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PHYSICS LETTERS B

Physics Letters B 597 (2004) 285–290

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One-loop corrections to radiative muon decay

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Received 28 April 2004; accepted 12 July 2004

Available online 23 July 2004

Editor: G.F. Giudice

Abstract

One-loop QED corrections to the differential width of radiative muon decay are considered. Results can be used to analyze high statistics data of modern and future experiments.

PACS: 13.35.Bv; 13.40.Ks

Keywords: Muon decay; Radiative corrections

1. Introduction

Since the discovery of muon in 1937, the studies of its properties were always very important for the progress of the elementary particle physics. Nowadays, such precision observations like the muon life time and the muon anomalous magnetic moment are important for the checks of the Standard Model and searches for *new physics*. Besides many others, the process of radiative muon decay,

$$\mu^+ \to e^+ + \nu_e + \bar{\nu}_\mu + \gamma, \tag{1}$$

is investigated in the modern experiments. In particular, the set of data from the PIBETA ($\pi\beta$) experiment [1] at the Paul-Scherrer Institute contains a considerable amount of these decays. Accurate measurements of the process provide interesting information about the structure of weak interactions.

In this Letter we construct an advanced theoretical prediction for the differential distribution of process (1). Our calculations of radiative corrections (RC) allow to reduce the theoretical uncertainty. That makes it possible to perform precision comparisons with the experimental data and potentially look for *new physics* or rule out certain extensions of the Standard Model.

In the limit of small energy loss (carried away by the neutrinos), radiative corrections to the process were considered in Ref. [2]. In this limit the standard

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doi:10.1016/j.physletb.2004.07.017

decay produces a background to the searches for the neutrinoless decay $\mu \to e\gamma$.

In this Letter we will consider the general kinematics assuming that the energies of the final state electron and photon are above of a certain threshold and the angle between their momenta is not small (see Section 3.3). The tree-level distribution and the notation are introduced in Section 2. Then we consider different RC contributions. In Conclusions we present some numerical results and estimate the theoretical uncertainty in description of the radiative muon decay.

2. The tree-level distribution

Within the Fermi model of four-fermion interaction, the differential width of radiative muon decay was first considered in Refs. [3,4]. Accurate formulae including the terms suppressed by the factor $(m_e/m_\mu)^2$ were recently presented in Ref. [5]. We checked that their results coincide with the relevant contribution, which have appeared in calculations of exact one-loop radiative corrections to the muon decay spectrum [6]. At the Born level the differential distribution of the electrons and photons of the process (1) has the form

$$\begin{aligned}
&\frac{\mathrm{d}^6\Gamma^{\mu^\pm\to e^\pm\nu\bar\nu\gamma}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}^2\Omega_e\,\mathrm{d}^2\Omega_\gamma}\\
&\quad=\Gamma_0\frac{\alpha}{64\pi^3 y}\beta\big[F(x,y,d)\mp\beta\vec{P}_\mu\hat{p}_e G(x,y,d)\\
&\qquad\qquad\mp\beta\vec{P}_\mu\hat{p}_\gamma H(x,y,d)\big],\\
&\Gamma_0=\frac{G_F^2 m_\mu^5}{192\pi^3},\qquad d=1-\beta c,\\
&\beta=\sqrt{1-\frac{m_e^2}{E_e^2}},
\end{aligned}\tag{2}$$

where G_F is the Fermi coupling constant; m_e and m_μ are the electron and muon masses, respectively; $\vec{P}_\mu$ is the muon polarization vector; x and y are the electron and photon energy fractions in the muon rest reference frame, $x = 2E_e/m_\mu$ and $y = 2E_\gamma/m_\mu$; by $\hat{p}_e$ and $\hat{p}_\gamma$ we denote the unit vectors in the directions of motion of the electron and photon, $\hat{p}_e = \vec{p}_e/|\vec{p}_e|$ and $\hat{p}_\gamma = \vec{p}_\gamma/|\vec{p}_\gamma|$; $c = \cos(\widehat{\vec{p}_e\vec{p}_\gamma})$. Functions $F(x,y,d)$, $G(x,y,d)$, and $H(x,y,d)$ can be found in Appendix of Ref. [5].

