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Radial Continuity of Set-Valued Metric Projections

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Some new continuity concepts for metric projections are introduced which are simpler and more general than the usual upper and lower semicontinuity. These concepts are strong enough to generalize a number of known results yet weak enough so that now the converses of many of these generalizations are also valid. In particular, in a large class of normed linear spaces, suns and Chebychev sets can be characterized by a certain continuity property of their metric projections.

1. INTRODUCTION

There has been much recent interest in studying various continuity criteria for the set-valued metric projection onto a set V. Particular interest has centered around the relationship between these criteria and either the structure of the set V itself or the geometry of the whole space. (See, for example, [3], [4], [7], [8], [10], [16], [17], [18], and [21].) In essentially all of these papers, the concepts of lower semicontinuity (l.s.c.) and/or upper semicontinuity (u.s.c.) for set-valued mappings (as defined, for example, in Hahn [12]) played the key role.

In this paper we consider some simpler and more general "radial" continuity criteria (called ORL, IRL, and ORU continuity). Roughly speaking, these criteria require that the restriction of the metric projection to certain prescribed line segments be l.s.c. or u.s.c. We will show that these criteria, which are formally much weaker than l.s.c. or u.s.c., are still strong enough to generalize a number of known results, and weak enough so that many of these theorems now have valid converses (which they did not have under the stronger hypotheses of l.s.c. or u.s.c.).

In particular, in a large class of spaces: suns are characterized by the ORL continuity of their metric projections (Corollary 2.4); Chebyshev sets are characterized by the IRL continuity of their metric projections (Corollary 3.8); and (in every space), those closed convex sets whose metric projections are compact-valued are characterized by the ORU continuity of their metric projections (Corollary 4.7). In what is probably the main result of Section 3, we prove (Theorem 3.6) the denseness of the set of those points whose set of best approximations is contained in a convex subset of a sphere. From this theorem we obtain Corollary 3.8 mentioned above as well as a theorem of Stechkin [19] which asserts—in a strictly convex space—the denseness of the set of points having unique best approximations. In Section 5, a set having both an IRL and ORU continuous metric projection is shown to be boundedly connected and have a "connected-valued" metric projection (Theorem 5.1). As a consequence (Corollary 5.4) a result of Wulbert ([23], [24]) is obtained to the effect that the set of rational functions $R_n^m[a, b]$ in $C[a, b]$ is boundedly connected.

Throughout this paper X will denote a (real or complex) normed linear space, X^* its dual space, and for every $x \in X$ and $r > 0$,

$$B(x, r) = \{y \in X: \| x - y \| < r\}, \qquad S(x, r) = \{y \in X: \| x - y \| = r\}.$$

We sometimes denote the unit sphere $S(0, 1)$ by $S(X)$. If $\varnothing \neq V \subset X$, the distance from a point x to V, denoted $d(x, V)$, is defined by $\inf\{\| x - v \|: v \in V\}$. The *metric projection* onto V is the mapping P_V which takes each element of X into its set of best approximations in V, i.e.

$$P_V(x) = \{v \in V: \| x - v \| = d(x, V)\}.$$

V is called *proximinal* if $P_V(x) \neq \varnothing$ for every $x \in X$. V is called *Chebyshev* if $P_V(x)$ is a single point for each $x \in X$. V is called a *sun* if for each $x \in X$ and $v \in P_V(x)$, $v \in P_V(v + \lambda(x - v))$ for every $\lambda \geqslant 0$. P_V is said to be l.s.c. (resp. u.s.c.) at x if for each open set W with $P_V(x) \cap W \neq \varnothing$ (resp. $P_V(x) \subset W$), there exists a neighborhood U of x such that $P_V(y) \cap W \neq \varnothing$ (resp. $P_V(y) \subset W$) for every $y \in U$. The kernel of the metric projection P_V is the set

$$P_V^{-1}(0) = \{x \in X: 0 \in P_V(x)\}.$$

The line segment joining the points x and y is the set

$$[x, y] = \{\lambda x + (1 - \lambda) y: 0 \leqslant \lambda \leqslant 1\}.$$

The line segment obtained by excluding the end points of $[x, y]$ is denoted by (x, y). The convex hull of a set A is denoted by $\mathrm{co}(A)$.

All other undefined notation or terminology is standard and can be found in [11].

2. ORL Continuity

The results of this section overlap some of those presented in [9]. For completeness we have included the results, but omitted the proofs.

Our first generalization of l.s.c. is the following.

DEFINITION 2.1. Let $V \subset X$ and $x_0 \in X$. P_V is said to be outer radially lower (abbrev. ORL) continuous at x_0 if for every $v_0 \in P_V(x_0)$ and each open set W with $W \cap P_V(x_0) \neq \varnothing$, there exists a neighborhood U of x_0 such that $P_V(x) \cap W \neq \varnothing$ for every x in $U \cap \{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$. P_V is called ORL continuous if it is ORL continuous at each point.

Remark. It is clear that every l.s.c. metric projection is ORL continuous. There are examples where the converse is false, however. E.g. in any space which does not have the property (P) of Brown [10], there exists a (finite-dimensional) subspace V such that P_V is not l.s.c. But from Theorem 2.3 below P_V is ORL continuous. It is easy to check that P_V is always ORL continuous on V as well as each point x where $P_V(x) = \varnothing$. Moreover, if V is a *subspace*, then P_V is ORL (resp. l.s.c.) if and only if P_V is ORL (resp. l.s.c.) on $P_V^{-1}(0)$.

