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Globally Analytic Simplification and the Levinson Theorem

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1. INTRODUCTION

The result of N. Levinson [25] plays an important role in the study of the asymptotic behavior of solutions of a linear system of differential equations

$$\frac{dy}{dt} = A(t)\, y, \tag{1.1}$$

as $t \to \infty$ where y is an n-dimensional vector, $A(t)$ is an $n \times n$ matrix. In order to state the Levinson theorem we need:

Assumption 1.1. The matrix $A(t)$ is continuous on $\mathscr{I}_0 = [t_0, \infty)$ (t_0, finite) and in the form

$$A(t) = \Lambda(t) + R(t), \tag{1.2}$$

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where $\Lambda(t)$ is an $n \times n$ diagonal matrix

$$\Lambda(t) = \begin{bmatrix} \lambda_1(t) & 0 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2(t) & 0 & \cdots & 0 & 0 \\ 0 & 0 & \lambda_3(t) & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & \lambda_n(t) \end{bmatrix} \tag{1.3}$$

and $R(t)$ is an $n \times n$ matrix satisfying

$$\int_{t_0}^{+\infty} |R(t)| \, dt < +\infty. \tag{1.4}$$

Let

$$\begin{aligned} \lambda_{jk}(t) &= \lambda_j(t) - \lambda_k(t) \\ D_{jk}(t) &= \Re(\lambda_{jk}(t)) \end{aligned} \qquad (j, k = 1, 2, ..., n). \tag{1.5}$$

The diagonal entries $\lambda_j(t)$ $(j = 1, 2, ..., n)$ of $\Lambda(t)$ satisfy

Assumption 1.2. The functions $\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)$ are continuous on the interval $\mathscr{I}_0$. Furthermore, for each fixed j, the set of positive integers $\{1, 2, ..., n\}$ is the union of two disjoint subsets $\mathscr{P}_{j1}$ and $\mathscr{P}_{j2}$, where

(i) $k \in \mathscr{P}_{j1}$ if

$$\lim_{t \to +\infty} \int_{t_0}^{t} D_{jk}(\tau) \, d\tau = -\infty,$$

$$\int_{s}^{t} D_{jk}(\tau) \, d\tau < K \qquad \text{for} \quad t_0 \leqslant s \leqslant t,$$

for some positive number K,

(ii) $k \in \mathscr{P}_{j2}$ if

$$\int_{s}^{t} D_{jk}(\tau) \, d\tau < K \qquad \text{for} \quad s \geqslant t \geqslant t_0$$

for some positive number K.

A version of the Levinson theorem can be stated as the following (e.g., see E. A. Coddington and N. Levinson [4], or M. S. P. Eastham [5]).

THEOREM 1.1. *Under Assumptions* 1.1 *and* 1.2, *there exists an* $n \times n$ *matrix* $Q(t)$ *such that*

(1) *the derivative* $dQ(t)/dt$ *exists and the entries* $Q(t)$ *and* $dQ(t)/dt$ *are continuous in* t *on the interval* $\mathscr{I}_0$,

(2) $\lim_{t \to +\infty} Q(t) = 0$,

(3) *the transformation*

$$y = [I + Q(t)]\, z \tag{1.6}$$

changes system (1.1) *to*

$$\frac{dz}{dt} = \Lambda(t)\, z \tag{1.7}$$

on the interval $\mathscr{I}_0$, *where* I *is the* $n \times n$ *identity matrix.*

Remark 1.1. Assume that the functions $\lambda_1(t), ..., \lambda_n(t)$ are continuous on the interval $\mathscr{I}_0$ and that

$$\lim_{t \to +\infty} \lambda_j(t) = \mu_j \qquad (j = 1, 2, ..., n)$$

exist. Then, if the real parts of $\mu_1, ..., \mu_n$ are mutually distinct, the functions $\lambda_1(t), ..., \lambda_n(t)$ satisfy Assumption 1.2.

Remark 1.2. Theorem 1.1 has been shown to be the basis of many important results for asymptotic integrations of differential equations. For instances see W. A. Harris, Jr. and D. A. Lutz [19] and M. S. P. Eastham [5].

Remark 1.3. A similar result of Theorem 1.1 for the system (1.1) with $A(t) = h(t)\, A_0(t)$, $A_0(t)$ being periodic, and $h(t)$ a scalar function tending to 0 as $t \to +\infty$, was obtained recently by W. A. Harris, Jr. and Y. Sibuya [20].

