Physics Letters B 750 (2015) 39–44

Contents lists available at ScienceDirect

Physics Letters B

www.elsevier.com/locate/physletb

A Chern–Simons gravity action in $d = 4$

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ARTICLE INFO

Article history:
Received 2 July 2015
Received in revised form 10 August 2015
Accepted 12 August 2015
Available online 14 August 2015
Editor: M. Cvetič

ABSTRACT

Recently, Antoniadis, Konitopoulos and Savvidy have introduced in Refs. [1–4] a procedure to construct background-free gauge invariants, using non-abelian gauge potentials described by forms of higher degree. Their construction is particularly useful because it can be used in both, odd- and even-dimensional spacetimes. Using their technique, we generalize the Chern–Weil theorem and construct a gauge-invariant, $(2n+2)$-dimensional transgression form, and study its relationship with the generalized Chern–Simons forms introduced in Refs. [1,2].

Using the methods for FDA manipulation and decomposition in 1-forms developed in Ref. [5] and applied in Refs. [6] and [7], we construct a four-dimensional Chern–Simons gravity action, which is off-shell gauge invariant under the Maxwell algebra.

1. Introduction

Many models in string theory predict an infinite tower of particles of arbitrary high spin in their spectrum (e.g., see Refs. [8,9]). For instance, in the low energy limit of open string theory with Chan–Paton charges [10], the massless states can be identified with Yang–Mills quanta.

A text book example of the situation is provided by the so-called Kalb–Ramond field (also known as NS–NSB field). It is a 2-form quantum field, i.e. an antisymmetric tensor field $B_{\mu\nu}$ with two indices. The gauge transformation $\delta B_{\mu\nu}(x) = \partial_\mu \varepsilon_\nu(x) - \partial_\nu \varepsilon_\mu(x)$ leaves invariant the 3-form field strength $H_{\mu\nu\rho}(x) = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}$. Therefore when a background metric is provided, it is possible to construct the invariant action principle [8] $S_{\text{KR}} = -\frac{1}{12}\int \mathrm{d}^D x\sqrt{|g|}H_{\mu\nu\rho}H^{\mu\nu\rho} = -\frac{1}{2}\int H \wedge *H$ for the fields.[1]

These fields of rank two and higher are interesting as part of the spectrum of a QFT. This motivates a generalization of Yang–Mills symmetry to include forms of higher degree as non-abelian gauge fields. In recent times, important research has been carried out in this direction, see for instance Refs. [11–15].

In particular, in Ref. [1] this idea of using forms of higher degree as non-abelian gauge fields was used to construct gauge invariant Lagrangian forms which are independent of the metric. These forms are analogous to the Pontryagin-forms in Yang–Mills gauge theory.

These results were generalized in Refs. [2–4]. There were found closed invariant forms, similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory. These forms are based on non-abelian tensor gauge fields and are polynomial on the corresponding curvature forms.

It is the purpose of this paper to extend these results to the case of the Chern–Weil theorem and transgression forms in $(2n+2)$ dimensions. Using this generalized case, we will construct a four-dimensional Chern–Simons gravity action invariant under the Maxwell algebra. This is accomplished using the formalism developed in Refs. [1–4] and afterwards using the methods developed in Ref. [5], and which were later applied in Refs. [6] and [7].

This paper is organized as follows. In Section 2 we briefly review the usual Chern–Simons theory and the non-abelian tensor gauge theory. In Section 3 we study a generalization of the Chern–Weil theorem. Taking this theorem as the starting point, it is possible to construct generalized $(2n+2)$-dimensional trangression forms, which allows us to reproduce the $(2n+2)$-dimensional Chern–Simons forms obtained in Refs. [1,2]. These mathematical results are used in Section 4 to study the construction of an off-shell gauge-invariant Chern–Simons–Antoniadis–Savvidy action for gravity in $d = 4$. We finish in Section 5 with some final remarks and some considerations on future possible developments.

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[1] From now on and to make the notation less cumbersome, the wedge '∧' product between differential forms will be implicitly assumed.

http://dx.doi.org/10.1016/j.physletb.2015.08.030
0370-2693/

2. Gravitation as a gauge theory

Quantum chromodynamics and the Weinberg–Salam electroweak unification are described by gauge theories. The same is true for GUTs. However, the standard theory of General Relativity for gravity does not correspond to a genuinely off-shell gauge invariant theory, even though General Relativity and Yang–Mills theories have similar geometric foundations.

Yang–Mills theories require a background metric in order to construct the action principle. On the contrary, an authentic gauge invariant theory of gravity requires a background-free action principle. An action for gravity fulfilling this condition is provided by Chern–Simons and Transgression form gravities. This kind of theories have been extensively studied (see for instance Refs. [16–25]), but the construction is possible only in the case of odd-dimensional spacetimes. In this work we consider the construction of a transgression form in even dimensions and a four-dimensional Chern–Simons gravity action. The resulting theory is off-shell gauge invariant under the Maxwell algebra.

