CHARACTERIZATION OF MONOIDS BY PROPERTIES OF REGULAR ACTS

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There has been done quite some research describing monoids by properties of their categories of left acts. This approach is usually called homological classification of monoids. In most cases the properties around projectivity and around injectivity have been used for homological classifications. The present article takes regularity of acts as a clue for homological classification of monoids. We analyze monoids over which all acts with one of the properties around projectivity or injectivity are regular and conversely monoids over which all regular acts have one of these properties.

Introduction

In module theory three different definitions of a regular module have been introduced. For Ware a regular module is projective such that all its cyclic submodules are direct summands [23]. Apparently this definition does not give anything new for acts, since all projective acts have this property. A bigger class of regular modules is defined by Zelmanowitz [24] and again a bigger class by Fieldhouse [3]. In this context one point of interest lies in the question whether the endomorphism ring of a regular module is a regular ring. For acts over a monoid the discussion of regularity is only in the beginning. We adopt the definition introduced by Tran [22], which is similar to the one of Zelmanowitz for modules.

This definition keeps regular acts close to projective acts and to regular monoids.

In this article we investigate the relations between regular acts and various concepts around projectivity and injectivity of acts, that is we add another aspect to homological classification of monoids, and thereby promote the research on regular acts. So, for example, we characterize monoids over which all projective acts are regular and vice versa.
1. Basic definitions and results

In the following, $S$ will always stand for a monoid. A left $S$-act is a set $A$ on which $S$ acts unitarily from the left in the usual way, that is to say

$$(st)a = s(ta), \quad 1a = a, \quad \text{for } a \in A, \ s, t \in S,$$

where $1$ denotes the identity of $S$.

By $S$-Act we denote the category of all left $S$-acts.

Note that the coproduct $\coprod$ in $S$-Act is the disjoint union. By $\mathbb{N}$ we denote the set of positive integers, by $\mathbb{N}_0$ the set of non-negative integers. Any semigroup concept not defined here can be found, for example, in Howie [6].

A left $S$-act is called **regular** if for any $a \in A$ there exists a homomorphism $f: Sa \to S$ such that $f(a)a = a$ (cf. [22]), or equivalently, a left $S$-act is regular if and only if all its cyclic subacts are projective (cf. 1.5).

It is clear that a (von Neumann) regular [6] monoid $S$ is a regular left $S$-act. The converse is not true. Take, for example, $S$ to be a right cancellative monoid. Then $S$ is a regular left $S$-act without being a regular monoid. Another natural example of a regular $S$-act is the set $X$ over the monoid $\text{P}(X)$ of all mappings from $X$ to $X$, since $\text{P}(X) \times \text{P}(X) \to \text{P}(X)$, where $c_x(y) = x$ for all $y \in X$, and thus $\text{P}(X) \times \text{P}(X)$ is projective for all $x \in X$.

In [14] there are given other examples of regular acts and, moreover, there is given a method to construct new regular acts starting from regular acts.

Tran (Proposition 8 of [22]) presents an example of a monoid with one idempotent over which no left $S$-act is regular. In [19] Sild gives necessary and sufficient conditions for a Rees matrix semigroup to be regular as an act over itself.

The definitions of free acts, projective generators, projective acts, strongly flat acts, torsion free acts, injective cogenerators, injective acts are well known and can be found, for example, in [13], [15], [18], [20], and [21]. For convenience we repeat them or, whenever possible we give lemmata formulating the originally categorical notions in terms of $S$-acts.

**1.1. Lemma** [12]. A left $S$-act $A$ is a generator in the category of left $S$-acts if and only if there exists an epimorphism $f: A \to S$.

**1.2. Lemma.** A left $S$-act $A$ is

(a) free if and only if $A \equiv \coprod S$, $S$ being considered as a left $S$-act;

(b) projective if and only if $A \equiv \coprod S e_i$ for $e_i^2 = e_i, e_i \in S$ [12];

(c) strongly flat if and only if $sa = tb$ with $a, b \in A$ and $s, t \in S$, implies the existence of elements $c \in A$ and $s', t' \in S$ such that $ss' = tt'$, $a = s'c$, and $b = t'c$. Moreover, if $a = b$ there exists $s' \in S$ such that $s'c = a$ and $ss' = ts'$ [21].