Theorem 1.1 can be proved in the following manner. From (1.1), (1.2), (1.6), and (1.7), we see that $Q(t)$ satisfies a linear differential equation

$$\frac{dQ}{dt} = \Lambda(t)\, Q - Q\Lambda(t) + R(t)[I + Q]. \tag{1.8}$$

As (1.8) is a linear equation, if a solution $Q(t)$ is shown to satisfy condition (2) in an interval $\mathscr{I} = [t_1, \infty)$, for a large t_1, then, $Q(t)$ exists on $\mathscr{I}_0$ and satisfies Theorem 1.1. Assumptions 1.1 and 1.2 are employed to show the existence of such $Q(t)$.

On the other hand, if the entries of matrix $A(t)$ (i.e., those of $A(t)$ and $R(t)$) are analytic on $\mathscr{I}_0$, in addition to satisfying Assumptions 1.1 and 1.2, then the solution $Q(t)$ of (1.8) satisfying (2) is analytic on $\mathscr{I}_0$. In this paper, we will apply recent results of global analytic simplifications (diagonalization and triangularization) of a matrix function by H. Gingold and P. F. Hsieh [14] to study a certain type of equations which can be reduced to one with analytic coefficients, in addition to satisfying Assumptions 1.1 and 1.2, to assure the *global analyticity* of the transformation matrix, i.e., that of $Q(t)$. In Section 2, we will discuss the results of global analytic simplification of matrix functions. In Section 3, we will state and prove the main results. As the matrix $Q(t)$ is obtained by the linear system (1.8), we will investigate in Section 4 the conditions for the coefficients $A(t)$ of (1.1) so that $Q(t)$ is analytic at $t = +\infty$. Four examples pertaining to the results of this paper are given in Section 5.

2. Simplification of Matrix Functions

In order to find the conditions which assure that the transformation matrix $I + Q(t)$ given in Theorem 1.1 is globally analytic on $\mathscr{I}_0$, we will investigate in this section the conditions of $A(t)$ which can be reduced to one wich is globally analytic on $\mathscr{I}_0$ besides satisfying Assumptions 1.1 and 1.2. There is a well-developed theory for canonical transformation of matrices with constant entries. The same cannot be said if the entries of the matrix are functions of one or several variables. We summarize recent results of H. Gingold and P. F. Hsieh [14] about the global simplification of a matrix $A(t)$ which is analytic in a finite or infinite real interval, when all of its eigenvalues are real valued in this real interval, to an upper-triangular matrix by a *unitary analytic* matrix. The main result of this section is the following

Theorem 2.1. *Let $A(t)$ be an n by n matrix function with analytic entries on $[a, b]$, where $-\infty \leqslant a < b \leqslant \infty$. Assume that every eigenvalue of $A(t)$ is analytic on $[a, b]$. Then, there exists a unitary matrix $U(t)$ analytic on $[a, b]$ such that*

$$B(t) = U^{-1}(t)\, A(t)\, U(t), \tag{2.1}$$

where $B(t)$ is an upper-triangular matrix whose entries are analytic functions of t on $[a, b]$.

Here the analyticity of a function on $[a, b]$ means the analyticity in an open interval containing $[a, b]$.

The labeling of the eigenvalues is important to assure the analyticity of these eigenvalues. For instance, for

$$F(t)=\begin{bmatrix} 0 & t \\ t & 0 \end{bmatrix}, \qquad t\in(-\infty,\infty), \tag{2.2}$$

we take $\pm t$ as eigenvalues, rather than $\pm |t|$. In this paper, we will presume that *the eigenvalues of a matrix are properly labeled such that each eigenvalue is a real analytic function of t on* $[a, b]$.

The analyticity of the eigenvalues of $A(t)$ plays an important role in the construction of $U(t)$. The following lemma gives a condition which assures the analyticity of an eigenvalue of $A(t)$ on $[a, b]$.

LEMMA 2.2. *Let $A(t)$ be an n by n matrix function with analytic entries on $[a, b]$. If every eigenvalue of $A(t)$ is real valued on $[a, b]$, then all the eigenvalues are analytic on $[a, b]$.*

This lemma is proved in H. Gingold and P. F. Hsieh [14] and F. Rellich [28, 29] by showing that the Puiseux expansion of an eigenvalue about a point on $[a, b]$, say $t=\alpha$, as a root of the characteristic polynomial of $A(t)$, when extended to a complex t-plane, involves only the integer powers of $(t-\alpha)$ and converges uniformly in a neighborhood of $t=\alpha$ (e.g., see E. Hille [21]). This fact also can be generalized as follows.

