A Crystalline Motion of Spiral-Shaped Curves with Symmetry

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We study the evolution of spiral-shaped polygonal curves by crystalline curvature. Crystalline curvature is known to extend the notion of ordinary curvature to the special class of nonsmooth curves. With the assumption of symmetry, we show that the motion of our spirals can be analyzed up to the rest state, possibly beyond singularities.

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1. INTRODUCTION

There has been much interest recently in the subject of motion by crystalline curvature. As is well known now, crystalline curvature is defined for a special class of nonsmooth curves and naturally extends ordinary curvature for smooth curves. The evolution of polygonal curves by crystalline curvature corresponds to the evolution of smooth curves by curva-

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ture, not only in the sense of the normal velocity formulation, but also from the viewpoint of variational structures. It is further claimed that some physical models of crystal growth give rise to such evolution. Concerning this area of research, including basic properties and other background materials, we refer to an excellent monograph by Gurtin [8].

Here we are concerned with the motion of spiral-shaped polygonal curves by their crystalline curvature, which is a continuation of our previous study [10]. In [10], we considered a crystalline motion of spirals; the polygonal curves we discussed were assumed to be spiral out to infinity with one end being fixed at the origin. We recall that the crystalline motion of polyhedral curves is described by a system of ordinary differential equations (ODEs). Our spirals satisfy an ODE system of infinite order. The short time existence and uniqueness for this infinite system are established; the method of proof involves a finite approximation and an application of comparison principle to the truncated systems. However, we have investigated the evolution only up to the first breakdown; the contact between the origin and the line segment might happen there.

In this paper, we deal with the polygonal spirals with symmetry and pursue the evolution possibly beyond singularities; at the expense of imposing the assumption of symmetry, we analyze the motion more in detail. See Theorem 1 in Section 2 for precise statements.

The motivations for considering the spirals are principally twofold and are the same as our previous work [10]. One reason is the fact that spiral patterns are commonly observed in real crystal growth [13, 15]; we want to understand the mechanism via the mathematical models, even though our attempt is a first step. The other reason is simply that spiral motion is mathematically interesting [12]. One cannot represent spirals easily by way of a level set or a graph of some functions. We recall that the so-called level set method is a popular technique in the research for curvature dependent motion; it is suitable especially to closed curves. See, for instance, [6, 11, 18] as to the connection with crystalline evolution. On the other hand, we also have to point out that the study on the motion of polyhedral graphs has attracted much attention; a practical use of the abstract analysis such as the nonlinear semigroup theory yields rather profound results. We refer to [2–4]. Our strategy of proof is somewhat elementary compared to the above articles; indeed, essentially the same observations are already provided in several works [11, 21]. However, the consequence is quite interesting and seems new. As to other mathematical treatments of spirals, we refer to [9, 14, 22] and references therein.

The paper is organized as follows: In Section 2 the basic setting and main result are presented. Section 3 is devoted to the proof. We give numerical examples in Section 4. Generalizations are discussed in Section 5.
2. FORMULATION AND MAIN RESULTS

We begin by recalling basic definitions and equations.

The dynamics of interfaces between the two-phase, as modern physical theory claims, is governed by an evolution law involving curvature and kinematic energies. If we neglect the difference of bulk energies between phases, the motion of interfaces is expressed by the equation

\[ b(\nu)V = (f(\nu) + f''(\nu))K. \] (2.1)

Here \( V \) and \( K \) denote the normal velocity and the curvature of the interface, respectively. The symbol \( \nu \) stands for the angle to the interface normal, \( b(\nu) > 0 \) is the kinetic coefficient, and \( f(\nu) \) is the interfacial energy. If \( f(\nu) \) is crystalline energy, that is, \( f \) has convexified Frank diagrams that are polygonal, then (2.1) reduces to the evolution of polygonal interfaces described by a system of ODEs. We recall that the Frank diagram \( \mathcal{F}(f) \) of \( f \) is defined by the one-level set of \( f \):

\[ \mathcal{F}(f) = \{ x \in \mathbb{R}^2 | f(x) = 1 \}. \]

A crystalline motion, which was introduced by Taylor [21] and independently by Angenent and Gurtin [1], refers to these ODE system. We exhibit explicit formula shortly.

