Lagrange and Hermite Interpolation in Banach Spaces*

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Let $f$ map a Banach space $X$ into itself, and let $x_1, x_2, \ldots, x_n$ be distinct points of $X$. Then there exists a polynomial $y(x)$ of degree $(n - 1)$ which interpolates $f$ at these points. Furthermore, $y(x)$ has a Lagrange representation

$$y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} f(x_i).$$

where $w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n)$, $w_i'(x_i)$ is the first Fréchet derivative of $w_i$ at $x_i$, and $L_i$, $i = 1, 2, \ldots, n$, is an appropriately chosen $n$-linear operator. In an analogous manner, an Hermite polynomial $\tilde{y}(x)$ of degree $(2n - 1)$ is derived, which interpolates $f$ and $f'$ at $x_1, x_2, \ldots, x_n$. Finally, if $X$ is a Hilbert space, the polynomials $y(x)$ and $\tilde{y}(x)$ are shown to have simple representations in terms of inner products.

1. INTRODUCTION

Let $X$ and $Y$ be Banach spaces and let $f$ be a function mapping $X$ into $Y$. If $X$ and $Y$ are the real line $\mathbb{R}$, the classical Lagrange and Hermite interpolation problems are, respectively, to find a polynomial $y(x)$ of degree $(n - 1)$ which interpolates $f$ at $n$ given distinct points $x_1, x_2, \ldots, x_n$, and to find a polynomial $\tilde{y}(x)$ of degree $(2n - 1)$ which interpolates $f$ at $x_1, x_2, \ldots, x_n$ while $\tilde{y}'$ interpolates $f'$ at these points. In this paper we solve the Banach space analogs of these two problems, using polynomial operators. That is, we exhibit polynomials $y(x)$ and $\tilde{y}(x)$, of degrees $n - 1$ and $2n - 1$, such that $y$ interpolates $f$ at the $n$ given distinct points $x_1, x_2, \ldots, x_n$, $\tilde{y}$ interpolates $f$, and $\tilde{y}'$ interpolates $f'$ at these points. In particular, we show that $y(x)$ has a Lagrange representation

$$y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} f(x_i).$$ (1)

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where \( w'_i(x_i) \) is the Fréchet derivative of \( w_i \) at \( x_i \), and \( w_i(x) \) is the \( n \)-th degree polynomial

\[
w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n),
\]

\( L_i \) being an appropriately chosen \( n \)-linear operator. In the event \( X \) is a Hilbert space, the polynomials \( \gamma(x) \) and \( \bar{\gamma}(x) \) are shown to have simple representations in terms of inner products.

### 2. Polynomials in a Linear Space

Let \( X \) be a linear space over the field of real (complex) numbers. For each \( k = 1, 2, \ldots \), let \( X^k \) denote the direct product

\[
\underbrace{X \times X \times \cdots \times X}_{k \text{ times}}.
\]

A \( k \)-linear operator \( M \) on \( X \), is a function on \( X^k \) into a linear space \( Y \) which is linear and homogeneous in each of its arguments separately. That is, for each \( i = 1, 2, \ldots, k \),

\[
M(x_1, x_2, \ldots, x_i + y_i, \ldots, x_k) = M(x_1, x_2, \ldots, x_i, \ldots, x_k) + M(x_1, x_2, \ldots, y_i, \ldots, y_k),
\]

and

\[
M(x_1, x_2, \ldots, ax_i, \ldots, x_n) = aM(x_1, x_2, \ldots, x_i, \ldots, x_n).
\]

A \( 0 \)-linear operator \( L_0 \), on \( X \), is a constant function. That is, for some fixed \( y \in Y \), \( L_0x = y \) for all \( x \in X \). We shall identify a \( 0 \)-linear operator \( L_0 \) with its range so that \( L_0x = L_0 \) for all \( x \in X \). In the event \( x_1 = x_2 = \cdots = x_k = x \) we shall adopt the notation

\[
M(x_1, x_2, \ldots, x_k) = Mx^k,
\]

where \( M \) is a \( k \)-linear operator.

For \( k = 0, 1, 2, \ldots, n \), let \( L_k \) be a \( k \)-linear operator on \( X \). Then the operator \( P \) on \( X \) into \( Y \) given by

\[
P(x) = L_0 + L_1x + L_2x^2 + \cdots + L_nx^n
\]

is called a polynomial of degree \( n \) on \( X \).