2.1. Chern–Weil theorem: Chern–Simons and transgressions forms

Chern–Simons $(2n+1)$-forms can be obtained as local potentials for the $(2n+2)$-Pontryagin–Chern forms

$$\mathcal{P}_{2n+2} = \langle F^{n+1} \rangle, \tag{1}$$

where $\langle \cdots \rangle$ stands for a multilineal symmetric invariant polynomial for the Lie algebra

$$\langle \cdots \rangle : \mathfrak{g}^{n+1} \to \mathbb{R},$$

as for instance the one provided by the symmetrized trace in some matrix representation of the Lie algebra (see [26]), i.e., $\langle F^{n+1} \rangle = \mathrm{Str}\,(\underbrace{F \wedge \cdots \wedge F}_{n+1})$.

The Pontryagin–Chern forms (1) which satisfy the condition $\mathrm{d}\mathcal{P}_{2n+2} = 0$, where $F = \mathrm{d}A + A^2$ is the 2-form Yang–Mills field-strength of the 1-form vector field A. From the Poincaré lemma, we know that *locally* there exists a $(2n+1)$-form $\mathcal{C}_{2n+1}$ such that $\mathcal{P}_{2n+2} = \mathrm{d}\mathcal{C}_{2n+1}$. This $(2n+1)$-form $\mathcal{C}_{2n+1}$ is called a Chern–Simons form.

It is easily checked that the Chern–Simons form is locally quasi-invariant under gauge transformations (i.e. invariant modulo closed forms) [32]. In order to find an explicit expression for the Chern–Simons form, we have to make use of the Chern–Weil theorem, which we will sketch briefly.

2.1.1. Chern–Weil theorem

Let A_0 and A_1 be two one-form gauge connections on a fiber bundle over a $(2n+1)$-dimensional base manifold M, and let F_0 and F_1 be the corresponding curvatures. Then, the difference of Pontryagin–Chern forms is exact,

$$\left\langle F_1^{n+1} \right\rangle - \left\langle F_0^{n+1} \right\rangle = \mathrm{d}\mathcal{T}^{(2n+1)}\left(A_1, A_0\right), \tag{2}$$

where

$$\mathcal{T}^{(2n+1)}\left(A_1, A_0\right) = (n+1) \int_0^1 \mathrm{d}t \left\langle \Theta F_t^n \right\rangle \tag{3}$$

is called a transgression $(2n+1)$-form, where $\Theta = A_1 - A_0$ and $A_t = A_0 + t\Theta$. The 2-form F_t stands for the field-strength of the 1-form connection A_t, $F_t = \mathrm{d}A_t + A_t A_t$.

Setting $A_0 = 0$ and $A_1 = A$ in (3), we obtain the well known Chern–Simons $(2n+1)$-form

$$\mathcal{C}_{2n+1}\left(A\right) = \mathcal{T}^{(2n+1)}\left(A, 0\right) = (n+1) \int_0^1 \mathrm{d}t \langle A \left(t\mathrm{d}A + t^2 A^2\right)^n \rangle. \tag{4}$$

From the Chern–Weil theorem it is straightforward to show that under gauge transformations $d\delta\mathcal{C}^{(2n+1)}\left(A\right) = 0$, i.e. the Chern–Simons form is quasi-invariant. However, it is important to stress that since a connection cannot be globally set to zero unless the bundle (topology) is trivial, Chern–Simons forms turn out to be only locally defined.

These properties imply that Chern–Simons forms have nice features as Lagrangians: (i) they lead to gauge theories with a fiber-bundle structure, whose only dynamical field is a one-form gauge connection A, and (ii) they do change by only a total derivative under gauge transformations. When we choose $\mathfrak{g} = \mathfrak{so}\left(2n+2\right)$ we can write $A = \frac{1}{l} e^a P_a + \frac{1}{2} \omega^{ab} J_{ab}$, and therefore the Chern–Simons form provides with a *background-free* gravity theory in $d = 2n+1$. The main drawback of the construction is that transgression and Chern–Simons Lagrangians seem to be intrinsically odd-dimensional. In the following sections we will show how this issue can be circumvented.

2.2. Non-abelian tensor gauge fields

The idea of extending the Yang–Mills fields to higher rank tensor gauge fields was used in Ref. [1] in order to construct gauge invariant and metric independent forms in higher dimensions. These forms are analogous to the Pontryagin–Chern forms in Yang–Mills gauge theory.

These results were generalized in Refs. [2–4], where the authors found closed invariant forms similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory. These forms are based on non-abelian tensor gauge fields and are polynomial on the corresponding curvature forms.

