956 \times 10^{-23}$	$p_7 = 7.651208305 \times 10^{-38}$
$q_7 = 3.966392956 \times 10^{-23}$	$\mu_8 = 1.368112398$	$\bar{r}_8 = 1.245017980$
$\bar{r}_8 \eta = 0.3310531129$		

We stop the process, since  $\bar{r}_8 = \bar{r}_7$ . Then, we set  $r \simeq \bar{r}_8 = 1.245017980$ . Consequently

$$\bar{r}_0 = 1.195858164.$$

The conditions of [Theorem 2.5](#) are satisfied. In consequence, Eq. (3.3) has a solution  $x^*$  in  $\{\varphi \in C[0, 1]; \|\varphi - 1\| \leq 0.3310531129\}$ .

To obtain a numerical solution of (3.3), we first discretize the problem and approach the integral by a Gauss–Legendre numerical quadrature with eight nodes,

$$\int_0^1 f(t) dt \approx \sum_{j=1}^8 w_j f(t_j).$$

If we denote  $x_i = x(t_i)$ ,  $i = 1, 2, \dots, 8$ , Eq. (3.3) is transformed into the following nonlinear system:

$$x_i = 1 + \frac{x_i}{4} \sum_{j=1}^8 a_{ij} x_j, \quad i = 1, 2, \dots, 8,$$

where  $a_{ij} = \frac{s_i w_j}{s_i + s_j}$ .

Denote now  $\bar{x} = (x_1, x_2, \dots, x_8)^T$ ,  $\bar{1} = (1, 1, \dots, 1)^T$ ,  $A = (a_{ij})$  and write the last nonlinear system in the matrix form:

$$\bar{x} = \bar{1} + \frac{1}{4} \bar{x} \odot (A\bar{x}), \quad (3.5)$$

where  $\odot$  represents the inner product. If we choose  $\bar{x}_0 = (1, 1, \dots, 1)^T$  and  $\bar{x}_{-1} = (0.99, 0.99, \dots, 0.99)^T$ , after eight iterations by applying method (CSTM) with  $a = b = 1/2$  and  $c = 1$ , and using the stopping criterion  $\|\bar{x}_n - \bar{x}^*\| < 10^{-100}$ , we obtain the numerical solution  $\bar{x}^* = (x_1^*, x_2^*, \dots, x_8^*)^T$  given in [Table 1](#).

Moreover, if we consider the computational order of convergence  $\rho$  (see [2]),

$$\rho \approx \ln \left( \frac{\|\bar{x}_{n+1} - \bar{x}^*\|_\infty}{\|\bar{x}_n - \bar{x}^*\|_\infty} \right) / \ln \left( \frac{\|\bar{x}_n - \bar{x}^*\|_\infty}{\|\bar{x}_{n-1} - \bar{x}^*\|_\infty} \right), \quad n \in \mathbb{N},$$

we obtain  $\rho = 1.6249 \dots$

**Table 1**

Numerical solution  $\bar{x}^* = (x_1^*, x_2^*, \dots, x_8^*)^T$  of system (3.5).

$j$	$x_j^*$	$j$	$x_j^*$	$j$	$x_j^*$	$j$	$x_j^*$
1	1.0217...	3	1.1259...	5	1.2035...	7	1.2420...
2	1.0733...	4	1.1701...	6	1.2269...	8	1.2499...

**Example 3.3.** The last example is devoted to illustrate the numerical behavior of the methods introduced in (CSTM). To do this, we consider as a function test the integral equation

$$x(s) = s - \frac{s}{2} \int_0^1 \cos(x(t)) dt, \quad s \in [0, 1], \tag{3.6}$$

introduced by Döring in [22].

The choice of the integral Eq. (3.6) as a function test is based in two points. First, it is known its exact solution:  $x^*(s) = \varrho s$ , where  $\varrho \approx 0.522$ , is a solution of the nonlinear equation

$$2t^2 - 2t + \sin t = 0.$$

Second, all the functional compositions derived from the application of (CSTM) can be explicitly computed. Actually, we rewrite Eq. (3.6) in the form  $F(x) = 0$ , where  $F : C[0, 1] \rightarrow C[0, 1]$  and

$$[F(x)](s) = x(s) - s + \frac{s}{2} \int_0^1 \cos(x(t)) dt \quad s \in [0, 1]. \tag{3.7}$$