Let \( \mathcal{L}_n[X, Y] \), \( n = 0, 1, 2, \ldots \), denote the set of \( n \)-linear operators on \( X \) into \( Y \). If \( X = Y \), we shall simply write \( \mathcal{L}_n[X] \); we shall identify \( \mathcal{L}_0[X] \) with \( X \).
If \( L \in \mathcal{L}_n[X, Y], n \geq 1 \), then for each \( x \in X \), \( L(x) \in \mathcal{L}_{n-1}[X, Y] \) is the \((n-1)\)-linear operator defined by
\[
(L(x))(x_2, x_3, \ldots, x_n) = L(x, x_2, x_3, \ldots, x_n).
\]

In general, if \( n > k \geq 1 \), then for each \( x_1, x_2, \ldots, x_k \in X \), \( L(x_1, x_2, \ldots, x_k) \in \mathcal{L}_{n-k}[X, Y] \) is the \((n-k)\)-linear operator defined by
\[
(L(x_1, x_2, \ldots, x_k))(x_{k+1}, \ldots, x_n) = L(x_1, x_2, \ldots, x_n).
\]

If \( L \) is \( n \)-linear \((n > 1)\), we shall let \( a_iL \) denote the \((n-1)\)-linear operator on \( X \) into \( \mathcal{L}[X, Y] \) defined by
\[
(a_iL)(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) = L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n),
\]
where
\[
(L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n))(x) = L(x_1, x_2, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n).
\]

In general, \( n \)-linear operators are not symmetric. That is, it need not be true that
\[
L(x_1, x_2, \ldots, x_n) = L(x_{i_1}, x_{i_2}, \ldots, x_{i_n})
\]
for all permutations \((i_1, i_2, \ldots, i_n)\) of \((1, 2, \ldots, n)\). For this reason, in general,
\[
\partial_i L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \neq \partial_i L(x_1, x_2, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n).
\]

Now let \( L \) be any \( n \)-linear operator and let \( x_1, x_2, \ldots, x_n \) be any points of \( X \). We define a function \( w \) on \( X \) into \( Y \) by
\[
w(x) = L(x - x_1, x - x_2, \ldots, x - x_n).
\]

Clearly, \( w(x) \) is a polynomial
\[
L_n x^n + L_{n-1} x^{n-1} + \cdots + L_1 x + L_0
\]
of degree \( n \) on \( X \), where \( L_n = L \), and \( L_0 = (-1)^n L(x_1, x_2, \ldots, x_n) \). For example, if \( L \) is bilinear,
\[
L(x - x_1, x - x_2) = L x^2 - L(x_1, x) - L(x, x_1) + L(x_1, x_2).
\]
Thus \( L_2 = L \), \( L_0 = L(x_1, x_2) \), and \( L_1 = -L(x_1, \cdot) - L(\cdot, x_2) \).

An \( n \)-linear operator \( L \) is said to be bounded provided there exists a constant \( M > 0 \) for which
\[
\| L(x_1, x_2, \ldots, x_n) \| \leq M \| x_1 \| \cdot \| x_2 \| \cdot \cdots \cdot \| x_n \|.
\]
Analogously to the 1-linear case, it can be proved that an n-linear operator \( L \) is bounded if and only if it is continuous. Continuity of \( L \) is defined in terms of the product topology on \( X^n \). If we define
\[
\| L \| = \inf \{ M : \| L(x_1, x_2, \ldots, x_n) \| \leq M \| x_1 \| \cdot \| x_2 \| \cdot \ldots \cdot \| x_n \| \},
\]
then
\[
\| L \| = \sup \{ \| L(x_1, x_2, \ldots, x_n) : \| x_i \| = 1, i = 1, 2, \ldots, n \}.
\]
(1)
Clearly, whenever \( Y \) is a Banach space, \( L_n[X, Y] \), with the norm (1), is also a Banach space.

Finally, it will be useful to note [3] that \( L_n[X, Y] \) is isometric to \( L_1[X, L_{n-1}[X]] \), which is isometric to \( L_1[X, L_1[X, L_1[X, \ldots, L_1[X, Y] \cdot \ldots ]]] \).

### 3. FRÉCHET DERIVATIVES OF OPERATORS

Let \( f \) be a function mapping an open subset \( V \) of a Banach space \( X \) into a Banach space \( Y \). Let \( x_0 \in V \). If there exists a linear operator \( U \in L_1[X, Y] \) such that
\[
\| f(x_0 + \Delta x) - f(x_0) - U(\Delta x) \| = o(\| \Delta x \|),
\]
then \( U = f'(x_0) \) is called the Fréchet derivative of \( f \) at \( x_0 \). Equivalently,
\[
U(x) = \lim_{t \to 0} \frac{f(x_0 + tx) - f(x_0)}{t},
\]
where the convergence is uniform on the sphere \( \{ x : \| x \| = 1 \} \). It follows from this definition that if \( L \) is a bounded, n-linear operator on \( X \), and \( f(x) = Lx^n \), then \( f'(x) = \sum_{i=1}^{n} \partial_i Lx^{n-1} \). In particular, if \( L \) is bilinear and \( f(x) = Lx^2 \), then \( f'(x) = L(x, \cdot ) + L(\cdot , x) \). If \( L \) is symmetric, then, clearly, \( f'(x) = nLx^{n-1} \).