In what follows we will concentrate on the case of unpolarized muon decay, since it is the one measured in the PIBETA experiment. In the unpolarized case only three variables are relevant and the tree-level distribution can be represented as

$$\frac{\mathrm{d}^3\Gamma^{\mathrm{Born}}_{\mathrm{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c}=\Gamma_0\frac{\alpha}{8\pi y}\beta F(x,y,d).\tag{3}$$

Model independent parameterization of four-fermion interaction (see Particle Data Group [7]) leads to the appearance of two additional contributions. One of them is proportional to the difference $(1-4\rho/3)$, which describes the deviation of the Michel parameter ρ from its value in the Standard Model. And the other one contains parameter $\bar\eta$, which is a positive semi-definite quantity (see Ref. [8])

$$\begin{aligned}
\bar\eta=&\big(|g^V_{RL}|^2+|g^V_{LR}|^2\big)\\
&+\frac{1}{8}\big(|g^S_{LR}+2g^T_{LR}|^2+|g^S_{RL}+2g^T_{RL}|^2\big)\\
&+2\big(|g^T_{LR}|^2+|g^T_{RL}|^2\big),
\end{aligned}\tag{4}$$

where $g^{S,V,T}_{RL,LR}$ are the right–left (RL) and left–right (LR) coupling constants, which parameterize non-standard scalar (S), vector (V) and tensor (T) four-fermion interactions. In principle, one can look also for other exotic interactions, e.g., for the ones mediated by antisymmetric tensor fields [9]. Extraction of $\bar\eta$ from the experimental data potentially can put strict limits on physics beyond the Standard Model.

3. Radiative corrections

New precision experiments call for an adequate level of accuracy in theoretical predictions within the Standard Model. Effects of higher orders of the perturbation theory become important. Here we will consider the first order QED radiative corrections. As usually, we separate them into three parts: (i) emission of an additional soft photon; (ii) effect due to one-loop virtual photonic correction; (iii) emission of an additional hard photon. Note that all the relevant pure week corrections (like loop insertions into the W-propagator) are included into the G_F coupling constant [10,11], which is measured directly from the muon lifetime. Effects of strong interactions in the

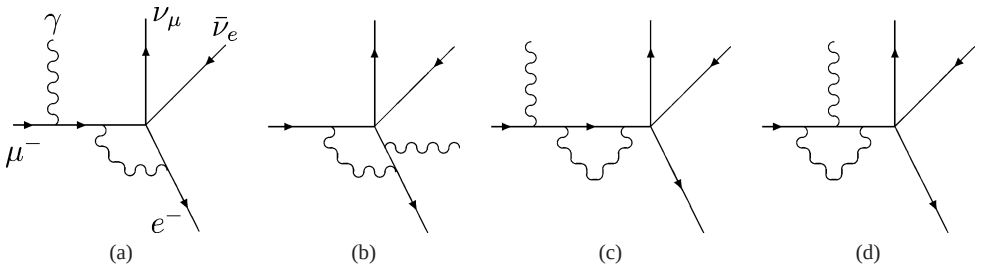

Fig. 1. Types of Feynman diagrams for radiative muon decay with one-loop RC.

process under consideration are negligible for the moment. They start to appear only at the order $\mathcal{O}(\alpha^2)$ through hadronic vacuum polarization.

3.1. Soft photon contribution

We assume, that emission of an additional soft photon of energy below certain threshold is not distinguished by the experiment from the tree-level process (1). The energy of the soft photon, ω_2, is limited by the parameter Δ:

$$\omega_2 \leqslant \Delta \frac{m_\mu}{2}, \quad \Delta \ll 1. \tag{5}$$

The corresponding correction can be factorized out in front of the tree-level differential distribution:

$$\frac{\mathrm{d}^3\Gamma^{\mathrm{Soft}}_{\mathrm{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c} = \delta^{\mathrm{Soft}} \frac{\mathrm{d}^3\Gamma^{\mathrm{Born}}_{\mathrm{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c},$$

$$\delta^{\mathrm{Soft}} = \frac{\alpha}{2\pi}\left\{\frac{1}{2}L^2 - (L - 2 + 2\ln x) \times \left(1 - 2\ln\Delta - \ln\frac{m_e^2}{\lambda^2}\right) - 2\ln^2 x + 4\ln x - 2\zeta(2)\right\} + \mathcal{O}\left(\frac{m_e^2}{m_\mu^2}\right),$$

$$\zeta(2) = \frac{\pi^2}{6}, \tag{6}$$

where λ is a fictitious photon mass; L is the so-called large logarithm, $L = \ln(m_\mu^2/m_e^2) \approx 10.66$. Quantity δ^{Soft} coincides with the corresponding factor, arising in the correction to the non-radiative muon decay (see, e.g., Ref. [6]).