Note that strongly flat acts are called weakly flat in [21] and flat in [4], [13], and [15]. For the definition of the tensor product $\otimes$ of acts see [7], [12] or [21]. Note
that for a fixed left $S$-act $A$ tensoring by $A$ is a functor from the category of right $S$-acts into the category of sets. The original definition of a strongly flat $S$-act $A$ is that $- \otimes A$ preserves equalizers and pullbacks in the usual categorical sense.

A left $S$-act $A$ is called flat (weakly flat, principally weakly flat) if the functor $- \otimes M$ preserves all monomorphisms (all embeddings of right ideals into $S$, all embeddings of principal right ideals into $S$) (cf. [7]).

Note that in the category of left $A$-modules these three notions coincide with strong flatness.

The left $S$-act $A$ is called torsion free if $sa = sb$ with $a, b \in A$, $s \in S$ left cancellable, implies $a = b$ [22].

1.3. Lemma (Proposition 2 of [18]). The left $S$-act $A$ is a cogenerator in $S$-$\text{Act}$ if and only if $A$ contains the injective envelope (cf. [1]) of any subdirectly irreducible left $S$-act which contains only itself or possibly 0 as subacts.

Let $X$ be a set. Define on the set $X^S$ of all mappings from $S$ to $X$ left multiplication by elements of $S$ in the following way:

$$(sf)(x) = f(xs) \quad \text{for all } s \in S, x \in X \text{ and } f \in X^S.$$  

Then $X^S$ becomes a left $S$-act.

A left $S$-act $A$ is called cofree if $A \cong X^S$ for some set $X$ [17].

The left $S$-act $A$ is called injective if, given a homomorphism of left $S$-acts $i : M \rightarrow N$, for any homomorphism $f : M \rightarrow A$ there exists a homomorphism $g : N \rightarrow A$ such that $f = gi$.

A left $S$-act $A$ is called (principally) weakly injective if for any inclusion $i : I \rightarrow S$ where $I$ is a (principal) left ideal of $S$ and for any homomorphism $f : I \rightarrow A$ there exists a homomorphism $g : S \rightarrow A$ such that $f = gi$ [1].

Note that principally weakly injective acts are called p-injective acts in [16]. All three notions coincide in the category of left modules over a ring.

A left $S$-act $A$ is called divisible if $dA = A$ for every right cancellable element $d$ of $S$ [2].

1.4. Proposition (cf. Proposition 1.6 of [11]). For any left $S$-act, we have the following implications:

$$\text{free} \Rightarrow \text{projective generator} \Rightarrow \text{projective}$$

$$\Rightarrow \text{strongly flat} \Rightarrow \text{flat} \Rightarrow \text{weakly flat}$$

$$\Rightarrow \text{principally weakly flat} \Rightarrow \text{torsion free}.$$  

$$\text{cofree} \Rightarrow \text{injective} \Rightarrow \text{weakly injective}$$

$$\Rightarrow \text{principally weakly injective} \Rightarrow \text{divisible}.$$
A left act with one generating element is called cyclic. If \( \rho \) is a left congruence on \( S \), then \( S/\rho \) is a cyclic left \( S \)-act (where, for example, the class of 1 is a generating element). Special cyclic left \( S \)-acts are Rees factors. Let \( I \) be a left ideal of \( S \). Then the Rees factor of \( S \) by \( I \) is the quotient act of \( S \) by the congruence one class of which coincides with \( I \) all other classes being singletons. The Rees factor of \( S \) by \( I \) is denoted by \( S/I \).

Next we give some more details on regular acts.

1.5. Proposition [22]. A left \( S \)-act \( A \) is regular if and only if all cyclic subacts of \( A \) are projective.

This proposition is in a certain analogy to the situation for regular modules (cf. [24]).

1.6. Proposition. If \( A \) is a regular left \( S \)-act and \( B \) is a subact of \( A \), then \( B \) is a regular act. If \( A_i, \ i \in I, \) are regular \( S \)-acts, then \( \bigsqcup_{i \in I} A_i \) is a regular act.

Proof. Follows immediately from the definition of regular acts.

Now we present some special (strongly) flat acts.