From here on, for simplicity, we confine ourselves to investigating crystalline motions whose Frank diagrams \( \mathcal{F}(f) \) are regular \( n \)-polygons and kinetic coefficient \( b = 1 \). The set \( N_n \) of angles corresponding to the corner point of \( \mathcal{F}(f) \) is then given by

\[ N_n = \left\{ \frac{2k\pi}{n} \mid k = 0, 1, \ldots, n-1 \right\}. \]

Now we specify a class of spirals we consider. Suppose the spiral-shaped polygonal curve \( S(t) \) is formulated for each \( t \) in some interval. The curve \( S(t) \) is assumed to be symmetric with respect to the origin \( O \), which is kept fixed during the motion. Let \( L_0(t), L_1(t), L_2(t), \ldots, L_n(t), \ldots \) be the line segments of \( S(t) \), numbering from the origin counterclockwise; the initial endpoint of \( L_i(t) \) is the final endpoint of \( L_{i-1}(t) \). Attaching the minus sign, we denote by \( L_{-i}(t) \) (\( i = 0, 1, 2, \ldots \)) the image of \( L_i(t) \) under reflection with respect to the origin \( O \). To be precise, \( L_{-0}(t), L_{-1}(t), L_{-2}(t), \ldots, L_{-n}(t), \ldots \) are the line segments of \( S(t) \), numbering from the origin clockwise. For each \( i \), let \( v_i \in N_i \) be the unit outward normal vector to \( L_i(t) \), which is independent of \( t \); it follows that \( v_i \cdot v_{i+1} = \cos(2\pi/n) \).

We remark that \( L_i(t) \) is contained in \( \{ x \in \mathbb{R}^2 | x \cdot v_i = d_i(t) \} \), where \( d_i(t) \)
FIG. 1. Spiral-shaped polygonal curves. The left figure shows the case in which $n$ is even, and the right figure shows the case in which $n$ is odd.

represents the distance from the origin to the line containing $L_i(t)$. The length of $L_i(t)$ will be denoted by $l_i(t)$. By symmetry, we see that $d_{-i}(t) = d_i(t)$, $l_{-i}(t) = l_i(t)$, and also $\nu_{-i} = \nu_i$. See Figure 1 for our arrangements.

The evolution (2.1) on each $L_i(t)$ now turns out to be [8, (12E)]

$$V_i(t) = -\frac{2(1 - \cos(\theta))}{\sin \theta} \frac{x_i}{l_i(t)}$$

$$=: K_i(t), \quad (2.2)$$

where $V_i(t)$ is the normal velocity of $L_i(t)$ and we have put $\theta := 2\pi/n$. The symbol $\chi_i \in \{-1, 0, 1\}$ is determined according to the rule $\chi_i = 1$ if the corner, which is built with $L_i(t)$ and both of its adjacent sides, is locally convex about the direction $\nu_i$; $\chi_i = -1$ if the corner is locally concave; $\chi_i = 0$ otherwise. See Figure 2. $K_i(t)$ is called the crystalline curvature of $L_i(t)$. We have to remark that $S(t)$ is shrinking toward the origin, contrary to real phenomena.

The formation of our $S(t)$ immediately implies that $\chi_i = 1$ for $i \geq 1$, $\chi_0 = 0$, and $\chi_{-i} = -1$ for $i \geq 1$. Since $V_i(t) = \chi_i \frac{d}{dt} L_i(t)$, we infer that
(2.2) becomes
\[
\frac{d}{dt} d_i(t) = -\frac{2\tan(\theta/2)}{l_i(t)}, \quad \text{if } |i| \geq 1,
\]
(2.3)
\[
\frac{d}{dt} d_0(t) = 0,
\]
where \(d_0(t) := d_{+0} + d_{-0}(t)\). We note that by geometry
\[
l_i(t) = \frac{1}{\sin \theta} (d_{i-1}(t) + d_{i+1}(t) - 2d_i(t) \cos \theta), \quad \text{if } |i| \geq 1,
\]
\[
l_0(t) := l_{+0}(t) + l_{-0}(t) = \frac{1}{\sin \theta} (d_1(t) + d_{-1}(t)),
\]
from which we conclude that (2.3) becomes an ODE system for \(d_i(t)\). In the next section, during the proof, we further derive another equivalent system of equations for the crystalline curvature \(K(t)\). See (3.3) below.

Before proceeding, basic assumptions which our initial spirals should satisfy are required.