We shall need the derivative of \( w \). Let \( L \) be n-linear and let \( x_1, x_2, \ldots, x_n \) be points of \( X \). We let \( \partial_i w \) or \( w/(x - x_i) \) denote the operator on \( X \) into \( L_i[X, Y] \) defined by
\[
\partial_i w(z) = L(z - x_1, z - x_2, \ldots, z - x_{i-1}, \cdot, z - x_{i+1}, \ldots, z - x_n).
\]
We set
\[
\partial_i w(z) = (w/(x - x_i))(z) = w(z)/(x - x_i).
\]
It should be noted that the operator \( w/(x - x_i) \) is completely independent of the \( x \) in the denominator; the denominator \( (x - x_i) \) is purely symbolic.
THEOREM 3.1. Let $L$ be a bounded, $n$-linear operator. Let $x_1, x_2, \ldots, x_n \in X$, and set

$$w(x) = L(x - x_1, x - x_2, \ldots, x - x_n).$$

Then $w'(x_0) = \sum_{i=1}^{n} w(x_0)/(x - x_i)$ and, in particular, $w'(x_i) = w(x_i)/(x - x_i) = \partial_i w(x_i)$.

Proof. Let $x_0$ be a fixed point of $X$. Then, using the multilinearity and boundedness of $L$,

$$\|w(x_0 + \Delta x) - w(x_0) - \sum_{i=1}^{n} \frac{w(x_0)}{(x - x_i)} (\Delta x)\| = \|L(x_0 - x_1 + \Delta x, x_0 - x_2 + \Delta x, \ldots, x_0 - x_n + \Delta x) - L(x_0 - x_1, \ldots, x_0 - x_n) - \sum_{i=1}^{n} L(x_0 - x_1, \ldots, x_0 - x_{i-1}, \Delta x, x_0 - x_{i+1}, \ldots, x_0 - x_n)\| \leq \sum_{k=2}^{n} M_k \|\Delta x\|^k = o(\|\Delta x\|),$$

where each $M_k$ is a positive constant arising from $\|L\|$ and from the norms $\|x_0 - x_i\|, i = 1, 2, \ldots, n$.

One can speak also of higher order Fréchet derivatives. If $f: X \to Y$ and if $f'$ exists on an open neighborhood $V$ of $x_0$ in $X$, then $f''(x_0) = (f')'(x_0)$ is a linear operator on $X$ into $\mathcal{L}_1[X, Y]$ for which

$$\|f'(x_0 + \Delta x) - f'(x_0) - f''(x_0)(\Delta x)\| = o(\|\Delta x\|).$$

Thus, $f''(x_0) \in \mathcal{L}_1[X, \mathcal{L}_1[X, Y]]$ and, since $\mathcal{L}_1[X, \mathcal{L}_1[X, Y]]$ is isometric to $\mathcal{L}_2[X, Y]$, it follows that $f''(x_0)$ can be considered a bilinear operator on $X$ into $Y$ which is usually not symmetric. In general, $f^{(n)}(x_0)$, the $n$-th Fréchet derivative of $f$ at $x_0$, is a linear operator on $X$ into $\mathcal{L}_{n-1}[X, Y]$; so that $f^{(n)}(x_0)$ can be considered as belonging to $\mathcal{L}_n[X, Y]$.

Some examples of Fréchet derivatives are instructive. Let $X = Y = R^n$, the (real) Euclidean $n$-space. Then, if $f: X \to Y$ and for $(x_1, x_2, \ldots, x_n) \in X$, $f(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$, where $y_i = f_i(x_1, x_2, \ldots, x_n), i = 1, 2, \ldots, n$, each $f_i$ is a real-valued function of $n$ real variables. It can then be shown that
if each \( f_i \) has continuous first partial derivatives on some open set \( V \) in \( X \),
then \( f'(x) \) exists on \( V \) and is given by the matrix
\[
\frac{\partial f_1}{\partial x_1} (x) \quad \frac{\partial f_1}{\partial x_2} (x) \quad \cdots \quad \frac{\partial f_1}{\partial x_n} (x)
\]
\[
\frac{\partial f_2}{\partial x_1} (x) \quad \frac{\partial f_2}{\partial x_2} (x) \quad \cdots \quad \frac{\partial f_2}{\partial x_n} (x)
\]
\[
\vdots
\]
\[
\frac{\partial f_n}{\partial x_1} (x) \quad \frac{\partial f_n}{\partial x_2} (x) \quad \cdots \quad \frac{\partial f_n}{\partial x_n} (x)
\]
\[
= \left( \frac{\partial f_i}{\partial x_j} (x) \right), \quad i, j = 1, 2, \ldots, n,
\]
which is the gradient of \( f \) at \( x \). Analogously, if each \( f_i \) has continuous second
partials on some neighborhood \( U \) of \( x \), then \( f''(x) \) is given by the three-way
matrix
\[
\frac{\partial^2 f_1}{\partial x_j \partial x_k} (x), \quad i, j, k = 1, 2, \ldots, n,
\]
which is the Hessian of \( f \) at \( x \).