1.7. Proposition [4]. Let \( \rho \) be a left congruence on \( S \). \( S/\rho \) is strongly flat if and only if \( s\rho t \) for some \( s, t \in S \) implies the existence of \( v \in S \) such that \( sv = tv \) and \( \rho v \).

1.8. Proposition (Proposition 1.9 of [11]). Let \( u \in S \) and let \( \rho \) be the left congruence on \( S \) defined by \( s\rho t \) if and only if \( su^k = tu^l \) for some \( k, l \in \mathbb{N}_0 \). Then \( S/\rho \) is flat.

A left \( S \)-act is called simple if it has no proper subacts. A left \( S \)-act is called completely reducible if it is a coproduct of simple acts.

A left \( S \)-act \( A \) is called faithful (strongly faithful) if from \( sa = ta, s, t \in S \), for all (some) \( a \in A \) it follows that \( s = t \).

2. All ... acts are regular

In this section we investigate monoids over which all left acts with one of the properties introduced in Section 1 are regular. We have complete descriptions of \( S \) in 15 cases (Proposition 2.1 and Theorems 2.2, 2.3, 2.7). It turns out that \( S = \{0, 1\} \) or \( S = \{1\} \) follows in the entire context derived from cofree or injective cogenerator and the same is true for the last two terms following from projectivity (see Proposition 1.4). Crucial are the remaining types of flatness (Theorems 2.4, 2.6).

2.1. Proposition [22]. Strongly faithful acts are regular.
2.2. Theorem. All completely reducible left $S$-acts are regular if and only if $S$ contains a right zero.

**Proof.** *Necessity.* The one-element left $S$-act 0 is obviously completely reducible. Hence 0 is regular. By Proposition 1.5, 0 is projective which implies that $S$ contains a right zero, using Lemma 1.2.

**Sufficiency.** From the existence of a right zero it follows that the only simple left $S$-acts are one-element. Obviously the one-element acts are projective and regular by Proposition 1.5. But then, by Proposition 1.6 every completely reducible left $S$-act is regular.

Now we consider the concepts from Proposition 1.4.

We recall the following definition. A monoid $S$ is called *left PP monoid* if all principal left ideals of $S$ are projective. PP monoids were investigated in [5] and [8].

The following theorem improves part of Theorem 6 in [22] where left cancellativity of $S$ is a sufficient condition for (iii) of our theorem.

2.3. Theorem. The following conditions on $S$ are equivalent.

(i) All free left $S$-acts are regular.

(ii) All projective generators in $S$-Act are regular.

(iii) All projective left $S$-acts are regular.

(iv) $S$ is a left PP monoid.

**Proof.** The implications (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from Proposition 1.4.

(iii) $\Rightarrow$ (iv). $S$ is a projective act by Lemma 1.2. Hence $S$ is regular. By Proposition 1.5 all cyclic subacts (or in other words, principal left ideals) of $S$ are projective. Hence $S$ is a PP monoid.

(iv) $\Rightarrow$ (i). If $S$ is a PP monoid, then $S$ is regular by Proposition 1.5. Then, by Lemma 1.2 and Proposition 1.6, an arbitrary free left $S$-act is regular.

A monoid $S$ is called *semiperfect* if all cyclic strongly flat acts are projective. Examples are monoids which satisfy the minimum condition for principal right ideals (cf. [4]).

The following theorem establishes a relation to semiperfect monoids comparable to the situation for rings (over a left perfect ring every projective module is regular (cf. [24])).

2.4. Theorem. If all strongly flat $S$-acts are regular, then $S$ is a semiperfect PP monoid. If $S$ is a semiperfect PP monoid, then all finitely generated strongly flat $S$-acts are regular.

**Proof.** If all strongly flat $S$-acts are regular, then all cyclic strongly flat $S$-acts are regular and therefore, by Proposition 1.5, all cyclic strongly flat $S$-acts are projec-
tive. Hence $S$ is semiperfect. If all strongly flat $S$-acts are regular, then, in particular, all free $S$-acts are regular. Hence $S$ is a PP monoid by Theorem 2.3.

Let $S$ be a semiperfect PP monoid and let $A$ be an arbitrary finitely generated strongly flat $S$-act. Then, by Proposition 5.5 of [21], $A$ is the coproduct of cyclic strongly flat $S$-acts $A_i$, $i \in I$. Since $S$ is semiperfect each $A_i$, $i \in I$, is projective. Since $S$ is a PP monoid each $A_i$, $i \in I$, is regular by Theorem 2.3. But then $A$ is regular by Proposition 1.6.