LEMMA 2.3. *Let $A(t)$ be an n by n matrix function with analytic entries on $[a, b]$. If every eigenvalue of $A(t)$ does not change its argument when t traverses along a closed curve in the complex t-plane, then all the eigenvalues are analytic on $[a, b]$.*

Theorem 2.1 is an extension of Schur's decomposition for a constant matrix (see I. Schur [31] or, e.g., R. Bellman [2], G. Strang [33]) to one with analytic entries.

As a special case of Theorem 2.1, we have the following theorem for a Hermitian matrix.

THEOREM 2.4. *Let $A(t)$ be an n by n matrix function with analytic entries on $[a, b]$. If $A(t)$ is Hermitian on $[a, b]$, then there exists a unitary matrix $U(t)$ analytic on $[a, b]$ such that*

$$B(t)=U^{-1}(t)\,A(t)\,U(t) \tag{2.3}$$

is a diagonal matrix whose elements are analytic on $[a, b]$.

This is the global version of the Rellich Theorem [28, 29]. In fact, since $A(t)$ is Hermitian, all of its eigenvalues are real. It is easy to see that $B(t)$

given by (2.1) satisfies the relation $B^*(t) = B(t)$ in addition to being upper-triangular. Thus $B(t)$ is diagonal.

The proofs of Theorems 2.1 and 2.4 also can be carried out for a matrix satisfying the required conditions in an interval (a, b) ($[a, b)$ or $(a, b]$). Thus, we have

COROLLARY 2.5. *Let $A(t)$ be an n by n matrix function with analytic entries in (a, b) ($[a, b)$ or $(a, b]$), where $-\infty \leqslant a < b \leqslant \infty$. Assume that every eigenvalue of $A(t)$ is analytic in (a, b) ($[a, b)$ or $(a, b]$). Then, there exists a unitary matrix $U(t)$ analytic in (a, b) ($[a, b)$ or $(a, b]$) such that*

$$B(t) = U^{-1}(t)\, A(t)\, U(t), \tag{2.4}$$

where $B(t)$ is an upper-triangular matrix whose entries are analytic functions of t in (a, b) ($[a, b)$ or $(a, b]$).

COROLLARY 2.6. *Let $A(t)$ be an n by n matrix function with analytic entries in (a, b) ($[a, b)$ or $(a, b]$). If $A(t)$ is assumed to be Hermitian in (a, b) ($[a, b)$ or $(a, b]$), then there exists a unitary matrix $U(t)$ analytic in (a, b) ($[a, b)$ or $(a, b]$) such that*

$$B(t) = U^{-1}(t)\, A(t)\, U(t) \tag{2.5}$$

is a diagonal matrix whose elements are analytic in (a, b) ($[a, b)$ or $(a, b]$).

In 1948, W. G. Leavitt [24] proved that if all the entries and the eigenvalues of $A(x)$ are analytic in a compact domain in the complex x-plane, then there exists a matrix $T(x)$ analytic in that domain such that $B(x) = T^{-1}(x)\, A(x)\, T(x)$ is upper-triangular. In 1962, W. Wasow [34] investigated the conditions that, when $A(x)$ and $B(x)$ are analytic functions in a domain, the pointwise existence of $T(x)$ assures the existence of analytic matrix $T(x)$ in a subdomain. In 1965, Y. Sibuya [32] proved that if $A(x)$ is a matrix of analytic functions in a domain and its characteristic polynomial can be factored, then there exists a matrix $T(x)$ analytic globally in this domain such that $B(x)$ is block-diagonal. The same assertion was also proved by P. F. Hsieh and Y. Sibuya [22] in 1966 with matrices that were functions of several variables. In 1975, B. L. J. Braaksma [3] proved also a result similar to that of W. G. Leavitt by means of algebroid functions. In 1978, H. Gingold [9, 10] developed methods to locally triangularize a continuous matrix of several variables and globally block-diagonalize a smooth matrix of several variables. Using algebraic methods, many studied and extended W. Wasow's results, for example, J. Ohm and H. Schneider [27], B. McDonald [26], S. Friedland [6, 7], and R. Guralnick [17, 18].

Even though the global existence of an analytic and unitary matrix $U(t)$ for the Rellich diagonalization [28, 29] has been claimed in the literature (e.g., H. Baumgärtel [1]) and used by H. Gingold [11], H. Gingold and P. F. Hsieh [12, 13], and H. Gingold and V. Trutzer [15], we find it useful to provide a constructive proof to that statement which will be useful as a numerical algorithm as well.