(A1) \(d_i(0) \leq d_{i+1}(0)\) for all \(i \geq 0\) and furthermore
\[
d_i(0) < d_{i+n/2}(0) \quad (d_{i+n}(0)), \quad \text{if } n: \text{even},
\]
\[
d_i(0) < \min\{d_{i+(n-1)/2}(0), d_{i+(n+1)/2}(0)\}, \quad \text{in } n: \text{odd}.
\]
(4.4)

(A2) For some \(N \geq 0\), there holds \(l_N(0) = \infty\), or equivalently \(d_{N+1}(0) = \infty\).
The assumption (A1) itself makes our initial spirals $S(0)$ embedded; $S(0)$ possesses no self-intersection. The influence of (A2) is that the ODE system (2.3) is of finite order, which simplifies the analysis compared to our previous work [10]. We return to this point later in Section 5.

We further remark that in (2.4) for odd $n$, less stringent assumption such as below would suffice to form a spiral properly:

$$d_i(0) \cos(\theta/2) < \min\{d_{i+(n-1)/2}(0), d_{i+(n+1)/2}(0)\}.$$

The effect of the restricted condition listed in (2.4) for odd $n$ will be clear in the course of the proof.

Now our main results read as follows.

**Theorem 1.** Assume (A1) and (A2). The spiral $S(t)$ defined through the solution of (2.3) exists and stays embedded for all $t \geq 0$ in the next sense: There are finite $0 < t_1 < t_2 < \cdots < t_J < \infty$ with $J \leq N$ such that at each $t_i$, we have $\lim_{t \to t_i} \lambda(t) = 0$, where the process of renumbering is made right after each $t_i$, $S(t)$ is a rest state for $t \geq t_i$. (See Fig. 3).

### 3. Proof of Theorem

The ODE system we are going to treat is, to be focused again,

$$\frac{d}{dt}d_i(t) = -\frac{2(1 - \cos \theta)}{d_{i-1}(t) + d_{i+1} - 2d_i(t) \cos \theta}$$

$$:= -\frac{1}{(\Delta^n d)_i(t) + d_i(t)} \quad \text{for } 1 \leq i \leq N \text{ with } d_{-i}(t) = d_i(t),$$

$$\frac{d}{dt}d_0(0) = 0, \quad d_{N+1} = \infty$$

FIG. 3. Rest state.
where the initial data \(d_{-1}^{N+1}(0), \ldots, d_{-1}(0), d_{0}(0), d_{1}(0), \ldots, d_{N}(0), d_{N+1}(0)\) fulfill (A2) and (A3), and we have introduced the notation

\[
(\Delta_{\theta} d)_{i}(t) = \frac{d_{i+1}(t) - 2d_{i}(t) + d_{i-1}(t)}{2(1 - \cos \theta)}
\]  

Equation (3.2) serves as the Laplacian in the discrete problem.

By virtue of \(d_{i}(0) > 0\) for \(i \neq 0\), which is deduced implicitly from our setting, the short time existence and uniqueness for (3.1) is straightforward. We will argue the long time behavior. First we show that a certain breakdown of solutions to (3.1) must occur. To do this, we transform (3.1) into the equivalent system of equations for \(K_{i}(t)\). Since there holds for \(i \geq 1\),

\[
\frac{d}{dt} l_{i}(t) = \frac{2(1 - \cos \theta)}{\sin^{2} \theta} \left( -\frac{1}{l_{i+1}(t)} - \frac{1}{l_{i-1}(t)} + \frac{2\cos \theta}{l_{i}(t)} \right),
\]

it is rather immediate to derive

\[
\frac{d}{dt} K_{i}(t) = (K_{i}(t))^{2}(\Delta_{\theta} K)_{i}(t) + K_{i}(t),
\]

for \(1 \leq i \leq N - 1\) with \(K_{-1}(t) = K_{N}(t) = K_{0}(t) = K_{N}(t) = 0\),

where the initial data \((K_{i}(0))_{i=-N}^{N}\) can be computed.

**Lemma 1.** There is a \(T < \infty\) such that \(\lim_{t \to T} \sup |K_{i}(t)| = \infty\).

**Proof.** We define, for sufficiently small positive \(\varepsilon\),

\[
\varphi_{i} := \begin{cases} 
-\varepsilon \cos(i\theta/2), & \text{if } 1 \leq |i| \leq n, \\
0, & \text{otherwise}.
\end{cases}
\]