3.2. One-loop virtual correction

Here we will consider the effect of one-loop photonic corrections. Some representatives of the relevant Feynman diagrams are given in Fig. 1. There are two diagrams of class (a) with photon emission from an external leg (electron or muon line). In the same way the two box-type diagrams of class (b) describe real photon emission from virtual electron and muon propagators. Diagrams of classes (c) and (d) give corrections to photon radiation from a single leg. To get the corresponding correction to the muon decay spectrum we have to multiply the complete set of amplitudes of classes (a)–(d) by two tree-level amplitudes, describing single photon emission. In our calculations we followed the procedure which has been applied in Ref. [12].

The standard technique for one-loop integration was used. The list of relevant integrals can be found in the Appendix of our preprint [13]. To eliminate the ultraviolet divergences we applied renormalization of the masses and wave functions of the electron and muon. Note that this is enough in the case of muon decay (see Refs. [14,15]), contrary to the general case of the Fermi four-fermion interaction. An analytical result for the virtual correction was obtained. We do not give the full formula here, since it is rather long.

3.3. Emission of an additional collinear hard photon

Events with registration of two hard photons are supposed to be rejected by the experimental event selection. But if the additional photon is emitted at a small angle with respect to the momentum of the outgoing electron (positron), the former is not recognized by a calorimetric detector as an independent particle (this can happen if there is no any considerable magnetic field in the detector volume). So, for the so-called collinear photon emission, one observes an effective electron with the energy and momentum composed by the sum of the corresponding quantities of the pho-

ton and the *bare* electron. Let us assume that this kind of calorimetric registration happens in the experiment, if the angle between the electron and photon momenta does not exceed a certain value θ_0, which plays the role of a small parameter. We demand $m_e/m_\mu \ll \theta_0 \ll 1$. Typical experimental values for this parameter, a few degrees, satisfy our conditions. On the other hand, the angle between the observed photon and the electron should satisfy the condition $\theta = \widehat{\vec{p}_e \vec{p}_\gamma} \geqslant \theta_0$.

According to the general factorization procedure, we can represent the result for the contribution of collinear photon radiation as the product of two factors:

$$\frac{\mathrm{d}^3\Gamma^{\text{H-coll}}_{\text{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c} = \frac{\mathrm{d}^3\Gamma^{\text{Born}}_{\text{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c} R_{\text{coll}},$$

$$R_{\text{coll}} = \frac{\alpha}{2\pi}\int_{\Delta/x}^{1}\frac{\mathrm{d}z}{z} \times\left\{\left[1+(1-z)^2\right]\left(L+2\ln x-1+\ln\frac{\theta_0^2}{4} + 2\ln(1-z)\right)+z^2\right\}. \tag{7}$$

The tree-level radiative muon decay (with photon emission at large angles with respect to the electron momentum) serves as a short-wave sub-process. Emission of a collinear photon by the outgoing electron serves as a long-wave sub-processes. The formula for the collinear radiation factor agrees with the one in Ref. [16].

Integration over the energy fraction of the collinear photon, z, gives

$$R_{\text{coll}} = \frac{\alpha}{2\pi}\left[\left(L+2\ln x-1+\ln\frac{\theta_0^2}{4}\right) \times\left(2\ln x-\frac{3}{2}-2\ln\Delta\right)-4\zeta(2) + \frac{11}{4}\right]. \tag{8}$$

Note that the lower limit of the collinear hard photon energy fraction is adjusted to the upper limit of soft photon emission.