A Lie algebra valued, 1-form connection A can be written making more or less explicit dependence on the Lie algebra generator basis T_a or the basis of 1-forms $\mathrm{d}x^\mu$,

$$A = A_\mu \otimes \mathrm{d}x^\mu = A^a{}_\mu T_a \otimes \mathrm{d}x^\mu.$$

The same is true for the gauge potential 2-form $B = \frac{1}{2!} B_{\mu\nu} \otimes \mathrm{d}x^\mu \mathrm{d}x^\nu = \frac{1}{2!} B^a{}_{\mu\nu} T_a \otimes \mathrm{d}x^\mu \mathrm{d}x^\nu$. The corresponding 2-form and 3-form "curvatures" are given by $F = \frac{1}{2!} F_{\mu\nu} \otimes \mathrm{d}x^\mu \mathrm{d}x^\nu$ and $H = \frac{1}{3!} H_{\mu\nu\lambda} \otimes \mathrm{d}x^\mu \mathrm{d}x^\nu \mathrm{d}x^\lambda$ respectively, where

$$F = \mathrm{d}A + A^2, \quad H = \mathrm{D}B = \mathrm{d}B + [A, B]. \tag{5}$$

The curvatures F and H satisfy the Bianchi identities,

$$\mathrm{D}F = 0, \tag{6}$$

$$\mathrm{D}H + [B, F] = 0. \tag{7}$$

The infinitesimal, non-abelian gauge transformations of the generalized gauge fields are given by

$$\delta A = \mathrm{D}\xi_0, \tag{8}$$

$$\delta B = \mathrm{D}\xi_1 + [B, \xi_0], \tag{9}$$

where ξ_0 is a 0-form gauge parameter $\xi_0 = \xi^a T_a$ and ξ_1 is a 1-form gauge parameter $\xi_1 = \xi^a{}_\mu T_a \otimes \mathrm{d}x^\mu$ [1]. Under these gauge transformations, the curvatures transform as [2]

$$\delta F = \mathrm{D}(\delta A) = [F, \xi_0] \tag{10}$$

$$\delta H = \mathrm{D}(\delta B) + [\delta A, B] \tag{11}$$

2.3. Chern–Simons forms in $(2n+2)$ dimensions

In Refs. [1,2] there were found closed invariant forms similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory. In particular, it was found that there exists a gauge invariant metric-independent invariant $\Gamma(A)$ in $(2n+3)$-dimensional space-time

$$\Gamma_{2n+3} = \langle F^n H \rangle \tag{12}$$

where $H = \mathrm{d}B + [A, B]$ is the 3-form field-strength tensor for the rank-2 gauge field B. By direct computation of the derivative it is possible to prove that Γ_{2n+3} is a closed form, $\mathrm{d}\Gamma_{2n+3} = 0$ (see the proof in Ref. [2]). According to the Poincaré lemma, this implies that Γ_{2n+3} can be locally written as an exterior differential of a certain $(2n+2)$-form. In order to find this potential $(2n+2)$-form, the variation of Γ_{2n+3} induced by a variation of A and B is computed. Since

$$\delta F = \mathrm{D}(\delta A), \quad \delta H = \mathrm{D}(\delta B) + [\delta A, B], \tag{13}$$

the variation $\delta\Gamma_{2n+3}$ is given by

$$\begin{aligned}\delta\Gamma_{2n+3} &= \langle \delta F F^{n-1} H + \cdots + F^{n-1}\delta F H + F^n \delta H \rangle \\ &= \mathrm{d}\langle \delta A F^{n-1} H + \cdots + F^{n-1}\delta A H + F^n \delta B \rangle\end{aligned} \tag{14}$$

Following Ref. [26], we introduce a one-parameter family of potentials and strengths through the parameter t, $0 \le t \le 1$:

$$A_t = tA, \quad F_t = tF + (t^2 - t)A^2,$$

$$B_t = tB, \quad H_t = tH + (t^2 - t)[A, B].$$

When a variation of the form $\delta = \delta t(\partial/\partial t)$ is chosen, we have $\delta A_t = \delta t A$ and $B_t = \delta t B$. From eq. (14), we have

$$\Gamma_{2n+3} = \langle F^n H \rangle = \mathrm{d}\mathfrak{C}^{(2n+2)}_{\mathrm{ChSAS}}, \tag{15}$$

where the $(2n+2)$-form $\mathfrak{C}^{(2n+2)}_{\mathrm{ChSAS}}$, is what we will call a "Chern–Simons–Antoniadis–Savvidy" form, and it is given explicitly by

$$\mathfrak{C}^{(2n+2)}_{\mathrm{ChSAS}}(A, B) = \int_0^1 \mathrm{d}t \langle A F_t^{n-1} H_t + \ldots + F_t^{n-1} A H_t + F_t^n B \rangle. \tag{16}$$

This result is analogous to the usual Chern–Simons form (4), but in even dimensions [2]. From eq. (16), we have for the case $n = 1$ [2],

$$\mathfrak{C}^{(4)}_{\mathrm{ChSAS}} = \int_0^1 \mathrm{d}t \langle A H_t + F_t B \rangle = \langle F B \rangle. \tag{17}$$

This means that the four-dimensional Chern–Simons–Antoniadis–Savvidy action is given by

$$S(A, B) = \int_{\mathcal{M}^4} \langle F B \rangle. \tag{18}$$

Using eqs. (10) and (9), it is direct to prove that the action eq. (18) is gauge invariant (modulo boundary terms) under the transformations eqs. (8) and (9).