Then for each  $x, y, z \in C[0, 1]$  the divided difference operator  $[x, y; F](z)$  is defined as follows

$$[x, y; F](z)(s) = z(s) + \frac{s}{2} \int_0^1 z(t)K(x, y; t) dt \quad s \in [0, 1],$$

where

$$K(x, y; t) = \frac{\cos(x(t)) - \cos(y(t))}{x(t) - y(t)}.$$

In addition,

$$[x, y; F]^{-1}(z)(s) = z(s) + \frac{\int_0^1 sz(t)K(x, y; t) dt}{2 + \int_0^1 tK(x, y; t) dt} \quad s \in [0, 1].$$

In the numerical experiment we compare some Chebyshev–Secant-type methods obtained for  $a = b = 1$  and  $c$  a free parameter,

$$\begin{cases} x_{-1}, x_0 \in C[0, 1], \\ y_k = x_k - A_k^{-1} F(x_k), & A_k = [x_{k-1}, x_k; F], \quad (k \geq 0) \\ x_{k+1} = y_k - cA_k^{-1} F(y_k), \quad (k \geq 0) \end{cases} \tag{3.8}$$

with the classical Secant method

$$x_{k+1} = x_k - A_k^{-1} F(x_k), \quad x_{-1}, x_0 \in C[0, 1], \quad (k \geq 0). \tag{3.9}$$

Notice that the Secant method (3.9) is included in the family (3.8). In fact it corresponds with the case  $c = 0$ .

In this example, we are not interested in checking if the convergence conditions are satisfied or not, but comparing the numerical behavior of the sequences obtained by applying methods (3.8) to the operator defined in (3.7).

If we consider two initial approximations in the form  $x_{-1}(s) = \zeta_{-1}s$ ,  $x_0(s) = \zeta_0s$ ,  $\zeta_{-1}, \zeta_0 \in \mathbb{R}$ , and we apply methods (3.8) to operator (3.7), we obtain the iterates

$$x_k(s) = \varrho_k s, \quad k \geq 1,$$

where the sequence  $\{\varrho_k\}$  is defined as follows:

$$\begin{cases} \varrho_{-1} = \zeta_{-1}, \quad \varrho_0 = \zeta_0, \\ \beta_k = \varrho_k - g(\varrho_k) \frac{\varrho_k - \varrho_{k-1}}{g(\varrho_k) - g(\varrho_{k-1})} \\ \varrho_{k+1} = \beta_k - cg(\beta_k) \frac{\varrho_k - \varrho_{k-1}}{g(\varrho_k) - g(\varrho_{k-1})}, \quad (k \geq 0), \end{cases} \tag{3.10}$$

**Table 2**

Errors obtained by different methods of (3.8) when the solution  $x^*(s)$  of Eq. (3.6) is approximated.

$e_k$	$c = 0$	$c = 0.25$	$c = 0.5$	$c = 0.75$	$c = 1$
1	$1.1 \times 10^{-2}$	$8.7 \times 10^{-3}$	$6.0 \times 10^{-3}$	$3.4 \times 10^{-3}$	$6.9 \times 10^{-4}$
2	$3.1 \times 10^{-4}$	$1.8 \times 10^{-4}$	$8.4 \times 10^{-5}$	$2.5 \times 10^{-5}$	$5.0 \times 10^{-7}$
3	$2.9 \times 10^{-7}$	$9.7 \times 10^{-8}$	$2.1 \times 10^{-8}$	$1.7 \times 10^{-9}$	$1.7 \times 10^{-15}$
4	$7.5 \times 10^{-12}$	$1.1 \times 10^{-12}$	$7.4 \times 10^{-14}$	$8.9 \times 10^{-16}$	0

where the auxiliary function  $g(t)$  is:

$$g(t) = t - t - \frac{\sin t}{2t}. \quad (3.11)$$

If we consider the max-norm  $\|x\|_\infty = \max_{t \in [0,1]} |x(t)|$ , the errors  $e_k = \|x_k - x^*\|_\infty$  can be obtained from the sequence (3.10):

$$e_k = |\varrho^* - \varrho_k|, \quad \varrho = 0.5224366093993514 \dots$$

These errors are shown, for different values of the parameter  $c$ , in Table 2. Notice that the smaller errors are obtained for  $c = 1$ .

## Conclusion

We provided a semilocal convergence analysis of (CSTM) for approximating a locally unique solution of an equation in a Banach space. Using a combination of Lipschitz and center-Lipschitz conditions, instead of only Lipschitz conditions [1], we provided an analysis with a larger convergence domain and weaker sufficient convergence conditions than in [1]. Note that these advantages are obtained under the same computational cost as in [1], since in practice the computation of the Lipschitz constant  $M$  requires the computation of  $M_0$ . Hence, the applicability of (CSTM) has been extended.

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