More generally, one can show that if \( L \) is a bounded, \( n \)-linear operator
on \( X \), then \( L^{(k)}(x) \) is a bounded, \( k \)-linear operator on \( X \).

4. The Interpolation Problem — Existence

Let \( c_1, c_2, \ldots, c_n \) be points of a Banach space \( X \). The interpolation problem
is that of finding, for each sequence \( \{x_1, x_2, \ldots, x_n\} \) of distinct points of \( X \),
a polynomial operator \( p \) which interpolates \( \{c_1, c_2, \ldots, c_n\} \) at \( \{x_1, x_2, \ldots, x_n\} \),
so that \( p(x_i) = c_i \). We shall prove that there always exists a polynomial of
degree \( (n - 1) \) which solves the interpolation problem.

To this end, let \( L \) be a bounded \( n \)-linear operator in \( \mathcal{L}_n[X] \); let \( x_1, x_2, \ldots, x_n \)
be distinct points of \( X \) and let \( w(x) = L(x - x_1, x - x_2, \ldots, x - x_n) \). Then
\( w \) is a polynomial of degree \( n \) mapping \( X \) into \( X \), and
\[
\frac{w(x)}{(x - x_i)} = \partial_i w(x) = L(x - x_1, x - x_2, \ldots, x - x_{i-1}, x - x_{i+1}, \ldots, x - x_n),
\]
is a polynomial of degree \( (n - 1) \) which maps \( X \) into \( \mathcal{L}_1[X] \). We have shown
that \( w'(x) = \sum_{i=1}^{n} w(x)/(x - x_i) \), so that
\[
\frac{w'(x_i)}{(x - x_i)} = \partial_i w(x_i) = w(x_i)/(x - x_i) \]
is a linear operator. Thus, should \( w'(x_i) \) be nonsingular for \( i = 1, 2, \ldots, n \),
then since \( l_i(x) = [w'(x_i)]^{-1} w(x)/(x - x_i) \), \( l_i \) would be a linear and operator-valued function having the property

\[
l_i(x_j) = \delta_{ij} I.
\]

Furthermore, for each \( x_0 \in X \), it is easily seen that \([l_i(x)](x_0) = l_i(x) \, x_0\) is a polynomial of degree \((n - 1)\). That is, we have proved

**THEOREM 4.1.** If there exists an \( n \)-linear operator \( L \) such that \([w'(x_i)]^{-1}\) exists for each \( i = 1, 2, \ldots, n \), where

\[
w(x) = L(x - x_1, x - x_2, \ldots, x - x_n),
\]

then the Lagrange polynomial \( y(x) \) of degree \((n - 1)\) given by

\[
y(x) = \sum_{i=1}^{n} l_i(x) \, c_i \left( \sum_{i=1}^{n} l_i(x) f(x_i) \right),
\]

where \( l_i(x) = [w'(x_i)]^{-1} w(x)/(x - x_i) = [w'(x_i)]^{-1} \partial_i w(x) \), solves the interpolation problem (interpolates the function \( f \) at the \( n \) distinct points \( x_1, x_2, \ldots, x_n \) of \( X \)).

Thus, to solve the interpolation problem, it is enough to prove that such an \( n \)-linear operator exists. It would actually suffice to prove the existence of a family \( \{L_1, L_2, \ldots, L_n\} \) of \( n \)-linear operators having the property that \([w_i'(x_i)]^{-1}\) exists for \( i = 1, 2, \ldots, n \), where \( w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n) \). If this were the case, we could take

\[
y(x) = \sum_{i=1}^{n} [w_i'(x_i)]^{-1} \frac{w_i(x)}{(x - x_i)} \, (c_i)
\]

as our interpolating polynomial. We shall prove the existence of such a family of \( L_i \)'s.

**THEOREM 4.2.** Let \( x_1, x_2, \ldots, x_n \) be distinct points of a Banach space \( X \). Then for each \( i = 1, 2, \ldots, n \) there exists an \( n \)-linear operator \( L_i \) for which \([w_i'(x_i)]^{-1}\) exists, where

\[
w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n).
\]

Furthermore, the \( L_i \)'s can be chosen so that \( w_i'(x_i) = I \), where \( I \) is the identity operator in \( \mathcal{L}_1[X] \).