LEMMA 2.2. *Let $V \subset X$ and $x_0 \in X$. The following statements are equivalent.*

(1) *P_V is* ORL *continuous at x_0.*

(2) *For each $v_0, v_1 \in P_V(x_0)$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \varnothing$ for every x in $\{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + \delta\}$.*

(3) *For each $v_0, v_1 \in P_V(x_0)$ and each sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$ with $x_n \to x_0$, $d(v_1, P_V(x_n)) \to 0$ (i.e. there exist $v_n \in P_V(x_n)$ such that $v_n \to v_1$).*

THEOREM 2.3. *Let $V \subset X$ and consider the following statements.*

(1) *V is a sun.*

(2) *P_V is* ORL *continuous.*

(3) *"Local best approximations are global," i.e. for each $x \in X$, every local minimum of the function $\Phi_x(v) = \| v - x \|$ on V is a global minimum.*

(4) *V is a moon (cf.* [1] *or* [9] *for the definition).*

Then (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4).

Remark. In general, the implications (3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (3) are false. To see that (3) $\not\Rightarrow$ (2), we need only to take V to be the complement of the open unit ball in the Euclidean plane. In this case, all best approximations are global but P_V is not ORL continuous at the origin. To see that (4) $\not\Rightarrow$ (3), let X be the Euclidean plane and

$$V = \{(\xi, \eta): (\tfrac{1}{4})\, \xi^2 + \eta^2 \geqslant 1\}.$$

This set is moon but the point $(0, -\tfrac{1}{2})$ has $(0, 1)$ as a local best approximation in V which is not a global best approximation.

We call a space X an *MS*-space if every moon in X is a sun. In such a space all the conditions of Theorem 2.3 are obviously equivalent. In particular, we have

COROLLARY 2.4. *Let X be an MS-space and $V \subset X$. Then V is a sun if and only if P_V is* ORL *continuous.*

Regarding this corollary, it should be mentioned that a large class of concrete spaces are *MS*-spaces. In [1] it was shown in particular that the *MS*-spaces include those of type $C_0(T)$, the real continuous functions vanishing at infinity on a locally compact Hausdorff space T, as well as those spaces of type $l_1(S)$. An even larger class of spaces which are *MS*-spaces was determined in [9]. On the negative side, no strictly convex space can be an *MS*-space [1].

It is interesting to compare Corollary 2.4 with a particular consequence of two results of Vlasov ([21; Theorem 7] and [20; Theorem 13]). These two results, when specialized to Hilbert space, yield the hard part of the following theorem (cf. also Asplund [2] for an alternate proof):

THEOREM. *A Chebyshev set V in a Hilbert space is a sun (i.e. is convex) if and only if P_V is continuous.*

It is still not known whether every Chebyshev set in a Hilbert space is convex. In fact, it is apparently unknown whether there exists a Chebyshev set in *any* space which is not a sun.[1] Finally, we do not know whether Corollary 2.4 is valid in non-*MS*-Spaces.

3. IRL CONTINUITY

A second generalization of l.s.c. is as follows.

DEFINITION 3.1. Let $V \subset X$ and $x_0 \in X$. P_V is said to be inner radially lower (abbrev. IRL) continuous at x_0 if for every $v_0 \in P_V(x_0)$ and each open

[1] Added in proof: C. B. Dunham ("Chebychev sets in $C[0, 1]$ which are not suns," to appear in Canadian Math. Bull.) has recently exhibited such an example.

set W with $W \cap P_V(x_0) \neq \varnothing$, there exists a neighborhood U of x_0 such that $P_V(x) \cap W \neq \varnothing$ for every x in $U \cap \{v_0 + \lambda(x_0 - v_0): 0 \leqslant \lambda \leqslant 1\}$. P_V is called IRL continuous if it is IRL continuous at each point.

Remark. Clearly, each l.s.c. metric projection is IRL continuous. The same example given in the remark following Definition 2.1 shows, using Theorem 3.3 below, that there are IRL continuous metric projections which are not l.s.c. Note that P_V is always IRL continuous on V as well as at each point x with $P_V(x) = \varnothing$. When V is a subspace, then P_V is IRL continuous if and only if it is IRL continuous on $P_V^{-1}(0)$.

LEMMA 3.2. *Let $V \subset X$ and $x_0 \in X$. The following statements are equivalent.*

(1) *P_V is* IRL *continuous at x_0.*

(2) *For each v_0, v_1 in $P_V(x_0)$ and each $\epsilon > 0$, there exists $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \varnothing$ for every x in $\{v_0 + \lambda(x_0 - v_0): 1 - \delta < \lambda \leqslant 1\}$.*

(3) *For each v_0, v_1 in $P_V(x_0)$ and each sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): 0 \leqslant \lambda \leqslant 1\}$ with $x_n \to x_0$, $d(v_1, P_V(x_n)) \to 0$ (i.e. there exist $v_n \in P_V(x_n)$ such that $v_n \to v_1$).*

Proof. (1) $\Rightarrow$ (2) is clear.

(2) $\Rightarrow$ (3). If the result were false, there would exist v_0, v_1 in $P_V(x_0)$ and a sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): 0 \leqslant \lambda \leqslant 1\}$ with $x_n \to x_0$ but $d(v_1, P_V(x_n)) \geqslant \epsilon > 0$ for every n. Choose $\delta > 0$ such that $P_V(x) \cap B(v_1, \epsilon) \neq \varnothing$ for every x in $\{v_0 + \lambda(x_0 - v_0): 1 - \delta < \lambda \leqslant 1\} \equiv R_\delta$. Then for n sufficiently large, $x_n \in R_\delta$ so $d(v_1, P_V(x_n)) < \epsilon$ which is a contradiction.