3. Transgressions forms in $(2n+2)$ dimensions

In this section we prove that it is possible to generalize the transgression form and the Chern–Weil theorem to the $(2n+2)$-dimensional case. The theorem ingredients are: (i) Two Lie-algebra valued, connection 1-forms A_0 and A_1. Their curvatures are given by $F_0 = \mathrm{d}A_0 + A_0^2$ and $F_1 = \mathrm{d}A_1 + A_1^2$, respectively. (ii) Two Lie-algebra valued, generalized connection 2-forms B_0 and B_1. Their generalized curvatures are given by $H_0 = \mathrm{d}B_0 + [A_0, B_0]$ and $H_1 = \mathrm{d}B_1 + [A_1, B_1]$, respectively (iii). In terms of these fundamental ingredients, it is possible to define the differences $\theta = A_1 - A_0$ and $\Phi = B_1 - B_0$, and the interpolating connections $A_t = A_0 + t\theta$ and $B_t = B_0 + t\Phi$. Their curvatures are given by

$$F_t = \mathrm{d}A_t + A_t^2, \tag{19}$$

$$H_t = \mathrm{D}_t B_t = \mathrm{d}B_t + [A_t, B_t]. \tag{20}$$

They satisfy the conditions

$$\frac{\mathrm{d}}{\mathrm{d}t} F_t = \mathrm{D}_t \theta \tag{21}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} H_t = \mathrm{D}_t \Phi + [\theta, B_t] \tag{22}$$

3.1. Generalized Chern–Weil theorem

Let A_0 and A_1 be two gauge connection 1-forms, and let F_0 and F_1 be their corresponding curvature 2-forms. Let B_0 and B_1 be two gauge connection 2-forms and let H_0 and H_1 be their corresponding curvature 3-forms. Then, the difference $\Gamma^{(1)}_{2n+3} - \Gamma^{(0)}_{2n+3}$ is an exact form,

$$\Gamma^{(1)}_{2n+3} - \Gamma^{(0)}_{2n+3} = \langle F_1^n H_1 \rangle - \langle F_0^n H_0 \rangle = \mathrm{d}\mathfrak{T}^{(2n+2)}(A_0, B_0; A_1, B_1), \tag{23}$$

where

$$\mathfrak{T}^{(2n+2)}(A_0, B_0; A_1, B_1) = \int_0^1 \mathrm{d}t \left(n \langle F^{n-1} \theta H_t \rangle + \langle F_t^n \Phi \rangle \right) \tag{24}$$

is what we call a "Antoniadis–Savvidy transgression form".

Proof. Let us start writing the LHS of eq. (23) as

$$\langle F_1^n H_1 \rangle - \langle F_0^n H_0 \rangle = \int_0^1 \mathrm{d}t \frac{\mathrm{d}}{\mathrm{d}t} \langle F_t^n H_t \rangle.$$

Using eqs. (21) and (22),

$$\begin{aligned}&\Gamma^{(1)}_{2n+3} - \Gamma^{(0)}_{2n+3} \\ &= \int_0^1 \mathrm{d}t \left(\left\langle n F_t^{n-1} \frac{\mathrm{d}F_t}{\mathrm{d}t} H_t \right\rangle + \left\langle F_t^n \frac{\mathrm{d}H_t}{\mathrm{d}\tau} \right\rangle \right), \\ &= \int_0^1 \mathrm{d}t \left(n \left\langle F_t^{n-1} \mathrm{D}_\tau \theta H_t \right\rangle + \mathrm{d} \left\langle F_t^n \Phi \right\rangle - (-1)^p \left\langle [B_t, \theta] F_t^n \right\rangle \right).\end{aligned}$$

Since

$$n \left\langle F_t^{n-1} \mathrm{D}_\tau \theta H_t \right\rangle = n \mathrm{d} \left\langle F_t^{n-1} \theta H_t \right\rangle - \left\langle \theta \left[B_t, F_t^n \right] \right\rangle,$$

we have

$$\langle F_1^n H_1\rangle - \langle F_0^n H_0\rangle = \int_0^1 \mathrm{d}t \left(n\mathrm{d}\left\langle F_t^{n-1}\theta H_t\right\rangle + \mathrm{d}\left\langle F_t^n \Phi\right\rangle - (-1)^p \left\langle \left[B_t, \theta F_t^n\right]\right\rangle \right), = \mathrm{d}\int_0^1 \mathrm{d}t \left(n\left\langle F_t^{n-1}\theta H_t\right\rangle + \left\langle F_t^n \Phi\right\rangle \right).$$