In the process of proving Theorem 2.1, the matrix $A(t)$ is extended to a simply connected complex domain which contains the original real interval such that the extended matrix $\tilde{A}(x)$ is analytic in this domain. By the fact that the eigenvalues are analytic in a domain containing $[a, b]$, we devise an algorithm to find a globally analytic eigenvector of $\tilde{A}(x)$ corresponding to an analytic eigenvalue with unit length on $[a, b]$. This is achieved by devising an algorithm to reduce an analytic matrix into the Smith form (see F. R. Gantmacher [8] or I. Gohberg, P. Lancaster, and L. Rodman [16]) of a meromorphic matrix. However, in the process to meet the normal condition (i.e., unit length) of the eigenvector, the analyticity in the complex sense is lost. Fortunately, the analyticity in the original real interval is recovered, with normality remaining intact, when the eigenvector is restricted to that interval. In the process of finding the orthonormal set of vectors which are analytic on the interval $[a, b]$, a modified Gram-Schmidt process is used in the complex domain and the resulting set of vectors is shown to be analytic on the interval $[a, b]$ and agrees with the straight Gram-Schmidt process on $[a, b]$. Then a mathematical induction process on the dimension n of the matrix $A(t)$ is used to prove Theorem 2.1. Details of the proof are given in [14].

Is noteworthy that if $A(t)$ is a periodic matrix and all the eigenvalues have the same periodicity, then the algorithm in the proof of Theorem 2.1 also assures the periodicity of $U(t)$. Thus we have the following

THEOREM 2.7. *Let $A(t)$ be a matrix defined for $t \in (-\infty, \infty)$. In addition to the assumptions of Theorem* 2.1, *if $A(t)$ and all of its eigenvalues are assumed to be periodic with period ω, then there exists a unitary matrix $U(t)$ analytic in $(-\infty, \infty)$ and periodic with period ω such that*

$$B(t) = U^{-1}(t)\, A(t)\, U(t) \tag{2.6}$$

is an upper-triangular matrix whose elements are analytic in $(-\infty, \infty)$ and periodic with period ω.

Similar to Theorem 2.4, we also can obtain the following theorem for a periodic Hermitian matrix.

THEOREM 2.8. *Let $A(t)$ be a matrix defined for $t \in (-\infty, \infty)$. In addition to the assumptions of Theorem* 2.4, *if $A(t)$ and all of its eigenvalues are* *periodic with period ω in $(-\infty, \infty)$, then there exists a unitary matrix $U(t)$ analytic and periodic with period ω in $(-\infty, \infty)$ such that*

$$B(t) = U^{-1}(t)\, A(t)\, U(t) \tag{2.7}$$

is a diagonal matrix whose elements are analytic and periodic with period ω in $(-\infty, \infty)$.

Theorem 2.1 is strong in the sense that if a matrix $A(t)$ has $C^{\infty}[0, 1]$ entries, there is no guarantee that it has even continuous eigenvectors. For instance (see H. Gingold [12], T. Kato [23, p. 111] or F. Rellich [28, 29]), let

$$A(t) = \exp(-t^{-2}) \begin{bmatrix} 0 & \cos t^{-1} \\ \sin t^{-1} & 0 \end{bmatrix}. \tag{2.8}$$

Then $A(t)$ is $C^{\infty}[0, 1]$ and has eigenvalues

$$\lambda_1(t) = -\lambda_2(t), \qquad \lambda_2(t) = \exp(-t^{-2}) \sqrt{\cos t^{-1} \sin t^{-1}}. \tag{2.9}$$

If

$$u(t) = \begin{bmatrix} w_1(t) \\ w_2(t) \end{bmatrix} \tag{2.10}$$

is a continuous eigenvector corresponding to $\lambda_2(t)$, then,

$$w_2(t) \cos t^{-1} = w_1(t) \sqrt{\cos t^{-1} \sin t^{-1}}, \tag{2.11}$$

$$w_1(t) \sin t^{-1} = w_2(t) \sqrt{\cos t^{-1} \sin t^{-1}}. \tag{2.12}$$

It is easily seen that

$$w_2(t_\nu) = 0 \qquad \text{for} \quad t_\nu = \frac{1}{(\nu\pi)}, \ \nu = 1, 2, ...,$$

and

$$w_1(t_\nu) = 0 \qquad \text{for} \quad t_\nu = \frac{2}{(2\nu - 1)\pi}, \ \nu = 1, 2, ...$$

Let $\nu \to \infty$ and we have $w_1(0) = w_2(0) = 0$, which is a contradiction.

Moreover, the triangular form of a matrix with analytic entries is the best simplification we can hope for since it is not always possible to find an analytic nonsingular matrix to reduce it to a Jordan canonical form. For instance, let

$$A(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}. \tag{2.13}$$

If it is reducible to a Jordan canonical form by $T(t)$, then

$$\begin{aligned} T^{-1}(t)\,A(t)\,T(t) &= \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \qquad (t \neq 0), \\ T^{-1}(0)\,A(0)\,T(0) &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad (t = 0). \end{aligned} \tag{2.14}$$

Thus $T(t)$ must be singular at $t=0$.