(3) $\Rightarrow$ (1). Suppose (3) holds but (1) fails. Then there exists $v_0 \in P_V(x_0)$ and an open set W with $P_V(x_0) \cap W \neq \varnothing$ such that for every neighborhood U of x_0 there exists an x in $U \cap \{v_0 + \lambda(x_0 - v_0): 0 < \lambda < 1\}$ such that $P_V(x) \cap W = \varnothing$. Choose v_1 in $P_V(x_0) \cap W$. Then for every n there exists $x_n = v_0 + \lambda_n(x_0 - v_0)$ with $1 - 1/n < \lambda_n < 1$ such that $P_V(x_n) \cap W = \varnothing$. Choose $\epsilon > 0$ such that $B(v_1, \epsilon) \subset W$. Then $P_V(x_i) \cap B(v_1, \epsilon) = \varnothing$ for $i = 0, 1, 2,\ldots$ Hence $x_n \to x_0$ but $d(v_1, P_V(x_n)) \geqslant \epsilon$ for every n, a contradiction.

THEOREM 3.3. *If $P_V(x)$ is convex, then P_V is* IRL *continuous at x.*

Proof. If $P_V(x) = \varnothing$, the result is trivially true. Let $v_0, v_1 \in P_V(x)$ and $x_n \in [x, v_0]$ with $x_n \to x_0$. Thus $x_n = v_0 + (1 - \epsilon_n)(x - v_0)$ where $0 \leqslant \epsilon_n \leqslant 1$ and $\epsilon_n \to 0$. Let $v_n = (1 - \epsilon_n) v_1 + \epsilon_n v_0$. Then $v_n \in P_V(x) \subset V$ and $v_n \to v_1$. Also,

$$\begin{aligned} \| x_n - v_n \| &= (1 - \epsilon_n) \| x - v_1 \| = (1 - \epsilon_n) \| x - v_0 \| \\ &= \| x_n - v_0 \| = d(x_n, V) \end{aligned}$$

so $v_n \in P_V(x_n)$.

Remark. The converse of Theorem 3.3 is false in general. For example, taking X to be the plane with the maximum norm and letting V be the two point set $\{(1, 0), (1, \frac{1}{2})\}$, one sees that $P_V(0) = V$ is not convex but P_V is IRL continuous at 0.

COROLLARY 3.4. *If V is convex or a Chebyshev set, then P_V is* IRL *continuous.*

Remark. Neither of the sufficient conditions of Corollary 3.4 are necessary. For example, by letting X denote the plane with the maximum norm, and

$$V = \{(\xi, \eta): \xi \leqslant 0\} \cup \{(\xi, \eta): \xi \leqslant \eta\},$$

it is seen that V is neither convex nor Chebyshev but P_V is IRL continuous (since $P_V(x)$ is convex for every x).

It will be useful, for proving some later results, to record the following fact. If $x \in S(X)$, then the minimal (necessarily convex) extremal subset of $S(X)$ which contains x is given by

$$\begin{aligned} E(x) &= \{v \in S(X): x = \lambda v + (1 - \lambda)u \text{ for some } 0 < \lambda < 1, u \in S(X)\} \\ &= \{v \in S(X): \| x - \lambda v \| = 1 - \lambda \text{ for some } 0 < \lambda < 1\}. \end{aligned}$$

This result is well-known and easy to prove.

As a consequence of this, we can give a brief proof of another useful result observed by Klee [13]:

LEMMA 3.5. *Let $v \in S(X)$ and $0 < \lambda < 1$. Then the set*

$$S = S(0, 1) \cap S(\lambda v, 1 - \lambda)$$

is star-shaped relative to v.

Proof. Let $x \in S$. Then $\| x - \lambda v \| = 1 - \lambda$ so $v \in E(x)$. Since $E(x)$ is convex, $[v, x] \subset S(0, 1)$. Also, since v, x, and $\lambda v + (1 - \lambda)x$ are in $S(\lambda v, 1 - \lambda)$, it follows that $[v, x] \subset S(\lambda v, 1 - \lambda)$. Hence $[v, x] \subset S$.

THEOREM 3.6. *Let $V \subset X$ be proximinal and suppose that every convex extremal subset of $S(X)$ is finite dimensional. Then for each $x \in X \backslash V$ there exists $v \in P_V(x)$ such that for every $y \in (v, x)$, $\text{co}(P_V(y)) \subset S(y, d(y, V))$. In particular, the set*

$$\{x \in X: \text{co}(P_V(x)) \subset S(x, d(x, V))\}$$

is dense in X.

Proof. Let $x \in X \backslash V$. We may assume $x = 0$ and $d(0, V) = 1$. Since $S(X) = \cup \{E(x): x \in S(X)\}$, we have

$$P_V(0) = \bigcup_{x \in S(X)} [V \cap E(x)] = \bigcup_{v \in S(X) \cap V} [V \cap E(v)].$$

Order the sets $\Psi = \{E(v): v \in S(X) \cap V\}$ by containment. If Φ is a totally ordered subset of Ψ, set $E = \bigcup \{E(v): E(v) \in \Phi\}$. Clearly, E is a convex extremal subset of $S(X)$. Further, since $\dim E < \infty$, it follows that E is the union of only finitely many sets $E(v)$. Thus there exists $E(v) \in \Phi$ such that $E = E(v)$. By Zorn's lemma Ψ has a maximal element $E(v_0)$, $v_0 \in S(X) \cap V$. Let $y = \lambda v_0$, $0 < \lambda < 1$. To complete the proof, it suffices to show that $P_V(y) \subset E(v_0)$. If not, then there is some $v_1 \in P_V(y) \backslash E(v_0)$. Hence $\| v_1 \| = 1$ and

$$\| v_1 - \lambda v_0 \| = \| v_1 - y \| = \| v_0 - y \| = 1 - \lambda$$

which implies $v_0 \in E(v_1)$ and hence $E(v_0) \subset E(v_1)$. But $E(v_0)$ was maximal so $E(v_1) = E(v_0)$ and $v_1 \in E(v_0)$, a contradiction.