Therefore, defining the $(2n+2)$-Antoniadis–Savvidy-transgression form as

$$\mathfrak{T}^{(2n+2)}(A_0, B_0; A_1, B_1) = \int_0^1 \mathrm{d}t \left(n\langle F^{n-1}\theta H_t\rangle + \langle F_t^n \Phi\rangle \right),$$

we have

$$\langle F_1^n H_1\rangle - \langle F_0^n H_0\rangle = \mathrm{d}\mathfrak{T}^{(2n+2)}(A_0, B_0; A_1, B_1). \qquad \square$$

Following the procedure followed in the case of the Chern–Simons forms, we define the $(2n+2)$-Chern–Simons–Antoniadis–Savvidy form as

$$\mathfrak{C}_{\mathrm{ChSAS}}^{(2n+2)} = \mathfrak{T}^{(2n+2)}(A, B; 0, 0) = \int_0^1 \mathrm{d}t \langle nAF_t^{n-1}H_t + BF_t^n\rangle.$$

This result agrees with the expression found by Antoniadis and Savvidy in Refs. [1,2]. It is interesting to notice that transgression forms (both, standard ones and the above generalization) are defined globally on the spacetime basis manifold of the principal bundle, and are off-shell gauge invariant. Chern–Simons forms (both, standard ones and the Antoniadis–Savvidy generalization) are locally defined and are off-shell gauge invariant only up to boundary terms (i.e., quasi-invariants). Physical consequences of this subtle difference between Chern–Simons and transgression forms has been studied in the literature for the case of standard odd-dimensional Chern–Simons gravity in Refs. [33,34]. What it could imply in the current approach is work in progress, as it will require a deeper exploration of the phenomenology of this kind of theories for specific symmetries. For this reason, in the next section we will study the construction of four-dimensional gravity using the Antoniadis and Savvidy [1,2] expression for $\mathfrak{C}_{\mathrm{ChSAS}}^{(2n+2)}$ with $n = 1$ and the Maxwell algebra as gauge symmetry.

4. Chern–Simons–Antoniadis–Savvidy form for the Maxwell algebra

We have seen that the four-dimensional Chern–Simons–Antoniadis–Savvidy action corresponds to

$$S_{\mathrm{ChSAS}}(A, B) = \int_{\mathcal{M}^4} \langle FB\rangle, \tag{25}$$

and it is invariant (modulo boundary terms) under the gauge transformations eqs. (8) and (9) [1,2]. Now we will use this construction for the particular case of the Maxwell algebra, in order to show the connection between eq. (25) and gravity in $d = 4$.

4.1. Maxwell algebra

The so-called Maxwell algebra was introduced in the early seventies (see Refs. [28,29]) as an algebra encoding the symmetries of a particle moving in a constant electromagnetic field. This algebra is generate by $\{P_a, J_{ab}, Z_{ab}\}$ where P_a are not common Poincaré translations. In fact, the commutation relations of the Maxwell algebra read

$$\begin{aligned}
[P_a, P_b] &= Z_{ab}, \\
[J_{ab}, P_c] &= \eta_{bc}P_a - \eta_{ac}P_b, \\
[J_{ab}, J_{cd}] &= \eta_{bc}J_{ad} + \eta_{ad}J_{bc} - \eta_{ac}J_{bd} - \eta_{bd}J_{ac}, \\
[J_{ab}, Z_{cd}] &= \eta_{bc}Z_{ad} + \eta_{ad}Z_{bc} - \eta_{ac}Z_{bd} - \eta_{bd}Z_{ac}.
\end{aligned}$$

This algebra and its invariant polynomials can be studied in the context of S-expansions (where it corresponds to the $\mathfrak{B}_4$ algebra, see Refs. [30,31]).

In order to write down a four-dimensional Chern–Simons–Antoniadis–Savvidy action for Maxwell algebra we start from the gauge connections A and B. The connection 1-form A is expressed in the Maxwell basis as

$$A = \frac{1}{l}e^a P_a + \frac{1}{2}\omega^{ab}J_{ab} + \frac{1}{2}k^{ab}Z_{ab}, \tag{26}$$

where e^a is identified as the vierbein 1-form, ω^{ab} is the spin connection 1-form, and k^{ab} is an extra antisymmetric bosonic 1-form field. The corresponding 2-form curvature $F = \mathrm{d}A + AA$ is given by

$$F = \frac{1}{l}T^a P_a + \frac{1}{2}R^{ab}J_{ab} + \frac{1}{2}F^{ab}Z_{ab}, \tag{27}$$

where T^a and R^{ab} are the standard torsion and Lorentz curvature 2-forms,

$$\begin{aligned}
T^a &= \mathrm{d}e^a + \omega^a{}_b e^b, \\
R^{ab} &= \mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb}, \\
F^{ab} &= \mathrm{D}_\omega k^{ab} + \frac{1}{l^2}e^a e^b.
\end{aligned} \tag{28}$$