It is noteworthy that the results in this section cannot be extended to a matrix of several variables. For instance (see T. Kato [23, p. 116] or F. Riesz and B. Sz.-Nagy [30, p. 379]), let

$$A(t_1, t_2) = \begin{bmatrix} t_1 & t_2 \\ t_2 & -t_1 \end{bmatrix}. \tag{2.15}$$

Then the eigenvalues of $A(t_1, t_2)$ are

$$\lambda_{\pm}(t_1, t_2) = \pm\sqrt{t_1^2 + t_2^2}, \tag{2.16}$$

which are analytic at (t_1, t_2) where $t_1^2 + t_2^2 \neq 0$ but not analytic where $t_1^2 + t_2^2 = 0$.

3. The Main Results

Consider a linear system of differential equations

$$\frac{dy}{dt} = A(t)\,y, \tag{3.1}$$

where y is an n-dimensional vector, $A(t)$ is an $n \times n$ matrix. Let $\lambda_1(t)$, $\lambda_2(t)$, ..., $\lambda_n(t)$ be eigenvalues of $A(t)$. We establish the following theorems which state the conditions under which (3.1) can be reduced to a diagonal system by a globally analytic transformation.

Assume the following:

Assumption 3.1. The matrix $A(t)$ is Hermitian and analytic on $\mathscr{I}_0 = [t_0, \infty)$. Then, by Theorem 2.4, there exists a unitary matrix $U(t)$ analytic on $\mathscr{I}_0$ such that

$$U^{-1}(t)\,A(t)\,U(t) = \Lambda(t) =: \operatorname{diag}\{\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)\}. \tag{3.2}$$

We assume that

Assumption 3.2. For the matrix $U(t)$ given in (3.2),

$$\int_{t_0}^{+\infty} |U'(t)|\, dt < +\infty. \tag{3.3}$$

Now we can establish

THEOREM 3.1. *Under the Assumptions* 1.2, 3.1, *and* 3.2, *there exists an* $n \times n$ *matrix* $Q(t)$ *analytic on* $\mathscr{I}_0$, *such that*

(i) $\lim_{t \to +\infty} Q(t) = 0$;

(ii) *the transformation*

$$y = U(t)[I + Q(t)]\, z \tag{3.4}$$

reduces (3.1) *to*

$$\frac{dz}{dt} = \Lambda(t)\, z, \tag{3.5}$$

where I *is the* $n \times n$ *identity matrix.*

If $A(t)$ is not necessarily Hermitian, but satisfies the following

Assumption 3.3. The matrix $A(t)$ and all of its eigenvalues are analytic on $\mathscr{I}_0$.

Then, by Theorem 2.1, there exists a unitary matrix $U(t)$ analytic on $\mathscr{I}_0$ such that

$$U^{-1}(t)\, A(t)\, U(t) = B(t) =: \Lambda(t) + B_1(t), \tag{3.6}$$

where $B_1(t)$ is a nilpotent upper-triangular matrix. We assume that

Assumption 3.4. For the matrices $U(t)$ and $B_1(t)$ given in (3.6),

$$\int_{t_0}^{+\infty} |U'(t)|\, dt < +\infty, \tag{3.7}$$

$$\int_{t_0}^{+\infty} |B_1(t)|\, dt < +\infty. \tag{3.8}$$

We also can establish

THEOREM 3.2. *Under the Assumptions* 1.2, 3.3, *and* 3.4, *there exists an* $n \times n$ *matrix* $Q(t)$ *analytic on* $\mathscr{I}_0$, *such that*

(i) $\lim_{t \to +\infty} Q(t) = 0$;

(ii) *the transformation*

$$y = U(t)[I + Q(t)]\, z \tag{3.9}$$

reduces (3.1) *to*

$$\frac{dz}{dt} = \Lambda(t)\, z, \tag{3.10}$$

where I *is the* $n \times n$ *identity matrix.*

To see Theorem 3.1, let

$$y = U(t)\, w. \tag{3.11}$$

Then, the vector ω satisfies the differential equation

$$\frac{dw}{dt} = [\Lambda(t) - U^{-1}(t)\, U'(t)]\, w. \tag{3.12}$$

Since $\Lambda(t)$ and $U^{-1}(t)\, U'(t)$ are analytic on $\mathcal{I}_0$ by applying Theorem 1.1 we can get Theorem 3.1.