Remark. Theorem 3.6 is false in general without the restriction on the finite-dimensionality of the faces of $S(X)$. For example, take $X = L_\infty([0, 1], \mu)$ where μ is Lebesgue measure, and $V = S(X)$. Then V is clearly proximinal. However, if $x \in X$, $\| x \| < \frac{1}{2}$, there exist $v_1, v_2 \in P_V(x)$ such that $\| \frac{1}{2}(v_1 + v_2) \| < 1$. To see this, define, for each $n \geqslant 3$, the set

$$M_n = \left\{ t \in [0, 1]: | x(t) | > \| x \| - \frac{1}{n} \right\}.$$

Then $\mu(M_n) > 0$ and $M_n \supset M_{n+1}$ for every n, and

$$M \equiv \{t \in [0, 1]: | x(t) | = \| x \| \} = \bigcap_{3}^{\infty} M_n .$$

Clearly, $\mu(M) = \lim_n \mu(M_n)$. We consider two cases:

Case 1. $\mu(M) > 0$.

Then we can choose disjoint sets A, B such that $\mu(A) > 0$, $\mu(B) > 0$, and $A \cup B = M$. Define $v_1 = (\operatorname{sgn} x) \chi_A$, $v_2 = (\operatorname{sgn} x) \chi_B$, where χ_E denotes the characteristic function of E. Then $\| v_i \| = 1$ and $\| v_i - x \| = 1 - \| x \|$, i.e. $v_i \in P_V(x)$, but $\| \frac{1}{2}(v_1 + v_2) \| = \frac{1}{2}$.

Case 2. $\mu(M) = 0$.

Define $E_n = M_n \backslash M_{n+1}$. Then (E_n) is a disjoint sequence. By passing to a subsequence, if necessary, we may assume $\mu(E_n) > 0$ for every n. Define

$$v_1 = (\operatorname{sgn} x) \sum_{2}^{\infty} \left(1 - \frac{1}{2n}\right) \chi_{E_{2n}},$$

and

$$v_2 = (\operatorname{sgn} x) \sum_{1}^{\infty} \left(1 - \frac{1}{2n+1}\right) \chi_{E_{2n+1}}.$$

Then $\| v_i \| = 1$, $\| v_i - x \| = 1 - \| x \|$, i.e. $v_i \in P_V(x)$, but $\| \frac{1}{2}(v_1 + v_2)\| = \frac{1}{2}$.

From Theorem 3.6 we immediately obtain the well-known result of Stechkin [19]:

COROLLARY 3.7 [19]. *Let V be a proximinal subset of a strictly convex space X. Then the set*

$$\{x \in X\colon x \text{ has a unique best approximation in } V\}$$

is dense in X.

COROLLARY 3.8. *Let V be a proximinal subset of a strictly convex space. Then P_V is* IRL *continuous if and only if V is Chebyshev.*

Proof. The "if" part follows from Corollary 3.4. Assume P_V is IRL continuous and let $x \in X \backslash V$. By Theorem 3.6 and the strict convexity of X there exists $v \in P_V(x)$ such that each $y \in (v, x)$ has a unique best approximation (viz. v). If $P_V(x)$ contained some $v_1 \neq v$, this would violate the IRL continuity.

One should observe that (the "only if" part of) Corollary 3.8 does *not* follow from Corollary 3.7, but that the stronger conclusion of Theorem 3.6 is necessary.

In the special case when P_V is Hausdorff continuous (resp. lower semicontinuous), the "only if" part of Corollary 3.8 had been established by Blatter, Morris, and Wulbert [4] (resp. Blatter [5]). It is interesting to note that the converses of their results, however, are not valid. This follows from the recent example of Kripke [14] of a Chebyshev subspace, having a discontinuous metric projection, in a strictly convex reflexive space.

A subset V is called *boundedly compact* if the intersection of V with each closed ball is compact.

COROLLARY 3.9. *Let X be a strictly convex and smooth Banach space and $V \subset X$ be boundedly compact. The following are equivalent.*

(1) *P_V is* l.s.c.

(2) *P_V is* IRL *continuous.*

(3) *V is Chebyshev.*

(4) *V is convex.*

(5) *P_V is convex-valued.*

(6) *V is a sun.*

This result follows using Corollary 3.8 and the result of Vlasov [20] that in a smooth Banach space every boundedly compact Chebyshev set is convex. The equivalence of (1), (3), and (4) had been observed earlier by Blatter, Morris, and Wulbert [4].

In the important case when X is smooth, the restriction on the finite-dimensionality of the faces of $S(X)$ in Theorem 3.6 may be dropped.

THEOREM 3.10. *Let X be smooth and $V \subset X$ be proximinal. Let $x_0 \in X \backslash V$ and $v_0 \in P_V(x_0)$. Then for each $x \in (x_0, v_0)$,*

$$\operatorname{co}(P_V(x)) \subset S(x, d(x, V)).$$

In particular, the set

$$\{x \in X: \operatorname{co}(P_V(x)) \subset S(x, d(x, V))\}$$

is dense in X.

Proof. Let H_{v_0} be the unique supporting hyperplane to $S(x_0, \| x_0 - v_0 \|)$ at v_0. Let $x \in (x_0, v_0)$. By Lemma 3.5, the set

$$S = S(x_0, \| x_0 - v_0 \|) \cap S(x, \| x - v_0 \|)$$

is star-shaped about v_0. Choose any $v_1 \in P_V(x)$. Then $v_1 \in S$ and so $[v_0, v_1] \subset S$. Let H be the unique supporting hyperplane to $S(x_0, \| x_0 - v_0 \|)$ at $\frac{1}{2}(v_0 + v_1)$. Then $H \supset [v_0, v_1]$ and so $H = H_{v_0}$. This shows that $P_V(x) \subset H_{v_0}$ and hence $\operatorname{co}(P_V(x)) \subset H_{v_0}$. This completes the proof.