**Proof.** We start with \( i = 1 \). We must produce an \( n \)-linear operator \( L_1 \) for which \( w_1'(x_1) \) exists and is nonsingular, where

\[
w_1(x) = L_1(x - x_1, x - x_2, \ldots, x - x_n).
\]
Recall that if such an $L_1$ exists, then

$$w_1'(x_1) = \frac{w_1(x_1)}{(x - x_1)} = \partial_1 w_1(x_1)$$

$$= L_1(\cdot, x_1, x_2, x_1 - x_3, \ldots, x_1 - x_n),$$

which belongs to $\mathcal{L}_1[X]$. Also, $L_1 : X^{n-1} \to \mathcal{L}_1[X]$. With this in mind, let $X_{ij} = \text{span}\{x_i - x_j\}$. Since each $X_{ij}$ ($j = 2, 3, \ldots, n$) is one-dimensional, there exist continuous projections $P_{1j}$ of $X$ onto $X_{ij}$. Define

$$T_1 : X_{12} \times X_{13} \times \cdots \times X_{1n} \to \mathcal{L}_1[X]$$

by linearity, through the equation

$$T_1(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n) = I.$$

Then $T_1$ is a bounded (continuous), $(n - 1)$-linear operator in

$$\mathcal{L}_1[X_{12} \times X_{13} \times \cdots \times X_{1n}, Y].$$

That is,

$$\| T_1(a_2(x_1 - x_2), a_3(x_1 - x_3), \ldots, a_n(x_1 - x_n)) \|
= \| a_2a_3 \cdots a_nT_1(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n) \|
= \| a_2a_3 \cdots a_n \| \| I \|
= \frac{1}{\| x_1 - x_2 \| \| x_1 - x_3 \| \cdots \| x_1 - x_n \|} \| a_1(x_1 - x_2) \| \cdots \| a_n(x_1 - x_n) \|,$$

so that $\| T_1 \| = 1/\| x_1 - x_2 \| \| x_1 - x_3 \| \cdots \| x_1 - x_n \|$.

We extend $T_1$ to a continuous, $(n - 1)$-linear operator $T_1 : X^{n-1} \to \mathcal{L}_1[X]$ through the projections $P_{1j}$. That is, we define

$$T_1(y_1, y_2, \ldots, y_{n-1}) = T_1(P_{12}y_1, P_{13}y_2, \ldots, P_{1n}y_{n-1}).$$

Since the projections $P_{1j}$ are linear and continuous, it follows that $T_1$ is $(n - 1)$-linear and continuous. In particular, the map $P$,

$$P : X^{n-1} \to X_{12} \times X_{13} \times \cdots \times X_{1n}$$

given by $P(y_2, y_3, \ldots, y_n) = (P_{12}y_2, P_{13}y_3, \ldots, P_{1n}y_n)$ is continuous, so that the composition $T_1 \circ P - T_1$ is continuous.

Now define the $n$-linear operator $L_1$ by

$$L_1(y_1, y_2, \ldots, y_n) = [T_1(y_2, y_3, \ldots, y_n)](y_1).$$
The $n$-linearity of $L_1$ follows directly from the $(n - 1)$-linearity of $T_1$ and the fact that $T_1$ is linear and operator-valued. The boundedness of $T_1$ is also apparent. If $P_{1k} y_k = a_k [ (x_1 - x_k) / ||x_1 - x_k|| ]$, then $|| P_{1k} y_k || = | a_k |$. Thus,

$$L_1(y_1, y_2, \ldots, y_{n-1}, y_n) = [ T_1(y_2, y_3, \ldots, y_n)](y_1) = [ T_1(P_{12} y_2, P_{13} y_3, \ldots, P_{1n} y_n)](y_1)$$

$$= \frac{a_2 \cdot a_3 \cdots a_n}{||x_1 - x_2|| \cdot ||x_1 - x_3|| \cdots ||x_1 - x_n||} [ \bar{T}_1(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n)](y_1)$$

$$= \frac{a_2 \cdot a_3 \cdots a_n}{||x_1 - x_2|| \cdot ||x_1 - x_3|| \cdots ||x_1 - x_n||} y_1.$$

Therefore, if $K = 1 / ||x_1 - x_2|| \cdot ||x_1 - x_3|| \cdots ||x_1 - x_n||$, then

$$||L_1(y_1, y_2, \ldots, y_n)|| = K ||a_1 \cdot a_2 \cdots a_n|| ||y_1||$$

$$= K ||P_{12} y_2|| \cdot ||P_{13} y_3|| \cdots ||P_{1n} y_n|| \cdot ||y_1||$$

$$\leq K ||y_1|| \cdot ||y_2|| \cdots ||y_n||,$$

since each $P_{1k}$ is a projection and $||P_{1k} y|| = ||P_{1k}|| \cdot ||y||$.

Now let $w_1(x) = L_1(x - x_1, x - x_2, \ldots, x - x_n)$. Since $L_1$ is a bounded, $n$-linear operator, $w_1(x)$ is differentiable and

$$w_1'(x_1) = \frac{w(x_1)}{(x - x_1)}$$

$$= L_1(' , x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n)$$

$$= \bar{T}_1(x_1 - x_2, x_1 - x_3, \ldots, x_1 - x_n)$$

$$= I.$$

Thus $w_1'(x_1)$ is a non-singular, linear operator.