In a similar manner, Theorem 3.2 follows from Theorem 1.1.

Remark 3.1. In the light of Lemma 2.2, Assumption 3.3 can be replaced by

Assumption 3.3′. The matrix $A(t)$ is analytic on $\mathcal{I}_0$ and all of its eigenvalues are real on $\mathcal{I}_0$.

4. Analyticity at Infinity

As the globally analytic simplification process given in Theorems 2.1 and 2.4 assures the analyticity of $U(t)$ on $\bar{\mathcal{I}}_0 = [t_0, \infty]$, consequently, the same for $\Lambda(t)$ and $B_1(t)$, if $A(t)$ and all of its eigenvalues are analytic on $\bar{\mathcal{I}}_0$, and the matrix $Q(t)$ is a solution of the linear system (1.8), we can find $Q(t)$ analytic on the entire interval $\bar{\mathcal{I}}_0$ if certain additional conditions are satisfied by the coefficient of (3.1).

For the system (3.1), we assume the following:

Assumption 4.1. The matrix $A(t)$ is Hermitian and analytic on $\bar{\mathcal{I}}_0 = [t_0, \infty]$. Then, by Theorem 2.4, there exists a unitary matrix $U(t)$ analytic on $\bar{\mathcal{I}}_0$ such that

$$U^{-1}(t)\, A(t)\, U(t) = \Lambda(t) =: \operatorname{diag}\{\lambda_1(t), \lambda_2(t), ..., \lambda_n(t)\}. \tag{4.1}$$

We assume that

Assumption 4.2. The matrix $\Lambda(t)$ given in (4.1) is of $O(t^{-2})$ at $t=+\infty$.

Now we can establish

THEOREM 4.1. *Under the Assumptions* 4.1 *and* 4.2, *there exists an* $n \times n$ *matrix* $Q(t)$ *analytic on* $\bar{\mathcal{J}}_0$, *such that*

(i) $\lim_{t \to +\infty} Q(t) = 0$;

(ii) *the transformation*

$$y = U(t)[I + Q(t)]\, z \tag{4.2}$$

reduces (3.1) *to*

$$\frac{dz}{dt} = \Lambda(t)\, z, \tag{4.3}$$

where I *is the* $n \times n$ *identity matrix.*

If $A(t)$ is not necessarily Hermitian, but satisfies the following

Assumption 4.3. The matrix $A(t)$ and all of its eigenvalues are analytic on $\bar{\mathcal{J}}_0$. Then, by Theorem 2.1, there exists a unitary matrix $U(t)$ analytic on $\bar{\mathcal{J}}_0$ such that

$$U^{-1}(t)\, A(t)\, U(t) = B(t) = \Lambda(t) + B_1(t), \tag{4.4}$$

where $B_1(t)$ is a nilpotent upper-triangular matrix. We assume that

Assumption 4.4. The matrices $\Lambda(t)$ and $B_1(t)$ given in (4.4) are of $O(t^{-2})$ at $t=+\infty$.

We also can establish

THEOREM 4.2. *Under the Assumptions* 4.3 *and* 4.4, *there exists an* $n \times n$ *matrix* $Q(t)$ *analytic on* $\bar{\mathcal{J}}_0$, *such that*

(i) $\lim_{t \to +\infty} Q(t) = 0$;

(ii) *the transformation*

$$y = U(t)[I + Q(t)]\, z \tag{4.5}$$

reduces (3.1) *to*

$$\frac{dz}{dt} = \Lambda(t)\, z, \tag{4.6}$$

where I *is the* $n \times n$ *identity matrix.*

Note that Assumption 4.2, as well as Assumption 4.4, implies that (ii) of Assumption 1.2 is satisfied. Moreover, the analyticity of $U(t)$ on $\bar{\mathscr{I}}_0$ implies that (3.7) is satisfied, while Assumption 4.4 implies that (3.8) is satisfied. Thus, Theorems 4.1 and 4.2 can be shown similarly to that for Theorems 3.1 and 3.2. The analyticity of $Q(t)$ at $t = +\infty$ follows immediately from the fact that it satisfies the linear equation (1.8) and Assumption 4.2, or Assumption 4.4, respectively. In fact, let $\Phi(t)$ and $\Psi(t)$ be fundamental matrices of

$$\frac{d\Phi}{dt} = [A(t) + R(t)]\,\Phi, \tag{4.7}$$

and

$$\frac{d\Psi}{dt} = A(t)\,\Psi, \tag{4.8}$$

respectively. Then a general solution of (1.8) can be given by

$$Q(t) = \Phi(t)\,C\Psi(t)^{-1} + \int^t \Phi(t)\,\Phi(s)^{-1}\,R(s)\,\Psi(s)\,\Psi(t)^{-1}\,ds, \tag{4.9}$$

where C is an arbitrary constant $n \times n$ matrix. In particular, C can be chosen so that

$$Q(+\infty) = 0. \tag{4.10}$$

By the uniqueness of the solution of (1.8) satisfying (4.10), $Q(t)$ given by (4.9) with such C agrees with that obtained by Theorem 1.1.