THEOREM 3.11. *Let $V \subset X$. If P_V is* IRL *continuous, then*

$$\operatorname{co}(P_V(x)) \subset S(x, d(x, V)) \qquad \textit{for every} \quad x \in X.$$

Proof. Let $x \in X$. If $P_V(x) = \varnothing$, the result is trivial. Thus assume $P_V(x) \neq \varnothing$ and let $v_1, ..., v_n$ in $P_V(x)$, $\lambda_i > 0$, and $\sum_1^n \lambda_i = 1$. We must show $\sum_1^n \lambda_i v_i \in S(x, d(x, V))$. We proceed by induction in n. For $n = 1$ the result is trivial. Assume the result is true for $n - 1$. We may take $x = 0$ and

$d(0, V) = 1$. Thus we need to show $\| \sum_1^n \lambda_i v_i \| = 1$, and for this it suffices to show that $\| \sum_1^n \lambda_i v_i \| > 1 - \epsilon$ for every $\epsilon > 0$. Write

$$\sum_1^n \lambda_i v_i = \lambda_1 v_1 + (1 - \lambda_1) u, \qquad u = \frac{1}{1 - \lambda_1} \sum_2^n \lambda_i v_i .$$

By the induction hypothesis, $\| u \| = 1$. By IRL continuity, there exists $\delta > 0$ such that

$$B(v_i, \epsilon) \cap P_V(\lambda v_1) \neq \varnothing \qquad \text{for every} \quad 0 < \lambda < \delta$$

$(i = 2,..., n)$. Take any $0 < \lambda < \min\{\epsilon, \delta\}$ and $y_i \in B(v_i, \epsilon) \cap P_V(\lambda v_1)$ $(i = 2,..., n)$. Then

$$\| y_i \| = 1, \qquad \| y_i - \lambda v_i \| = 1 - \lambda, \qquad \| y_i - v_i \| < \epsilon.$$

By the induction step,

$$\frac{1}{1 - \lambda_1} \sum_2^n \lambda_i y_i \in S(0, 1) \cap S(\lambda v_1, 1 - \lambda) = S.$$

Since S is star-shaped relative to v_1 (Lemma 3.5),

$$\lambda_1 v_1 + \sum_2^n \lambda_i y_i = \lambda_1 v_1 + (1 - \lambda_1) \left[\frac{1}{1 - \lambda_1} \sum_2^n \lambda_i y_i \right] \in S.$$

Thus

$$1 - \left\| \sum_1^n \lambda_i v_i \right\| = \left\| \lambda_1 v_1 + \sum_2^n \lambda_i y_i \right\| - \left\| \sum_1^n \lambda_i v_i \right\| \leqslant \sum_2^n \lambda_i \| y_i - v_i \| < \epsilon$$

so $\| \sum_1^n \lambda_i v_i \| > 1 - \epsilon$.

In the special case when P_V is Hausdorff continuous, Theorem 3.11 was established by Blatter, Morris, and Wulbert [4]. Morris (oral communication) gave another proof of their theorem which essentially used only the IRL continuity of P_V. Our only excuse for including our own proof is that it is brief and direct.

The next result was first established in [6] as a consequence of the main "intersection theorem" (Satz 12) of that paper. Since it was shown to have some useful corollaries, and since the proof of the "intersection theorem" of [6] was quite lengthy, it seems worthwhile to record here a short direct proof.

THEOREM 3.12 [6; Satz 13]. *Let V be a sun. Then*

$$\text{co}(P_V(x)) \subset S(x, d(x, V)) \quad \textit{for every} \quad x \in X.$$

Proof. Let $x \in X$. If $P_V(x) = \varnothing$, the result is trivial. Thus assume $P_V(x) \neq \varnothing$ and let $v_1, ..., v_n$ in $P_V(x)$, $\lambda_i > 0$, $\sum_1^n \lambda_i = 1$. To show $\sum_1^n \lambda_i v_i \in S(x, d(x, V))$. [Use induction on n]. For $n = 1$, it is clear. Assume true for $n - 1$. Write

$$\sum_1^n \lambda_i v_i = \lambda_1 v_1 + (1 - \lambda_1) u, \qquad \text{where} \quad u = \frac{1}{1 - \lambda_1} \sum_2^n \lambda_i v_i .$$

Since V is a sun, $v_1 \in P_V(v_1 + \lambda(x - v_1))$ for every $\lambda > 1$ and so

$$d(v_1 + \lambda(x - v_1), V) = \lambda \| x - v_1 \|.$$

It follows that $v_i \in P_V(v_1 + \lambda(x - v_1))$ for every $\lambda > 1$ for $i = 2, ..., n$. By the induction hypothesis, for every $\lambda > 1$

$$u = \frac{1}{1 - \lambda_1} \sum_2^n \lambda_i v_i \in S(x, \| x - v_1 \|) \cap S(v_1 + \lambda(x - v_1), \lambda \| x - v_1 \|) \equiv S_\lambda .$$

Since S_λ is star-shaped relative to v_1, we have

$$\sum_1^n \lambda_i v_i = \lambda_1 v_1 + (1 - \lambda_1) u \text{ in } S_\lambda \subset S(x, \| x - v_1 \|).$$

It follows immediately that *in a strictly convex space every proximinal sun is Chebyshev.*

4. ORU CONTINUITY

Next we give a generalization of u.s.c.

DEFINITION 4.1. Let $V \subset X$ and $x_0 \in X$. P_V is called outer radially upper (abbrev. ORU) continuous at x_0 if for each $v_0 \in P_V(x_0)$ and each open set $W \supset P_V(x_0)$, there exists a neighborhood U of x_0 such that $P_V(x) \subset W$ for every x in $U \cap \{v_0 + \lambda(x_0 - v_0)\colon \lambda \geqslant 1\}$. P_V is called ORU continuous if it is ORU continuous at each point.