Recall that a monoid $S$ is called periodic if for every $s \in S$ there exist $k, l \in \mathbb{N}$, $k \neq l$, such that $s^k = s^l$. $S$ is called combinatorial if its maximal subgroups are one-element. It is easy to see that a monoid $S$ is periodic combinatorial if and only if for every $s \in S$ there exists $m \in \mathbb{N}$ such that $s^m = s^{m+1}$.

2.5. Lemma. Let $S$ be a PP monoid. For $u \in S$ consider the cyclic left $S$-act $S/\varrho$ where $\varrho$ is the left congruence on $S$ defined by $s \varrho t$ if and only if $su^k = tu^l$ for some $k, l \in \mathbb{N}_0$. Let all such $S/\varrho$ be regular and also all Rees factors $S/I$ where $I = Se$ for some idempotent $e \in S$ or $I$ consists of idempotents only. Then $S = \{1\}$ or $S = \{0, 1\}$.

Proof. Let $u \in S$ be an arbitrary element and let $\varrho$ be defined as above, Then $S/\varrho$ is regular and, by Proposition 1.5, projective. Obviously $1gu$. As any projective act is strongly flat, then, by Proposition 1.7, there exists $v \in S$ such that $u = uv$ and $1gv$. From $1gv$ it follows that $u^k = uv^l$ for some $k$ and $l$. Now

$$uv^l = uvu^l ~ \text{or} ~ u^k = u^{k+1}$$

which means that $S$ is periodic combinatorial. Consequently for every $u \in S$ there exists $m \in \mathbb{N}$ such that $u^m$ is idempotent. Let now $1 \neq e \in S$ be an arbitrary idempotent. By assumption $S/Se$ is then regular and, by Proposition 1.5, projective. Then, by Lemma 5 of [8], $e$ is a right zero of $S$. Let now $I$ be the set of all right zeros of $S$. If $I \neq \emptyset$, then $I$ is left ideal of $S$ consisting of idempotents. By assumption $S/I$ is regular. Then $S/I$ is projective by Proposition 1.5. Now, $|I| = 1$ by Lemma 5 of [8]. Then the single element of $I$ is the zero of $S$. Hence, for $u \in S$ and $m$ as before $u^m = 1$ or $u^m = 0$. Suppose that for all $u \in S$ the first possibility takes place. Then $S$ is a group and, in particular, the one-element left $S$-act $0 = S/S1$ is regular. Hence, 0 is projective by Proposition 1.5 which implies that $S$ must contain a right zero. Hence $S = \{1\}$. In the other case we get that $S = G \cup N$ where $G$ is the group of units of $S$ and $N$ is a nil semigroup. Let $g \in G$, $g \neq 1$, and let $\varrho$ be the left congruence on $S$ defined by

$$s \varrho t ~ \leftrightarrow ~ sg^k = tg^l ~ \text{for some} ~ k, l \in \mathbb{N}_0.$$ 

Then $S/\varrho$ is regular by assumption. Hence $S/\varrho$ is projective by Proposition 1.5. Obviously $1gg$. By Proposition 1.6 there exists $v \in S$ such that $v = gv$ and $1gv$. From $1gv$ it follows that $g^k = vg^l$ for some $k$ and $l$. The last equality implies $v \in G$. Now the equality $v = gv$ gives $g = 1$, a contradiction. Hence $S = N^1$. Let now $t \in N$, $t \neq 0$. As
$S$ is a PP monoid, $S$ is regular and $St$ is regular by Proposition 1.6. From the definition of regular acts it follows that there exists a homomorphism $f: St \rightarrow S$ such that $f(t)t = t$. Then $f(t) \neq 0$ as $t \neq 0$ and $f(t) = f(f(t)t) = f(t)f(t)$, thus $f(t)$ is an idempotent. Hence $f(t) = 1$. Since $t \in \mathbb{N}$ there exists $m \geq 2$ such that $t^m = 0$ but $t^{m-1} \neq 0$. Now

$$0 \neq t^{m-1} t^{m-1} 1 = t^{m-1} f(t) = f(t^m) = f(0) = 0,$$

a contradiction. Hence $S = \{0, 1\}$.