Taking \(\varepsilon\) smaller if necessary, we may assume \(K_{i}(0) \leq \varphi_{i}\) for all \(i\),

thanks to \(K_{i} \leq 0\) in our convention. We calculate

\[
(\Delta_{\theta} \varphi)_{i} + \varphi_{i} = \lambda \varphi_{i} \leq 0,
\]

where \(\lambda := (\cos(\theta/2) - \cos \theta)/(1 - \cos \theta) > 0\). Invoking the discrete comparison lemma, we infer that

\[
K_{i}(t) \leq \varphi_{i} \quad \text{for all } i
\]
as long as $K_i(t)$ exists. Putting

$$J(t) = \sum_{i=1}^{n} \frac{\varphi_i}{K_i(t)} > 0,$$

we compute

$$\frac{d}{dt}J(t) = - \sum_{i=1}^{n} (\Delta \varphi_i + K_i) \varphi_i$$

$$= - \sum_{i=1}^{n} (\Delta \varphi + \varphi_i) K_i - \frac{K_{n+1} \varphi_n}{2(1 - \cos \theta)}$$

$$\leq - \lambda \sum_{i=1}^{n} \varphi_i^2,$$

from which we conclude that $K_i(t)$ blows up before the time $T$ bounded by

$$T \leq \frac{J(0)}{\lambda \sum_{i=1}^{n} \varphi_i^2} < \infty.$$

This completes the proof. □

Lemma 1 means merely that the solution under the initial situation does not persist forever, and forces us to examine the character of breakdown of solutions. Enlarging somewhat the category of singularity, we let $t_1 > 0$ be the first time such that one of next three possibilities takes place:

1. $S(t_1)$ is no longer embedded; i.e., some $L_i(t)$ and $L_j(t)$ ($i \neq j$) intersect.
2. There is an $i > 0$ such that $l_i(t_1) = 0$, while $l_i(t_1) > 0$.
3. There holds $l_i(t_1) = 0$.

We show that cases 1 and 2 can be discarded; only option 3 can happen. Note that essentially the similar observation that a breakdown appears on the portion where $\chi_i = 0$ is already addressed in foregoing literature [11, 21]. We further remark that in case 3, it may be possible that $l_i(t_1) = l_j(t_1) = \cdots = l_N(t_1) = 0$ with $i < N$ at the same time $t_1$.

First we try to eliminate possibility 1. To achieve this, it is customary to employ the comparison–exclusion principle [5, 11]. Here we intend to take a different approach, which is to compute the equation for the distance; the same method was undertaken in our previous work [10].
We define for $0 \leq i \leq N - \frac{n}{2} + 1$,

$$
\delta_i(t) := d_{i+n/2}(t) - d_i(t)
$$

and

$$
\tilde{\delta}_i(t) := d_{-i-n/2}(t) - d_i(t).
$$

Then $\delta_i$ satisfy

$$
\frac{d}{dt} \delta_i(t) = K_i(t)K_{i+n/2}(t)((\Delta_y \delta)_i(t) + \delta_i(t)),
$$

if $1 \leq i \leq N - \frac{n}{2}$,

$$
\delta_0(0) > 0,
$$

$$
\delta_{N-n/2+1}(t) = \infty,
$$

$$
\tilde{\delta}_i(0) > 0, \quad \text{if } 1 \leq i \leq N - \frac{n}{2}.
$$

The discrete maximum principle implies $\delta_i(t) > 0$ as long as the solution exists. Similar argument shows that $\tilde{\delta}_i(t) > 0$ as long as the solution exists. These facts exclude case 1 for even $n$.

If $n$ is odd, we calculate the equation for the next two distances

$$
\delta_i^1(t) := d_{i+(n-1)/2}(t) - d_i(t),
$$

$$
\delta_i^2(t) := d_{i+(n+1)/2}(t) - d_i(t).
$$

Both $\{\delta_i^1(t)\}$ and $\{\delta_i^2(t)\}$ obey the system similar to (3.4) and we conclude that $\delta_i^1(t) > 0$ and $\delta_i^2(t) > 0$, taking into account the initial assumption (2.4). The case 1 is discarded.

Now we turn our attention to situation 2. As already noted, this part is known to specialists. We present a little different approach. First we remark that the monotonicity $d_i(t) \leq d_{i+1}(t) \ (i \geq 1)$ is preserved under the evolution. To see this, put $\eta_i(t) := d_{i+1}(t) - d_i(t) \ (0 \leq i \leq N)$ and write down the equations similar to (3.4):

$$
\frac{d}{dt} \eta_i(t) = K_i(t)K_{i+1}(t)((\Delta_y \eta)_i(t) + \eta_i(t)),
$$

$$
\eta_0(t) > 0,
$$

$$
\eta_N(t) = \infty,
$$

$$
\eta_k(0) \geq 0.
$$
We immediately obtain \( \eta_i(t) > 0 \) for \( i > 0 \) and \( 0 < t < t_1 \), thanks to the discrete maximum principle. Since we have, at the time \( t_1 \) when \( l_i(t_1) = 0 \) with \( i \neq 0 \),

\[
d_{i-1}(t_1) = d_i(t_1) \cos \theta = d_{i+1}(t_1),
\]

the monotonicity just verified would be violated. This is absurd and we have finished eliminating case 2.