Remark. Clearly, every u.s.c. metric projection is ORU continuous. P_V is obviously ORU continuous on V and at each point x with $P_V(x) = \varnothing$. When V is a subspace, P_V is ORU (resp. u.s.c.) if and only if P_V is ORU (resp. u.s.c.) on $P_V^{-1}(0)$.

LEMMA 4.2. *Let $V \subset X$ and $x_0 \in X$. Consider the following statements.*

(1) *P_V is ORU continuous at x_0*

(2) *For each $v_0 \in P_V(x_0)$ and each $\epsilon > 0$ there exists $\delta > 0$ such that*

$$\sup_{v \in P_V(x)} d(v, P_V(x_0)) < \epsilon$$

for every $x \in \{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + \delta\}$.

(3) *For each $v_0 \in P_V(x_0)$ and each sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$ with $x_n \to x_0$,*

$$\sup_{v \in P_V(x_n)} d(v, P_V(x_0)) \to 0$$

(4) *For each $v_0 \in P_V(x_0)$, each sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$ with $x_n \to x_0$, and each sequence (v_n) with $v_n \in P_V(x_n)$,*

$$d(v_n, P_V(x_0)) \to 0$$

(5) *For each $v_0 \in P_V(x_0)$, each sequence (x_n) in $\{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$ with $x_n \to x_0$, and each sequence (v_n) with $v_n \in P_V(x_n)$ and $v_n \to v$, $v \in \overline{P_V(x_0)}$.*

Then (1) $\Rightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5). *Moreover, if $P_V(x_0)$ is compact,* (4) $\Rightarrow$ (1) *and the first four statements are equivalent. If V is compact, then* (5) $\Rightarrow$ (1) *and all five statements are equivalent.*

Proof. (1) $\Rightarrow$ (2). Choose $v_0 \in P_V(x_0)$ and let

$$W = \cup \{B(v, \epsilon/2): v \in P_V(x_0)\} \supset P_V(x_0).$$

Then there exists a $\delta > 0$ such that $P_V(x) \subset W$ for every $x \in \{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + \delta\}$. Let $x \in \{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + \delta\}$ and $v \in P_V(x)$. Then there exists $v' \in P_V(x_0)$ such that $\| v' - v \| < \epsilon/2$ and so $d(v, P_V(x_0)) < \epsilon/2$. It follows that

$$\sup\{d(v, P_V(x_0)): v \in P_V(x)\} \leqslant \epsilon/2 < \epsilon$$

The proofs of the implications (2) $\Rightarrow$ (3) $\Leftrightarrow$ (4) $\Rightarrow$ (5) are routine.

Next assume that $P_V(x_0)$ is compact. If (4) holds but (1) fails then there is an open set $W \supset P_V(x_0)$ such that for every n there is an $x_n \in \{v_0 + \lambda(x_0 - v_0): 1 \leqslant \lambda < 1 + 1/n\}$ such that $P_V(x_n)\backslash W \neq \varnothing$. Choose $v_n \in P_V(x_n)\backslash W$. Then $x_n \to x_0$ so $d(v_n, P_V(x_0)) \to 0$. Choose $y_n \in P_V(x_0)$ such that $\| v_n - y_n \| \to 0$. By passing to a subsequence we may assume $y_n \to y_0$, $y_0 \in P_V(x_0)$. Hence $v_n \to y_0$ also. Since $y_0 \in W$ is open, $v_n \in W$ for n large. But this is a contradiction.

Finally, let V be compact. If (5) holds but (1) fails, then a similar argument yields a contradiction.

Remark. In general, the implications (4) $\Rightarrow$ (1) and (5) $\Rightarrow$ (1) are false. The following example invalidates both implications. Let V be the subset of the Euclidean plane defined by

$$V = \{(\xi, \eta): \xi \geqslant 1\} \cup \{(\xi, \eta): \xi \leqslant -1\}$$
$$\cup \{(\xi, \eta): |\xi| < 1, \eta \geqslant \sqrt{1-\xi^2}\}\backslash\{(1,0), (-1,0)\}.$$

Taking $x_0 = (0, 0)$, we have

$$P_V(x_0) = \{(\xi, \eta): \xi^2 + \eta^2 = 1, \eta > 0\}.$$

Then (4), and hence (5), is satisfied. However, taking $v_0 = (0, 1) \in P_V(x_0)$ and $W = \{(\xi, \eta): \eta > 0\}$, W is open and $W \supset P_V(x_0)$. Now every $x = (\xi, \eta)$ in the set $\{v_0 + \lambda(x_0 - v_0): \lambda > 1\}$ has the property that $\eta < 0$ and

$$P_V(x) = \{(1, \eta), (-1, \eta)\}$$

so $P_V(x) \cap W = \varnothing$. Thus P_V is not ORU continuous at x_0.

LEMMA 4.3. *If V is closed, then* (5) *of Lemma* 4.2 *holds.*

Proof. Let $v_0 \in P_V(x_0)$, $x_n \in \{v_0 + \lambda(x_0 - v_0): \lambda \geqslant 1\}$, $x_n \to x_0$, $v_n \in P_V(x_n)$, and $v_n \to v$. Then $v \in V$ and

$$\begin{aligned} \| x_0 - v \| &\leqslant \| x_0 - x_n \| + \| x_n - v_n \| + \| v_n - v \| \\ &= \| x_0 - x_n \| + d(x_n, V) + \| v_n - v \| \\ &\to d(x_0, V), \end{aligned}$$

i.e. $\| x_0 - v \| \leqslant d(x_0, V)$ so $v \in P_V(x_0)$.