A similar line of argument proves the existence, for each $i = 1, 2, \ldots, n$, of an $n$-linear operator $L_i$ for which $w_i'(x_i) = I$, where

$$w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n).$$

This completes the proof of the theorem.

As a direct result of Theorem 4.2 we have

**Theorem 4.3.** The interpolation problem can always be solved by a polynomial $y(x)$ of degree $(n - 1)$ having a Lagrange representation

$$y(x) = \sum_{i=1}^{n} l_i(x) c_i ,$$
where \( l_i(x) = [w_i'(x_i)]^{-1} w_i(x)/(x - x_i) = [w_i'(x_i)]^{-1} \partial_i w_i(x) \) and \( w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_n) \) for appropriately chosen \( n \)-linear operators \( L_1, L_2, \ldots, L_n \).

In the event \( X \) is a Hilbert space with inner product \((x, y)\), Theorem 4.2 also yields a representation theorem. Consider the projection \( P_{1j} \) of \( X \) onto \( X_{1j} \) given in the proof of Theorem 4.2. If \( X \) is a Hilbert space, then

\[
P_{1j} y_j = \left( y_j, \frac{x_1 - x_j}{\| x_1 - x_j \|} \right) \frac{x_1}{\| x_1 - x_j \|}.
\]

Thus

\[
L_1(y_1, y_2, \ldots, y_n) = \frac{(y_2, x_1 - x_2) \cdot (y_3, x_1 - x_3) \cdots (y_n, x_1 - x_n)}{\| x_1 - x_2 \|^2 \| x_1 - x_3 \|^2 \cdots \| x_1 - x_n \|^2} I(y_1).
\]

In particular, since \( w_1'(x_1) = 1 \),

\[
l_1(x) = I \circ \frac{w_1(x)}{(x - x_1)} = L_1(*, x - x_2, x - x_3, \ldots, x - x_n)
\]

\[
= \frac{(x - x_2, x_1 - x_2) \cdot (x - x_3, x_1 - x_3) \cdots (x - x_n, x_1 - x_n)}{\| x_1 - x_2 \|^2 \| x_1 - x_3 \|^2 \cdots \| x_1 - x_n \|^2} I.
\]

Analogously, one can prove that

\[
l_j(x) = \left[ \prod_{k=1 \atop k \neq j}^{n} (x - x_k, x_j - x_k) \right] \left[ \prod_{k=1 \atop k \neq j}^{n} \| x_j - x_k \| \right]^{-1} I.
\]

Thus we arrive at

**Theorem 4.4.** Let \( X \) be a Hilbert space with inner product \((x, y)\) and let \( c_1, c_2, \ldots, c_n \) be points of \( X \). Then, for any distinct points \( x_1, x_2, \ldots, x_n \) of \( X \), the polynomial \( y(x) \) of degree \( n - 1 \), given by

\[
y(x) = \sum_{i=1}^{n} \frac{\pi_i(x)}{\pi_i(x_i)} c_i,
\]

where

\[
\pi_i(x) = \prod_{k=1 \atop k \neq i}^{n} (x - x_k, x_i - x_k),
\]

satisfies \( y(x_i) = c_i \), \( i = 1, 2, \ldots, n \).

This theorem is evident by inspection; however, it is interesting to note how it followed naturally from the theory of Theorems 4.2 and 4.3.
5. HERMITE INTERPOLATION IN BANACH SPACES

Recall the classical Hermite polynomial \( y(x) \) of degree \((2n - 1)\) which interpolates a real-valued function \( f \) of a real variable at the \( n \) distinct points \( x_1, x_2, \ldots, x_n \) and for which \( y'(x) \) interpolates \( f' \) at these points. This \( y(x) \) is given by the formula

\[
y(x) = \sum_{i=1}^{n} \left( H_i(x) f(x_i) + \overline{H}_i(x) f'(x_i) \right),
\]

where \( H_i(x) = [1 - 2l_i'(x_i)(x - x_i)] l_i^w(x) \), and \( \overline{H}_i(x) = (x - x_i) l_i^w(x) \). Here \( l_i(x) \) is the polynomial \( w(x)/w'(x_i)(x - x_i) \) occurring in the classical Lagrange formula, and \( w(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \). It follows that

\[
H_i(x_j) = \delta_{ij} = \overline{H}_i(x_j),
\]

and

\[
H_i'(x_j) = 0 = \overline{H}_i(x_j), \quad \text{for } i, j = 1, 2, \ldots, n.
\]