From eq. (28) in the case $T^a = R^{ab} = F^{ab} = 0$, we recover the Maurer–Cartan equations for the Maxwell algebra,

$$\mathrm{d}e^a + \omega^a{}_b e^b = 0, \tag{29}$$

$$\mathrm{d}\omega^{ab} + \omega^a{}_c\omega^{cb} = 0, \tag{30}$$

$$\mathrm{D}_\omega k^{ab} + \frac{1}{l^2}e^a e^b = 0. \tag{31}$$

For the two-form B, we can write

$$B = B^a P_a + \frac{1}{2}B^{ab}J_{ab} + \frac{1}{2}\beta^{ab}Z_{ab}, \tag{32}$$

where B^a, B^{ab}, β^{ab} are 2-forms that we must determine. The corresponding 3-form curvature $H = \mathrm{D}B = \mathrm{d}B + [A, B]$ is given by

$$H = H^a P_a + \frac{1}{2}H^{ab}J_{ab} + \frac{1}{2}\Xi^{ab}Z_{ab}$$

where,

$$\begin{aligned}
H^a &= \mathrm{D}_\omega B^a - \frac{1}{l}B^a{}_b e^b \\
H^{ab} &= \mathrm{D}_\omega B^{ab} \\
\Xi^{ab} &= \mathrm{D}_\omega \beta^{ab} + k^a{}_c B^{cb} + k^b{}_c B^{ac} + \frac{1}{l}[e^a B^b - e^b B^a]
\end{aligned}$$

These equations are analogous to equation (2.13) of Ref. [5], or to equation (III.6.47) of Ref. [27], and therefore it is not a free differential algebra (FDA). But when the condition $H^a = H^{ab} = \Xi^{ab} = 0$ is imposed, we get the corresponding FDA

$$\mathrm{D}_\omega B^a - \frac{1}{l} B^a{}_b e^b = 0, \tag{33}$$

$$\mathrm{D}_\omega B^{ab} = 0, \tag{34}$$

$$\mathrm{D}_\omega \beta^{ab} + k^a{}_c B^{cb} + k^b{}_c B^{ac} + \frac{1}{l}[e^a B^b - e^b B^a] = 0. \tag{35}$$

The set of equations (29), (30), (31), (33), (34), (35) correspond to an FDA for the fields $\{e^a, \omega^{ab}, k^{ab}, B^a, B^{ab}, \beta^{ab}\}$.

The problem now is to express the form B defined by the FDA relations (33), (34) and (35) in terms of the one-forms $\{e^a, \omega^{ab}, k^{ab}\}$ of the Maxwell algebra.

To express the 2-forms $\{B^a, B^{ab}, \beta^{ab}\}$ as the wedge product of the 1-forms $\{e^a, \omega^{ab}, k^{ab}\}$ we follow a procedure developed in Refs. [5,6]. We impose the ansatz

$$B^a = \frac{a_1}{2l}\omega^a{}_b e^b + \frac{a_2}{2l} k^a{}_b e^b, \tag{36}$$

$$B^{ab} = \frac{b_1}{2l^2} e^a e^b + \frac{b_2}{2}\omega^a{}_c k^{cb} + \frac{b_3}{2} k^a{}_c k^{cb} + \frac{b_4}{2}\omega^a{}_c \omega^{cb}, \tag{37}$$

$$\beta^{ab} = \frac{c_1}{2l^2} e^a e^b + \frac{c_2}{2}\omega^a{}_c k^{cb} + \frac{c_3}{2} k^a{}_c k^{cb} + \frac{c_4}{2}\omega^a{}_c \omega^{cb}, \tag{38}$$

where $a_1, a_2, b_1, \ldots, b_4, c_1, \ldots, c_4$ are arbitrary constants. In order to fix them, we impose that the fields must satisfy the FDA conditions given by eqs. (29), (30), (31), (33), (34), (35).

Introducing eqs. (36) and (37) in eqs. (33) and (34) we find

$$a_1 = b_4, \quad a_2 = -b_1, \quad b_2 = b_3 = 0, \tag{39}$$

and using now eqs. (36), (37) and (38), we obtain

$$c_2 = 2a_1, \quad c_3 = 2a_2. \tag{40}$$

It means that the FDA fields are given by

$$B^a = \frac{a_1}{2l}\omega^a{}_b e^b + \frac{a_2}{2l} k^a{}_b e^b, \tag{41}$$

$$B^{ab} = \frac{a_1}{2}\omega^a{}_c \omega^{cb} - \frac{a_2}{2l^2} e^a e^b, \tag{42}$$

$$\beta^{ab} = \frac{c_1}{2l^2} e^a e^b + \frac{a_1}{2}\omega^a{}_c k^{cb} + \frac{a_1}{2}\omega^b{}_c k^{ac} + \frac{a_2}{2} k^a{}_c k^{cb} + \frac{a_2}{2} k^b{}_c k^{ac} + \frac{c_4}{2}\omega^a{}_c \omega^{cb}. \tag{43}$$

There are four arbitrary constants in the FDA expansion in terms of 1-forms; the fields given by eqs. (41), (42), (43) represent the most general solution that can be built from the fields $\{e^a, \omega^{ab}, k^{ab}\}$. Any choice of the constants represent a solution to the FDA.