THEOREM 4.4. *Let $V \subset X$ be a closed set and suppose $P_V(x)$ is convex for each x. If P_V is* ORU *continuous, then $P_V(x)$ is compact for each x.*

Proof. If not, there exists $x_0 \in X\backslash V$ and a sequence (y_n) in $P_V(x_0)$ which has no accumulation point. We may assume, by translating, that $y_1 = 0$. Also, by passing to a subsequence if necessary, we may assume $\| y_n \| \geqslant \epsilon$ for every $n \geqslant 2$, and some $0 < \epsilon < 1$. Choose $0 < \eta < \min\{1, \epsilon/(2 \| x_0 \|)\}$. Then $0 \in P_V(\eta x_0)$ and

$$P_V(\eta x_0) \subset \overline{B(0, 2\eta \| x_0 \|)}.$$

Also, $\eta y_n \in P_V(\eta x_0)$ for every n. Further, the sequence (ηy_n) has no accumulation point since (y_n) does not. Define, for each $n \geqslant 2$,

$$\lambda_n = \sup\{\lambda: \lambda \eta y_n \in P_V(\eta x_0)\}.$$

Then $\lambda_n \geqslant 1$. If $0 < \lambda < \lambda_n$, then by the convexity of $P_V(\eta x_0)$, $\lambda \eta y_n = \lambda n y_n + (1 - \lambda) \cdot 0 \in P_V(\eta x_0)$. Since $P_V(\eta x_0)$ is closed, $\lambda_n \eta y_n \in P_V(\eta x_0)$. Clearly, $[(n+1)/n]\, \lambda_n \eta y_n \notin P_V(\eta x_0)$ for each n and $([(n+1)/n]\, \lambda_n \eta y_n)$ has no accumulation point since $[(n+1)/n]\, \lambda_n \geqslant 1$. Since

$$\frac{n+1}{n} \lambda_n \eta \epsilon < \left\| \frac{n+1}{n} \lambda_n \eta y_n \right\| = \frac{n+1}{n} \eta \| x_0 \| < \epsilon,$$

it follows that $[(n+1)/n]\, \lambda_n \eta < 1$ for each n. Since $[0, y_n] \subset P_V(x_0)$ for each n, we have $[(n+1)/n]\, \lambda_n \eta y_n \in P_V(x_0) \subset V$. Hence, from the relation

$$\left\| \frac{n+1}{n} \eta x_0 - \frac{n+1}{n} \lambda_n \eta y_n \right\| = \frac{n+1}{n} \| \eta x_0 - \lambda_n \eta y_n \|$$
$$= \frac{n+1}{n} \| \eta x_0 \| = d\left(\frac{n+1}{n} \eta x_0 , V\right),$$

it follows that $[(n+1)/n]\, \lambda_n \eta y_n \in P_V((n + 1/n)\, \eta x_0)$. Let

$$W = V \Big\backslash \bigcup_{n=2}^{\infty} \left\{\frac{n+1}{n} \lambda_n \eta y_n\right\}.$$

Then W is open and $W \supset P_V(\eta x_0)$. By ORU continuity, $P_V([(n+1)/n]\, \eta x_0) \subset W$ for n sufficiently large. But this contradicts the fact that $[(n+1)/n]\, \lambda_n \eta y_n \notin W$ for every n.

Singer [18] had recently proved Theorem 4.4 in the particular case when V is a subspace and P_V is u.s.c. The proof given above is a refinement of his proof.

A close inspection of the proof of Theorem 4.4 reveals that it is not necessary that $P_V(x)$ be convex for each x but only that each of these sets be star-shaped.

Remark. The theorem is false in general if P_V is not star-shaped-valued. For example, taking $X = l_2$ and $V = X \backslash B(0, 1)$, then P_V is u.s.c. (hence ORU continuous), but $P_V(0) = S(X)$ is not compact.

There is a "converse" to Theorem 4.4.

THEOREM 4.5. *Let V be a sun such that $P_V(x)$ is compact for every $x \in X$. Then P_V is* ORU *continuous.*

Proof. Fix an arbitrary $x_0 \in X$ and $v_0 \in P_V(x_0)$. Let

$$x_n = v_0 + (1 + \epsilon_n)(x_0 - v_0), \qquad \epsilon_n > 0, \quad \epsilon_n \to 0,$$

i.e. $x_n \to x_0$. We need the following.

LEMMA. *If V is a sun and (x_n) is as above, then $P_V(x_0) = \bigcap_1^\infty P_V(x_n)$.*

Proof of Lemma. Let $v \in P_V(x_0)$. Then for each n,

$$\begin{aligned} \| x_n - v \| &\leqslant \| x_n - x_0 \| + \| x_0 - v \| \\ &= \| x_n - x_0 \| + \| x_0 - v_0 \| = \| x_n - v_0 \| \\ &= d(x_n, V) \end{aligned}$$

so $v \in P_V(x_n)$ and $P_V(x_0) \subset \bigcap_1^\infty P_V(x_n)$.

Conversely, if $v \in \bigcap_1^\infty P_V(x_n)$, then $\| x_n - v \| = d(x_n, V)$ for each n implies $\| x_0 - v \| = d(x_0, V)$ so $v \in P_V(x_0)$. This proves the lemma.

Now let W be an open set with $W \supset P_V(x_0)$. Since $P_V(x_0) = \bigcap_1^\infty P_V(x_n)$ and $P_V(x_n)$ is a decreasing sequence of compact sets, there is an integer N such that $P_V(x_n) \subset W$ for all $n \geqslant N$. Thus, for some $\delta > 0$,

$$P_V(x) \subset W \qquad \text{for all} \quad x \in \{v_0 + \lambda(x_0 - v_0) \colon 1 \leqslant \lambda < 1 + \delta\}.$$

It follows that there is a neighborhood U of x_0 such that if $x = v_0 + \lambda(x_0 - v_0)$, $\lambda \geqslant 1$, and $x \in U$, then $1 \leqslant \lambda < 1 + \delta$. Hence P_V is ORU continuous at x_0.