5. Examples

In this section we will see four differential equations which are amenable to theorems given in Sections 3 and 4 for the assurance of the existence of globally analytic $Q(t)$.

Example 1. Given

$$\frac{dy}{dt} = A(t)\,y, \tag{5.1}$$

where

$$A(t) = \begin{bmatrix} 2t^2 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.2}$$

which is symmetric and analytic on $[0, +\infty)$. The eigenvalues of $A(t)$ are

$$\lambda_1(t) = t^2 + \sqrt{1+t^4}, \qquad \lambda_2(t) = t^2 - \sqrt{1+t^4}, \tag{5.3}$$

which are real on $[0, +\infty)$, thus analytic there. By the algorithm given in [14], we have a unitary matrix

$$U(t) = \frac{1}{\sqrt{2}(1+t^4)^{1/4}} \begin{bmatrix} (\sqrt{1+t^4}+t^2)^{1/2} & (\sqrt{1+t^4}-t^2)^{1/2} \\ (\sqrt{1+t^4}-t^2)^{1/2} & -(\sqrt{1+t^4}+t^2)^{1/2} \end{bmatrix}, \tag{5.4}$$

which is analytic on $[0, +\infty)$. Then, $U^{-1} = U^T = U$ and

$$U(t)^{-1} A(t)\, U(t) = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix}. \tag{5.5}$$

Furthermore,

$$U'(t) = \frac{1}{\sqrt{2}(1+t^4)^{5/4}} \begin{bmatrix} \sqrt{1+t^4}-t^2 & -(\sqrt{1+t^4}+t^2) \\ -(\sqrt{1+t^4}+t^2) & -(\sqrt{1+t^4}-t^2) \end{bmatrix}. \tag{5.6}$$

Thus, (3.3) is satisfied. Note that

$$D_{12}(t) = \Re[\lambda_1(t) - \lambda_2(t)] = 2\sqrt{1+t^4}. \tag{5.7}$$

For $j=1$, both 1 and $2 \in \mathscr{P}_{12}$, while for $j=2$, $1 \in \mathscr{P}_{21}$ and $2 \in \mathscr{P}_{22}$. Hence, Theorem 3.1 can be applied to obtain the desirable $Q(t)$ on $[0, +\infty)$.

Example 2. Given

$$\frac{dy}{dt} = A(t)\, y, \tag{5.8}$$

where

$$A(t) = \begin{bmatrix} 1 & \dfrac{t+1}{t(t+2)} \\ \dfrac{t+1}{t(t+2)} & 0 \end{bmatrix}, \tag{5.9}$$

which is symmetric and analytic on $[t_0, +\infty]$ $(t_0 > 0)$. The eigenvalues of $A(t)$ are

$$\lambda_1(t) = \frac{(t+1)^2}{t(t+2)}, \qquad \lambda_2(t) = \frac{-1}{t(t+2)}, \tag{5.10}$$

which are real on $[t_0, +\infty]$, thus analytic there. By the algorithm given in [14], we have a unitary matrix

$$U(t) = \frac{1}{\sqrt{t^2+2t+2}} \begin{bmatrix} t+1 & -1 \\ 1 & t+1 \end{bmatrix}, \tag{5.11}$$

which is analytic on $[t_0, +\infty]$. Then, $U^{-1} = U^T$ and

$$U(t)^{-1} A(t)\, U(t) = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix}. \tag{5.12}$$

Moreover,

$$U'(t) = \frac{1}{(t^2+2t+2)^{3/2}} \begin{bmatrix} 1 & t+1 \\ -(t+1) & 1 \end{bmatrix}, \tag{5.13}$$

and it is integrable over $[t_0, +\infty]$. Note that

$$D_{12}(t) = \Re[\lambda_1(t) - \lambda_2(t)] = 1 + \frac{2}{t(t+2)}. \tag{5.14}$$

For $j=1$, both 1 and $2 \in \mathscr{P}_{12}$, while for $j=2$, $1 \in \mathscr{P}_{21}$ and $2 \in \mathscr{P}_{22}$. As Assumption 4.2 is not satisfied by $\lambda_1(t)$, Theorem 4.1 is not applicable to (5.8). However, Theorem 3.1 can be applied to obtain the desirable $Q(t)$ on $[t_0, +\infty)$.