It is interesting to note that if c_1 is a constant then it is possible to write, $c_1 = a_1 + \gamma$, where γ is another constant. Choosing $a_2 = c_4 = \gamma = 0$ leads to the solution given by

$$B = \frac{a_1}{2}[A, A]. \tag{44}$$

4.2. Chern–Simons–Antoniadis–Savvidy Lagrangian

Using the invariant tensor found in Ref. [31],

$$\langle J_{ab} J_{cd}\rangle = \alpha_0 l^2 \varepsilon_{abcd}, \quad \langle J_{ab} Z_{cd}\rangle = \alpha_2 l^2 \varepsilon_{abcd}, \tag{45}$$

being α_0 and α_2 arbitrary constants, the Chern–Simons–Antoniadis–Savvidy Lagrangian 4-form $\mathcal{L}^{(4)}_{\mathrm{ChSAS}} \equiv \mathfrak{C}^{(4)}_{\mathrm{ChSAS}}$ in 4D eq. (17) is explicitly given by

$$\mathcal{L}^{(4)}_{\mathrm{ChSAS}} = \frac{1}{4}\alpha_0 l^2 \varepsilon_{abcd} R^{ab} B^{cd} + \frac{1}{4}\alpha_2 l^2 \varepsilon_{abcd} R^{ab}\beta^{cd} + \frac{1}{4}\alpha_2 l^2 \varepsilon_{abcd}\mathrm{D}_\omega k^{ab} B^{cd} + \frac{1}{4}\alpha_2 \varepsilon_{abcd} B^{ab} e^c e^d. \tag{46}$$

Introducing the FDA expansion given by eqs. (41), (42) and (43) in (46), the Chern–Simons–Antoniadis–Savvidy Lagrangian for the Maxwell algebra takes the form

$$\begin{aligned}\mathcal{L}^{(4)}_{\mathrm{ChSAS}} = {} & \frac{\mu}{8}\varepsilon_{abcd} R^{ab} e^c e^d + \frac{\nu l^2}{8}\varepsilon_{abcd} R^{ab}\omega^c{}_f \omega^{fd} \\ & - \frac{\sigma}{8l^2}\varepsilon_{abcd}(e^a e^b e^c e^d + 2l^2 k^{ab} T^c e^d - 2l^4 R^{ab} k^c{}_f k^{fd}) \\ & + \frac{\tau}{8}\varepsilon_{abcd}(\omega^a{}_f \omega^{fb} e^c e^d + l^2 \mathrm{D}_\omega k^{ab}\omega^c{}_f \omega^{fb}) \\ & - \frac{\sigma}{8} d(\varepsilon_{abcd} k^{ab} e^c e^d),\end{aligned} \tag{47}$$

where $\mu = \alpha_2 c_1 - a_2\alpha_0$, $\nu = (\alpha_0 + 2\alpha_2)a_1 + c_4\alpha_2$, $\sigma = a_2\alpha_2$ and $\tau = a_1\alpha_2$.

From eq. (47), we can see that when $\mu \neq 0$ i.e., $\alpha_2 c_1 \neq \alpha_0 a_2$, the Chern–Simons–Antoniadis–Savvidy Lagrangian for the Maxwell algebra contains the Einstein–Hilbert term.

An interesting solution can be obtained choosing $a_1 = a_2 = 0$. In this case the fields of eqs. (41), (42), and (43) take the form

$$B^a = 0, \tag{48}$$

$$B^{ab} = 0, \tag{49}$$

$$\beta^{ab} = \frac{c_1}{2l^2} e^a e^b + \frac{c_4}{2}\omega^a{}_c \omega^{cb}. \tag{50}$$

Under this choice, the Chern–Simons–Antoniadis–Savvidy Lagrangian for Maxwell algebra takes the compact form

$$\mathcal{L}^{(4)}_{\mathrm{ChSAS}} = \frac{\mu}{8}\varepsilon_{abcd} R^{ab} e^c e^d + \frac{\nu}{8} l^2 \varepsilon_{abcd} R^{ab}\omega^c{}_f \omega^{fd},$$

where we can see that in the limit $l \to 0$, we obtain the Einstein–Hilbert Lagrangian,

$$\mathcal{L}^{(4)}_{\mathrm{ChSAS}} = \frac{\mu}{8}\varepsilon_{abcd} R^{ab} e^c e^d. \tag{51}$$