4. Results and conclusions

Summing up the contributions of soft, virtual, and hard collinear photonic corrections we receive the final answer for the first order radiative correction to the process (1). Here is our result for the corrected distribution, which substitutes function $F(x,y,d)$ from Eq. (3)

$$\begin{aligned} F^{\text{Corr.}}(x,y,d) &= F(x,y,d)\left(1+\frac{\alpha}{2\pi}A(x,y,d)\right) + \frac{\alpha}{2\pi}B_F(x,y,d), \\ A(x,y,d) &= 2\ln\frac{\theta_0^2}{4}(\ln x-\ln\Delta)-2\ln\Delta-\frac{3}{2}\ln\frac{\theta_0^2}{4} + \frac{1}{2}\left(\ln\frac{xyd}{2}-2\ln x\right)^2. \end{aligned} \tag{9}$$

We presented explicitly only the factorized part of the correction. The remaining non-factorizable part, $B_F(x,y,d)$, is rather long. We use it in a FORTRAN code for numerical estimates. Expressions for the radiatively corrected functions $G^{\text{Corr.}}(x,y,d)$ and $H^{\text{Corr.}}(x,y,d)$ have exactly the same form as Eq. (9) with the trivial substitutions: $F \to G(H)$ and $B_F(x,y,d) \to B_{G(H)}(x,y,d)$. The most important factorized part of the correction, $A(x,y,d)$, is universal for all the three functions.

It is worth to note that all the leading logarithm terms were factorized in each of the contributions, but they cancel out in the sum in accord with the Kinoshita–Lee–Nauenberg theorem [17,18]. Moreover, all the dependence on the parameters Δ and θ_0 is contained in $A(x,y,d)$.

In Fig. 2 we plotted the Born-level differential branching ratio of the radiative muon decay for a fixed value of c,

$$R(x,y,c) \equiv \frac{1}{\Gamma_0}\frac{\mathrm{d}^3\Gamma^{\text{Born}}_{\text{unpol.}}}{\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}c}. \tag{10}$$

The relative contribution of radiative corrections is illustrated by Fig. 3,

$$\delta^{\text{RC}} = \frac{F^{\text{Corr.}}(x,y,d)-F(x,y,d)}{F(x,y,d)}\times 100\%. \tag{11}$$

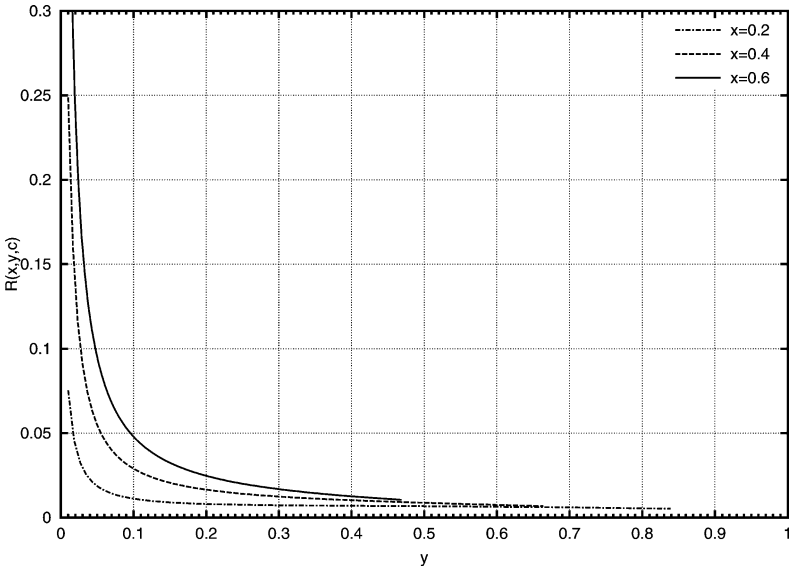

Fig. 2. Differential branching ratio versus electron energy fraction y for three different x-values with fixed $c = 0.5$, $\Delta = 0.01$, $\theta_0 = 3°$.