COROLLARY 4.6. *If V is a Chebyshev sun, then P_V is* ORU *continuous.*

Combining Theorems 4.4 and 4.5 we obtain:

COROLLARY 4.7. *Let V be a closed sun with $P_V(x)$ convex for each x. Then P_V is* ORU *continuous if and only if $P_V(x)$ is compact for every x.*

Remark. It is worth noticing that Corollary 4.7 is false with u.s.c. in place of ORU continuity even if V is a subspace. This follows since there exist Chebyshev subspaces with discontinuous metric projections. (The first such example was given by J. Lindenstrauss [15; pp. 87–88]).

There is one case where u.s.c. and ORU continuity coincide.

COROLLARY 4.8. *Let V be a closed hyperplane. The following are equivalent.*

(1) *P_V is* u.s.c.

(2) *P_V is* ORU *continuous*

(3) *$P_V(x)$ is compact for every x.*

The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) follow from above while the implication (3) $\Rightarrow$ (1) is a result of Singer [18].

5. IRL AND ORU CONTINUITY

DEFINITION. A subset V of X is called *boundedly connected* if $V \cap B(x, r)$ is connected for every $x \in X$ and $r > 0$.

This concept was introduced by Wulbert [24]. Observe that every boundedly connected set is connected, but not conversely in general.

THEOREM 5.1. *Let V be a proximinal set such that P_V is both* IRL *and* ORU *continuous. Then V is boundedly connected and $P_V(x)$ is connected for each x.*

Proof. If V were not boundedly connected, there would exist $x_0 \in X$ and $r > 0$ such that $B(x_0, r) \cap V$ is not connected. We may assume $x_0 = 0$. Thus $B(0, r) \cap V = A \cup B$, where A and B are nonempty disjoint sets which are open in V. Clearly, $P_V(0) \subset A \cup B$. We may assume $P_V(0) \cap A \neq \varnothing$. Let $y \in B$. Then there is a $\lambda_0 \in (0, 1)$ such that for every $\lambda \in [\lambda_0, 1]$, $P_V(\lambda y) \subset B$. Let

$$\beta = \inf\{\lambda \in [0, 1]: P_V(\lambda y) \subset B\}.$$

We first note that $P_V(\beta y) \subset B$. For if not, then $P_V(\beta y) \cap A \neq \varnothing$. Choose $v_0 \in P_V(\beta y) \cap A$. For any sequence $x_n \in (\beta y, y)$ such that $x_n \to \beta y$, there exists (by IRL continuity) $v_n \in P_V(x_n) \subset B$ such that $v_n \to v_0 \in A$. But this is impossible since A is open in V and $v_n \in B \backslash A$ for every n. Thus $P_V(\beta y) \subset B$.

On the other hand, since P_V is ORU continuous, it follows that there exists $\epsilon > 0$ such that $P_V(\lambda y) \subset B$ for every $\lambda \in (\beta - \epsilon, \beta)$. But this contradicts the definition of β and proves that V is boundedly connected.

The proof that $P_V(x)$ is connected for each x is virtually the same.

Remark. In the particular case when P_V is l.s.c., u.s.c., and $P_V(x)$ is compact for every x, Theorem 5.1 was established by Blatter, Morris, and Wulbert [4]. Pollul [17a] proved Theorem 5.1 in the particular case when P_V is both l.s.c. and u.s.c. The proof above is an obvious modification of Pollul's proof.

COROLLARY 5.2. *Let V be a Chebyshev set such that P_V is* ORU *continuous. Then V is boundedly connected.*

Proof. By Corollary 3.4, every Chebyshev set has IRL continuous metric projection.

COROLLARY 5.3. *Let V be a Chebyshev sun. Then V is boundedly connected.*

Proof. By Theorem 4.5, every Chebyshev sun has an ORU continuous metric projection.

COROLLARY 5.4 (Wulbert [23], [24]). *The set of rational functions* $R_n^m[a, b]$ *in* $C[a, b]$ *is boundedly connected.*

Proof. It is well-known that $R_n^m[a, b]$ is a Chebyshev sun.

Remark. From the results of this paper, it follows that each Chebyshev subspace V has a metric projection which is ORL, IRL, and ORU continuous. However, P_V may still be discontinuous.

Some Open Questions. The following questions arose naturally during this study. Let $V \subset X$ be proximinal and P_V be ORL continuous.

(1) Must P_V be IRL continuous?

(2) Must V be a sun?

(3) Must $\{x \in X: \text{co}(P_V(x)) \subset S(x, d(x, V))\}$ be dense in X?

We conjecture that the answer to each of these questions is negative. Note however that an affirmative answer to (2) in the case when X is a Hilbert space would have interesting consequences with regard to the convexity of Chebyshev sets. In particular, we could conclude that a Chebyshev subset V of a Hilbert space is convex and only if P_V is ORL continuous.[2]

Note Added in Proof. A preliminary preprint of this paper, with the same title, was GWDG-Bericht Nr. 3, Göttingen, Jan. 1972. Also, an announcement of some of these results appeared as "Some new continuity concepts for metric projections," in *Bull. Amer. Math. Soc.* **78** (1972), 974–978.

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[2] Added in proof: L. P. Vlasov has kindly informed us that his proof of a result in [22] actually shows that in a reflexive locally uniformly convex Banach space, every Chebyshev set with an ORL continuous metric projection is a sun. In particular, *a Chebyshev set in a Hilbert space is convex if and only if its metric projection is* ORL *continuous.*

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