Recall that a monoid $S$ is called left reversible if any two principal right ideals of $S$ intersect.

Lemma 2.5 helps to characterize left reversible monoids over which all flat left $S$-acts are regular. So far the problem of characterizing such monoids in general remains open.

Remark. From Lemma 2.5 it follows that $S$ is a periodic combinatorial PP monoid if all flat left $S$-acts are regular. This condition is not sufficient as can be seen from the next theorem.

2.6. Theorem. Let $S$ be a left reversible monoid. All flat left $S$-acts are regular if and only if $S = \{1\}$ or $S = \{0, 1\}$.

Proof. Sufficiency is proved in [22].

Necessity. If all flat left $S$-acts are regular, then all free left $S$-acts are regular and, by Theorem 2.3, $S$ is a PP monoid. All cyclic left $S$-acts $S/Q$ defined as in Proposition 1.8 are flat. Hence all of them are regular by assumption. From Proposition 6 of [9] it follows that all Rees factors $S/Se$ where $e$ is an idempotent and $S/I$ where $I$ is a left ideal of $S$ consisting of idempotents only are flat. Hence all of them are regular by assumption. Now it follows from Lemma 2.5 that $S = \{1\}$ or $S = \{0, 1\}$.

2.7. Theorem. The following conditions on $S$ are equivalent.

(i) All principally weakly flat left $S$-acts are regular.
(ii) All torsion free left $S$-acts are regular.
(iii) All cofree left $S$-acts are regular.
(iv) All injective cogenerators in $S$-Act are regular.
(v) All injective left $S$-acts are regular.
(vi) All weakly injective left $S$-acts are regular.
(vii) All principally weakly injective left $S$-acts are regular.
(viii) All divisible left $S$-acts are regular.
(ix) All faithful left $S$-acts are regular.
(x) All left $S$-acts are regular.
(xi) $S = \{1\}$ or $S = \{0, 1\}$. 
Proof. From Proposition 1.4 the implications

\[(ii) \Rightarrow (i) \text{ and } (viii) \Rightarrow (vii) \Rightarrow (vi) \Rightarrow (v) \Rightarrow (iv)\]

follow and, of course, any statement except (xi) follows trivially from (x), so, in particular, (xi), (viii), and (ii). The equivalence of (x) and (xi) is proved in [22]. It remains to show

\[(i) \Rightarrow (xi), \quad (iii) \Rightarrow (x), \text{ and } (ix) \Rightarrow (x).\]

(i) \Rightarrow (xi). If all principally weakly flat left S-acts are regular, then, of course, all free left S-acts are regular and S is a PP monoid by Theorem 2.3. All cyclic left S-acts S/\mathcal{O} defined as in Proposition 1.8 are flat and, hence, principally weakly flat.

By assumption they must be regular. It follows from Proposition 6 of [9] that all Rees factors S/Se where e is an idempotent and S/I where I is a left ideal consisting of idempotents only are principally weakly flat. By assumption they are regular.

Now, it follows by Lemma 2.5 that S = \{1\} or S = \{0, 1\}.

(iii) \Rightarrow (x). Let A be an arbitrary S-act. In Theorem 6 of [1] there is constructed an injective S-act B \supset A which actually is cofree. By assumption B is regular. But then A is also regular by Proposition 1.6.

(iv) \Rightarrow (x). Let A be an arbitrary S-act. Let C be an injective cogenerator in S-\text{Act} and D be the injective envelope of A (which exists by [1]). Then B = C \amalg D is again an injective cogenerator by Proposition 2 of [18]. Hence B is regular. Obviously, A \subset B. Then A is regular by Proposition 1.6.

(ix) \Rightarrow (x). Let A be an arbitrary S-act. Let B = S \amalg A. Then B is faithful. Hence, B is regular. From A \subset B and Proposition 1.6 it follows that A is regular.