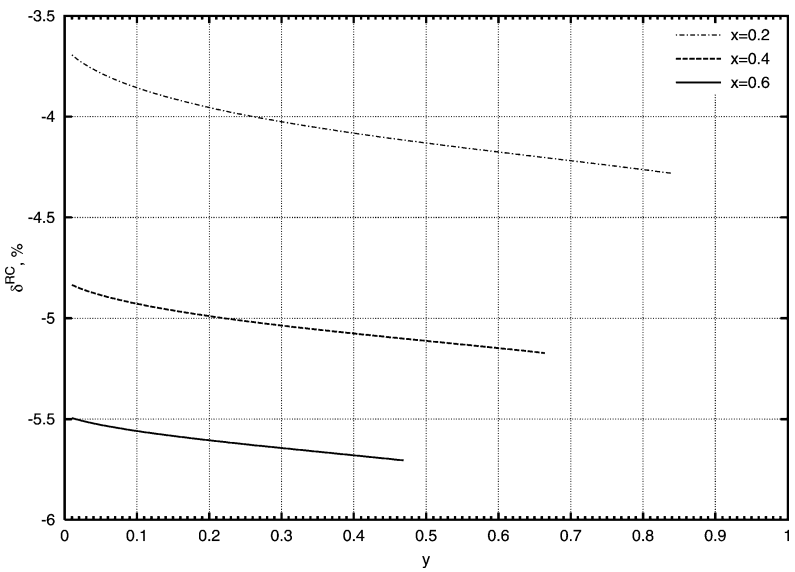

Fig. 3. Relative contribution of radiative corrections versus the electron energy fraction; parameters are the same as in Fig. 2.

The dependence on c value of δ^{RC} is rather weak. For given values of c and x the maximal value of y is defined by the kinematics:

$$y_{\max} = \frac{1 - x + m_e^2/m_\mu^2}{1 - x(1 - \beta c)/2}. \tag{12}$$

For the given set of parameters, the factorized part of the correction dominates and gives about 4/5 of the total effect.

To illustrate also the case of 100% polarized muon decay we present in Fig. 4 a plot for the relative contribution of radiative corrections for a set of fixed

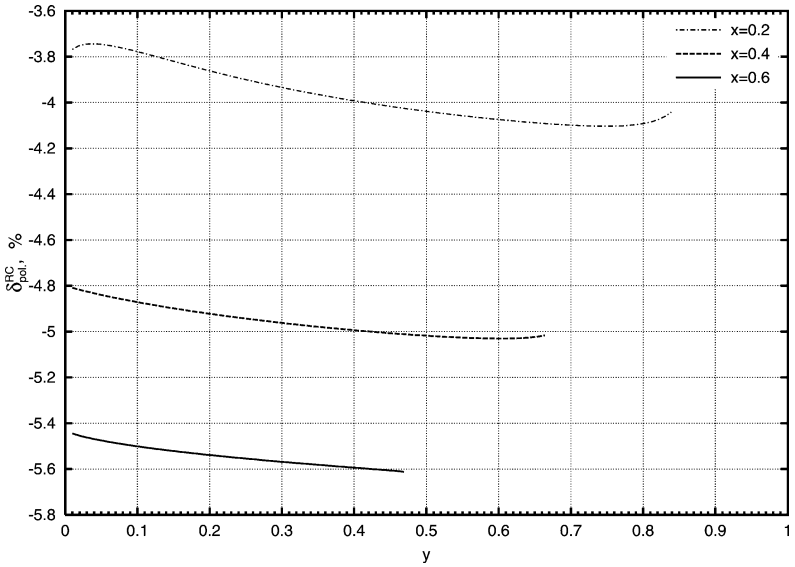

Fig. 4. Relative contribution of radiative corrections for the case of polarized muon decay versus the electron energy fraction.

variables. Namely, $\widehat{\vec{P}_\mu \vec{p}_e} = 30°$, $\widehat{\vec{P}_\mu \vec{p}_\gamma} = 60°$; c, θ_0 and Δ are the same as in Fig. 2. Quantity $\delta^{\mathrm{RC}}_{\mathrm{pol.}}$ is defined in analogy to Eq. (11) by adding the relevant contributions of G and H functions according to Eq. (2).

Thus we presented the calculation of one-loop QED corrections to the differential distribution of unpolarized muon decay. Our FORTRAN code is available upon request from the authors. The results can be applied also for the decays $\tau \to \mu \bar{\nu}_\mu \nu_\tau \gamma$ and $\tau \to e \bar{\nu}_e \nu_\tau \gamma$. The theoretical uncertainty of the spectrum description is defined by higher order QED radiative corrections (EW and QCD effects are negligible compared to the QED ones). As a rough upper estimate we can consider the relative contribution of the omitted higher order terms to be about $(\delta^{\mathrm{RC}})^2 \lesssim 3 \times 10^{-3}$, which is small compared to the present experimental precision. If ordered, one can easily get the most important higher order terms with logarithms of Δ and θ_0 by means of the soft and collinear approximations.

Acknowledgements

We thank V. Baranov, E. Kuraev, E. Velicheva for fruitful discussions. This work was supported by RFBR grant 04-02-17192. One of us (E. Sch.) is grateful to the "Dynasty" foundation and ICFPM.

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