Now suppose \( X \) is a Banach space and \( f \) is a function from \( X \) into \( X \) which has a continuous Fréchet derivative at \( n \) distinct points \( x_1, x_2, \ldots, x_n \) of \( X \). Referring to Theorem 4.2, let \( l_i(x) = [w_i'(x_i)]^{-1} w_i(x)/(x - x_i) = [w_i'(x_i)]^{-1} \delta_{ij} w_i(x) \). Since \( l_i(x) \) is linear and operator-valued, \( l_i^w(x) = l_i(x) \circ l_i(x) \), being the composition of two linear operators, is itself linear and operator-valued. Furthermore, \( l_i' : X \to \mathcal{L}_1[X, \mathcal{L}_1[X]] \) so that \( [l_i'(x)](y) \) is linear and operator-valued. It is thus obvious that, for each \( x \in X \), \( l_i'(x_j)(x - x_i) \) is linear and operator-valued. We now define the Banach space analog of the above function \( H_i(x) \) to be the linear operator-valued function on \( X \):

\[
H_i(x) = [I - 2l_i'(x_i)(x - x_i)] l_i^w(x),
\]

where \( I \) is the identity in \( \mathcal{L}_1[X] \). Since \( l_i(x_i) = \delta_{ij} I \), it is evident that

\[
H_i(x_i) = \delta_{ij} I. \quad (1)
\]

Furthermore, we can show that \( H_i'(x_j) = 0 \), the zero linear operator from \( X \) to \( \mathcal{L}_1[X] \), for \( i, j = 1, 2, \ldots, n \). A proof of this requires some basic facts about Fréchet derivatives [5].

If \( A \) is a linear operator from \( X \) into \( Y \), then \( A'(x) = A \) for all \( x \in X \). If \( F : X \to Y \) and \( F(x) = L_0 \), a constant, for all \( x \in X \), then \( F'(x) = 0 \in \mathcal{L}_1[X, Y] \) for all \( x \in X \). Let \( X, Y \) and \( Z \) be Banach spaces, and let \( F : X \to Y \) and \( G : Y \to Z \) be functions such that \( F \) is differentiable at \( x_0 \) and \( G \) is differentiable at \( y_0 = F(x_0) \). Then \( GF \) is differentiable at \( x_0 \), and \( (GF)'(x_0) = G'(y_0) F'(x_0) \). In particular, if \( G \) is linear, \( (GF)'(x_0) = GF'(x_0) \). Finally,
LEMMA 5.1. Let A and B be functions from X into \( \mathcal{L}_1[X] \) which are bounded, linear and operator-valued. If both A and B are differentiable at \( x_0 \) and if \( F(x) = A(x)B(x) \), then

\[
F'(x_0)(x) = A(x_0)B'(x_0)(x) + A'(x_0)(x)B(x_0).
\]

Proof. The proof follows directly from the continuity of A and B at \( x_0 \) and the definition of the Fréchet derivative.

Now let \( A_i(x) = I - 2l_i'(x_i)(x - x_i) \), \( B_i(x) = l_i^2(x) \). Then \( A_i'(x_0) = -2l_i'(x_i) \) since \( I \) and \( -2l_i'(x_i)(x_i) \) are constant and \( l_i'(x_i) \) is a linear operator. Using Lemma 5.1 we see that

\[
B_i'(x_j)(x) = l_i(x_j)l_i'(x_j)(x) + l_i'(x_j)(x)l_i(x_j)
\]

so that

\[
B_i'(x_j)(x) = \begin{cases} 0 \in \mathcal{L}_1[X] & \text{if } j \neq i, \\ -2l_i'(x_i)(x) & \text{if } j = i. \end{cases}
\]

But \( H_i(x) = A_i(x)B_i(x) \) so that, invoking again Lemma 5.1,

\[
H_i'(x_j)(x) = A_i'(x_j)(x)B_i(x_j) + A_i(x_j)B_i'(x_j)(x)
\]

\[
= -2l_i'(x_i)(x)l_i^2(x_j) + [I - 2l_i'(x_i)(x_j - x_i)]B_i'(x_j)(x)
\]

\[
= -2l_i'(x_i)(x)\delta_{ij}l + [I - 2l_i'(x_i)(x_i - x_j)][(\delta_{ij}l)l_i'(x_i)(x_j) + l_i'(x_j)(x)(\delta_{ij}l)]
\]

\[
= \begin{cases} 0 \in \mathcal{L}_1[X] & \text{if } j \neq i, \\ -2l_i'(x_i)(x) - 2l_i'(x_i)(x) = 0 \in \mathcal{L}_1[X] & \text{if } j = i. \end{cases}
\]

That is, \( H_i'(x_j) = 0 \in \mathcal{L}_1[X, \mathcal{L}_1[X]] \) for all \( i, j = 1, 2, \ldots, n \).