Example 3. Given

$$\frac{dy}{dt} = A(t)\, y, \tag{5.15}$$

where

$$A(t) = \frac{1}{t^2(1+t^2)} \begin{bmatrix} 1+t+t^4 & -1+t-t^3 \\ t+t^2-t^3 & -t+2t^2 \end{bmatrix}, \tag{5.16}$$

which is analytic on $[t_0, +\infty]$ $(t_0 > 0)$. The eigenvalues of $A(t)$ are

$$\lambda_1(t) = \frac{1}{t^2}, \qquad \lambda_2(t) = 1, \tag{5.17}$$

which are real on $[t_0, +\infty]$, thus analytic there. By the algorithm given in [14], we have a unitary matrix

$$U(t) = \frac{1}{\sqrt{1+t^2}} \begin{bmatrix} 1 & t \\ t & -1 \end{bmatrix}, \tag{5.18}$$

which is analytic on $[t_0, +\infty]$. Then, $U^{-1}=U^T=U$ and

$$U(t)^{-1}\,A(t)\,U(t)=\begin{bmatrix}\lambda_1(t) & \dfrac{1}{t^2}\\ 0 & \lambda_2(t)\end{bmatrix}. \tag{5.19}$$

Furthermore,

$$U'(t)=\frac{1}{(1+t^2)^{3/2}}\begin{bmatrix}-t & 1\\ 1 & t\end{bmatrix}. \tag{5.20}$$

Note that

$$D_{12}(t)=\Re[\lambda_1(t)-\lambda_2(t)]=\frac{1}{t^2}-1. \tag{5.21}$$

For $j=1$, $1\in\mathscr{P}_{12}$ and $2\in\mathscr{P}_{11}$, while for $j=2$, both 1 and 2 are $\in\mathscr{P}_{22}$. As Assumption 4.2 is not satisfied by $\lambda_2(t)$, Theorem 4.1 is not applicable to (5.15). However, noting that the off-diagonal entry $1/t^2$ of (5.19) is integrable over $[t_0, +\infty)$, Theorem 3.2 can be applied to obtain the desirable $Q(t)$ on $[t_0, +\infty)$.

Example 4. Given

$$\frac{dy}{dt}=A(t)\,y, \tag{5.22}$$

where

$$A(t)=\frac{1}{t^3}\begin{bmatrix}2t & 1\\ 1 & 0\end{bmatrix}, \tag{5.23}$$

which is symmetric and analytic on $[t_0, +\infty]$ $(t_0>0)$. The eigenvalues of $A(t)$ are

$$\lambda_1(t)=\frac{1}{t^3}(t+\sqrt{1+t^2}), \qquad \lambda_2(t)=\frac{1}{t^3}(t-\sqrt{1+t^2}), \tag{5.24}$$

which are real on $[t_0, +\infty]$, thus analytic there. By the algorithm given in [14], we have an unitary matrix

$$U(t)=\frac{1}{\sqrt{2}(1+t^2)^{1/4}}\begin{bmatrix}(\sqrt{1+t^2}+t)^{1/2} & (\sqrt{1+t^2}-t)^{1/2}\\ (\sqrt{1+t^2}-t)^{1/2} & -(\sqrt{1+t^2}+t)^{1/2}\end{bmatrix}, \tag{5.25}$$

which is analytic on $[t_0, +\infty]$. Then, $U^{-1} = U^T = U$ and

$$U(t)^{-1} A(t) U(t) = \begin{bmatrix} \lambda_1(t) & 0 \\ 0 & \lambda_2(t) \end{bmatrix}. \tag{5.26}$$

Furthermore,

$$U'(t) = \frac{(\sqrt{1+t^2} - t)^{1/2}}{2\sqrt{2}(1+t^2)^{5/4}} \begin{bmatrix} 1 & -\sqrt{1+t^2} - t \\ -\sqrt{1+t^2} - t & -1 \end{bmatrix}. \tag{5.27}$$

Note that

$$D_{12}(t) = \Re[\lambda_1(t) - \lambda_2(t)] = \frac{2}{t^3}\sqrt{1+t^2}. \tag{5.28}$$

For $j = 1$, both 1 and $2 \in \mathscr{P}_{12}$, while for $j = 2$, both 1 and $2 \in \mathscr{P}_{22}$. Also, since $\lambda_1(t)$ and $\lambda_2(t)$ satisfy Assumption 4.4, Theorem 4.1 can be applied to obtain the desirable $Q(t)$ on $[t_0, +\infty]$.

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