Another case particularly interesting choice is given by $a_1 = c_1 = c_4 = 0$. In this case the fields of eqs. (41), (42) and (43) are given by

$$B^a = \frac{a_2}{2l} k^a_b e^b, \tag{52}$$

$$B^{ab} = -\frac{a_2}{2l^2} e^a e^b, \tag{53}$$

$$\beta^{ab} = \frac{a_2}{2} k^a{}_c k^{cb} + \frac{a_2}{2} k^b{}_c k^{ac}, \tag{54}$$

and the Chern–Simons–Antoniadis–Savvidy Lagrangian for the Maxwell algebra is given by

$$\begin{aligned}\mathcal{L}^{(4)}_{\mathrm{ChSS}} = {} & \frac{\mu}{8}\varepsilon_{abcd} R^{ab} e^c e^d \\ & - \frac{\sigma}{8l^2}\varepsilon_{abcd}(e^a e^b e^c e^d + 2l^2 k^{ab} T^c e^d - 2l^4 R^{ab} k^c{}_f k^{fd}) \\ & - \frac{\sigma}{8} d(\varepsilon_{abcd} k^{ab} e^c e^d),\end{aligned} \tag{55}$$

where the case $k^{ab} = 0$ leads to the standard Einstein–Hilbert Lagrangian with cosmological constant.

5. Concluding remarks

In Refs. [1,2] there were found invariants similar to the Pontryagin–Chern forms in non-abelian tensor gauge field theory [11,12]. The first series of exact $(2n+3)$-forms are given by $\Gamma_{2n+3} = \langle F^n H_3 \rangle = \mathrm{d}\mathfrak{C}^{(2n+2)}_{\text{ChSAS}}$ where $H_3 = \mathrm{d}B + [A, B]$ is the 3-form field-strength tensor for the rank-2 gauge field B. The second series of invariant forms are defined in $2n+4$ dimensions and are given by $\Gamma_{2n+4} = \langle F^n H_4 \rangle = \mathrm{d}\mathfrak{C}^{(2n+3)}_{\text{ChSAS}}$ where the corresponding secondary $(2n+3)$-form $\mathfrak{C}^{(2n+3)}_{\text{ChSAS}}$ is defined in terms of the 4-form $H_4 = \mathrm{d}C + [A, C]$ as the field-strength tensor for the rank-3 gauge field C. The third series of forms is defined in $(2n+6)$ dimensions [3] $\Gamma_{2n+6} = \langle F^n H_6 \rangle + n\langle F^{n-1} H_4^2 \rangle = \mathrm{d}\mathfrak{C}^{(2n+5)}_{\text{ChSS}}$. The fourth series of invariant closed forms Γ_{2n+8} in $(2n+8)$ dimensions is given by [4] $\Gamma_{2n+8} = \langle F^n H_8 \rangle + 3n\langle F^{n-1} H_4 H_6 \rangle + n(n-1)\langle F^{n-2} H_4^3 \rangle = \mathrm{d}\mathfrak{C}^{(2n+7)}_{\text{ChSS}}$.

All forms Γ_{2n+3}, Γ_{2n+4}, Γ_{2n+6} and Γ_{2n+8} are analogous to the Pontryagin–Chern invariants $\mathcal{P}_{2n}$ in the Yang–Mills gauge theory in the sense that they are gauge invariant, closed and metric independent.

In Refs. [2–4] there were found explicit expressions for these invariants in terms of higher order polynomials of the curvature forms on a vector bundle. As with standard Chern–Simons forms, the secondary forms $\mathfrak{C}^{(2n+m)}_{\text{ChSAS}}$ are background-free but quasi-invariant and only locally defined (and therefore defined only up to boundary terms, $\mathfrak{C}^{(2n+m)}_{\text{ChSAS}} \sim \mathfrak{C}^{(2n+m)}_{\text{ChSAS}} + \mathrm{d}\sigma^{(2n+m-1)}$). In the present article we have constructed the $(2n+2)$-dimensional analogue of transgression forms and the Chern–Weil theorem. These transgression forms are defined globally and are off-shell gauge invariant, but the price to pay is the doubling in the number of fields. From this theorem is straightforward to recover the generalized $(2n+2)$-dimensional Chern–Simons–Antoniadis–Savvidy forms from Refs. [1,2] setting to zero half of the fields. The 2-form field B can be discomposed in terms of components of the 1-form A. It is performed in a self-consistent way by considering the generalization of Maurer–Cartan approach to forms of higher order, i.e. free differential algebras, and by following the procedure used in Refs. [5,6] and [7].

The final result is a four-dimensional gravity action principle equation (47), which is gauge quasi-invariant under the generalized gauge transformations eqs. (8), (9) for the Maxwell algebra.

The dynamics of the system will be presented elsewhere, but it is clear that the non-linear couplings with the k^{ab} field does generate in general non-vanishing torsion, in a way similar to the one presented in Ref. [35]. A non-vanishing torsion may lead to highly non-trivial consequences in cosmology (see Refs. [35–37]), where at the very end it plays the role of an extra stress-energy tensor in Einstein field equations.

Acknowledgements

This work was supported in part by FONDECYT grants 1130653 and 1150719 from the Government of Chile. One of the authors (I.M.) was supported by grants from Universidad de Concepción, Chile.

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