If \( \overline{H}_i(x) \) were a polynomial of degree \( 2n - 1 \) from \( X \) into \( X \) for which \( \overline{H}_i(x_j) = 0 \) for all \( i, j = 1, 2, \ldots, n \), and for which \( \overline{H}_i'(x_j) = \delta_{ij}l \), then

\[
y(x) = \sum_{i=1}^{n} \{H_i(x) f(x_i) + f'(x_i) \overline{H}_i(x)\}
\]

would be a polynomial of degree \( 2n - 1 \) interpolating \( f \) at \( x_1, x_2, \ldots, x_n \), with \( y' \) interpolating \( f' \) at these points. This follows directly from

\[
y'(x) = \sum_{i=1}^{n} H_i'(x) f(x_i) + f'(x_i) \overline{H}_i'(x).
\]

Note that, since \( H_i(x) \in X \) and \( f'(x_i) \in \mathcal{L}_1[X] \), \( f'(x_i) \) must precede \( H_i(x) \) in formula (2). Looking at the proof of Theorem 4.2, we find it can be readily
adapted to produce a \((2n - 1)\)-linear operator \(L_i\), for each \(i = 1, 2, \ldots, n\), for which \([w'_i(x_i)]^{-1}\) exists and equals \(I\), where
\[
w_i(x) = L_i(x - x_1, x - x_2, \ldots, x - x_i, \ldots, x - x_n, x - x_n) = L_i((x - x_1)^2, \ldots, (x - x_i)^2, \ldots, (x - x_n)^2).
\]

It follows easily that
\[
\Pi_i(x) = [w'_i(x_i)]^{-1} w_i(x) = w_i(x)
\]
obey the following relations:
\[
\Pi_i(x) = 0 \quad \text{for all} \quad i, j = 1, 2, \ldots, n,
\]
\[
\Pi'_i(x) = \delta_{ij} I.
\]

Thus we arrive at

**Theorem 5.2.** Let \(x_1, x_2, \ldots, x_n\) be distinct points of a Banach space \(X\) and let \(f : X \to X\) be differentiable at \(x_1, x_2, \ldots, x_n\). Then there exists a polynomial \(y\) of degree \((2n - 1)\),
\[
y(x) = \sum_{i=1}^{n} \{H_i(x)f(x_i) + f'(x_i) \Pi_i(x)\},
\]
which interpolates \(f\) at \(x_1, x_2, \ldots, x_n\), with \(y'(x)\) interpolating \(f'\) at these points. Furthermore,
\[
H_i(x) = [I - 2l_i(x)(x - x_i)] l_i^2(x), \quad \text{and} \quad \Pi_i(x) = [w'_i(x_i)]^{-1} w_i(x),
\]
where \(w_i(x) = L_i((x - x_1)^2, \ldots, (x - x_i)^2, \ldots, (x - x_n)^2), L_i\) being an appropriately chosen \((2n - 1)\)-linear operator. \(I\) is the identity in \(L_1[X]\).

In the event \(X\) is a Hilbert space, we can obtain a simple representation of \(y(x)\) in terms of inner products. First, one can show
\[
\Pi_i(x) = \frac{\pi_i^2(x)}{\pi_i^2(x_i)} (x - x_i), \quad \text{where} \quad \pi_i(x) = \prod_{k=1}^{n} (x - x_k, x_i - x_k)
\]
and (,) denotes inner product. Then, since \(l_i(x) = \pi_i(x)/\pi_i(x_i) I\), it follows upon differentiation that
\[
l'_i(x)(y) = \sum_{j=1}^{n} \frac{\pi_i(x) \cdot (y, x - x_j)}{(x - x_j, x_i - x_j) \cdot \pi_i(x_i)} I.
\]
Thus

$$I_i'(x_i)(x - x_i) = \sum_{j=1 \atop j \neq i}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)} I$$

and

$$H_i(x) = \left[ 1 - \sum_{j=1 \atop j \neq i}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)} \right] \frac{\pi_i^2(x)}{\pi_i^2(x_i)} I.$$ 

Therefore, we arrive at

**Theorem 5.3.** Let $X$ be a Hilbert space with inner product $(x, y)$ and let $x_1, x_2, \ldots, x_n$ be distinct points of $X$. Then the polynomial of degree $2n - 1$ given by

$$y(x) = \sum_{i=1}^{n} \{H_i(x)f(x_i) + f'(x_i)\bar{H}_i(x)\},$$

where

$$H_i(x) = \left[ 1 - \sum_{j=1 \atop j \neq i}^{n} \frac{(x - x_i, x_i - x_j)}{(x - x_j, x_i - x_j)} \right] \frac{\pi_i^2(x)}{\pi_i^2(x_i)} I$$

and

$$\bar{H}_i(x) = \left[ \frac{\pi_i^2(x)}{\pi_i^2(x_i)} \right] (x - x_i),$$

interpolates the function $f: X \to X$, while $y'$ interpolates $f'$ at $x_1, x_2, \ldots, x_n$. 

**References**