Now that the only breakdown is expressed by case 3, we are able to continue the motion beyond \( t_1 \) after renumbering the line segments appropriately. The proof of our main theorem is thus completed.

4. Numerical Examples

Here we present some numerical examples. These examples are computed using the crystalline algorithm which was mathematically investigated by several authors [6, 7, 11, 17, 24]. Their algorithm, however, does not seem to be directly applicable to the motion of the spiral, because of the nonconvexity and the non-closedness of the curve. We employ the scheme which is explained in [23].

Figure 4 shows the evolution of the spiral in the case where the Frank diagram is a regular hexagon. The initial polygonal curve satisfies our assumptions. More precisely, \( \theta = 2\pi / 6 \) and the length of the each side of the initial curve \( l_i(0) \) is given by \( l_i(0) = a \times l_{i-1}(0) \), \( a = 1.5 \), \( l_0(0) = 1.0 \). The center side which contains the origin does not move and the other sides move toward the origin. We plot figures when the center side disappears and the last figure shows the Frank diagram. We can observe

**FIG. 4.** The Frank diagram is a regular \( n \)-polygon \((n = 6)\).
that the center sides are extinct repeatedly and no other singularity occurs during the evolution.

Figure 5 shows the evolution of the spiral in the case where the Frank diagram is not a regular hexagon. Although we do not obtain any mathematical result for this situation, by using our scheme we can simulate the evolution numerically. The corresponding Frank diagram is shown at the end of Figure 5. The length of each side of the initial curve $l_i(0)$ is given by $l_i(0) = a \times l_{i-1}(0)$, $a = 1.9$, $l_0(0) = 0.5$. We plot figures when the center side disappears. We can also observe that the center sides are extinct repeatedly and no other singularity occurs during the evolution.

5. CONCLUDING REMARKS

In this final section, we discuss generalizations and topics for near future investigation.

As in our previous work [10], it is possible to send $N \to \infty$ if we impose necessary additional assumptions on the initial data such as

$$\text{(A3) } \liminf_{i \to \infty} l_i(0) = \infty \text{ and } \lim_{i \to \infty} d_i(0) = \infty.$$ 

We can prove the short time existence and uniqueness. Moreover, the extension of the solution beyond the first breakdown is permitted. Let us briefly explain the procedure.

First we note that the breakdown occurs within finite time even if $N \to \infty$; the time $T$ in Lemma 1 can be taken as independent of $N$. Also, when $N$ tends to infinity, the duration of existence is strictly positive as we saw in [10]. Let $t_1$ be the first breakdown time. At $t_1$, the only possible
singularity consists of the vanishing of the 0th segment; that is, \( \lim_{t \to t_1} l_0(t) = 0 \). Since (3.4) asserts that \( \min_{|i| \leq n/2} \delta_i(t) \) is non-decreasing, it is impossible that all the line segments shrink to the origin simultaneously for all large \( N \). As a byproduct, in particular, we learn that the number of line segments which vanish at the same time \( t_1 \) is less than \( (n + 1)/2 \). To summarize, we arrive at the next theorem.

**Theorem 2.** Assume (A1) and (A3). The spiral \( S(t) \) obtained as the solution of (2.3) exists globally and stays embedded. There exist at most countable \( 0 < t_1 < t_2 < \cdots < t_i < \cdots \) such that at each \( t_i \) there holds \( \lim_{t \to t_1} l_0(t) = \lim_{t \to t_1} l_1(t) = \cdots = \lim_{t \to t_1} l_i(t) = 0 \). The solution should be renumbered just after each \( t_i \).

The restriction that the underlying Frank diagram is a regular \( n \)-polygon can be weakened; there is no need to be regular. Although the curvature equation (3.3) turns very complicated in this case, the comparison principle established in Giga and Gurtin [5] works well instead.

If we include the difference \( F \) of bulk energy between phases, the original equation (2.1) becomes

\[
b(v_i)V_i = (f(v_i) + f''(v_i))K_i - F.
\]

It would be fascinating and important to consider the evolution of spirals under the general motion law (2.1'). However, it may be related to the problem of the creation of new facets, in other words, to the growth of crystals, and we believe that it calls for further research.

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