3. All regular acts are ...

In this section we investigate monoids over which all regular left acts have one of the properties introduced in Section 1. We have complete descriptions of S in 11 cases (Theorems 3.1, 3.3, 3.4, 3.9-3.13) and a partial answer for principally weakly flat acts (3.6 and Theorem 3.7). The problems of characterization of monoids over which all left regular acts are (weakly) flat, (weakly) injective, or faithful remain open. First we recall

3.1. Theorem (Theorem 6 of [22]). A regular left S-act is strongly faithful if and only if S is right cancellative.

In view of Proposition 2.1 it is clear that exactly for right cancellative S both classes coincide which actually is the formulation in Theorem 6 of [22].

Tran showed in [22] that there exist monoids over which no left act is regular. In
Characterization of monoids

In this section, naturally, we are interested in the situation when there exist regular acts. Suppose $A$ is a regular left $S$-act, $a \in A$. Then, by Definition 1.1, there exists a homomorphism $f : Sa \to S$ such that $f(a)a = a$. Then $f(a) = e$ is an idempotent as $f(a) = f(f(a)a) = f(a)f(a)$. It turns out that $f : Sa \to Se$ actually is an isomorphism, for suppose $f(sa) = f(ta)$, $s, t \in S$, then

$$sa = sf(a)(a) = sea = tea = tf(a)a = ta.$$ 

Now $Sa$ is regular by Proposition 1.6. Hence, $Se$ is a regular left ideal of $S$. By this we have shown that if there exist regular left $S$-acts, then there exist regular left ideals of $S$. Let now $T$ be the union of all regular left ideals of $S$. It follows from the definition of regular acts that $T$ is regular.

For convenience we collect these observations in the following

3.2. Lemma. If there exists a regular left $S$-act $A$, then for $a \in A$ we have an isomorphism $f : Sa \to Se$ such that $f(a) = e$, $e^2 = e \in S$. Then $Se$ is a regular left ideal of $S$ and $T$ which denotes the union of all regular left ideals of $S$ is the largest regular left ideal of $S$.

3.3. Construction. Let $Se$ be a regular left ideal of $S$, $u \in S$ and $I \subseteq S$ a left ideal of $S$ such that $I \subseteq Sue$, $I \neq Sue$. Let $x, y, z$ be elements not in $S$ and let

$$M = \{(x, sue) | s \in S, sue \notin I\} \cup \{(y, sue) | s \in S, sue \notin I\} \cup \{(z, sue) | s \in S, sue \in I\}.$$ 

Define left multiplication on $M$ by elements $t$ of $S$ as follows:

$$t(w, sue) = \begin{cases} 
(w, tsue) & \text{if } tsue \notin I \\
(z, sue) & \text{if } tsue \in I
\end{cases} \quad \text{and } w \in \{x, y\},$$

$$t(z, sue) = (z, tsue), \quad \text{for all } t \in S.$$ 

It is easy to check that $M$ is a left $S$-act with two generating elements $(x, ue)$ and $(y, ue)$. As $\{(x, sue)\} \cup \{(z, sue)\}$ and $\{(y, sue)\} \cup \{(z, sue)\}$ are subacts of $M$ isomorphic to $Sue \subset Se$, $M$ is regular. A similar construction has been used in [11] and in [15].

Now we consider the concepts of Proposition 1.4 in ascending order including also completely reducible.

3.4. Theorem. All regular left $S$-acts are torsion free if and only if for every idempotent $e \in T$, for every element $u \in S$, and for every left cancellable element $r \in S$ there exists an element $t \in S$ such that $true = ue$.

Proof. Necessity. Suppose that there exist an element $u \in S$, an idempotent $e \in T$ and a left cancellable element $r \in S$ such that the left ideal $I = Sue$ is strictly contained
Let $M$ be the regular $S$-act constructed in 3.3. By assumption $M$ must be torsion free. But now from $r(x, ue) = (z, rue)$ and $r(y, ue) = z(rue)$ we get $r(x, ue) = r(y, ue)$ which must imply $(x, ue) = (y, ue)$, a contradiction. Hence, $Srue = Sue$ which means that there exists $t \in S$ such that $true = ue$.

**Sufficiency.** To prove that all regular $S$-acts are torsion free it suffices to prove that all regular $S$-acts with two generating elements are torsion free. Let now $A = Sm \cup Sn$ be a regular $S$-act. By Lemma 3.2 there exists an isomorphism $f: Sm \rightarrow Se$, where $f(m) = e \in T$, $e^2 = e$. Let $r$ be a left cancellable element and let $rm_1 = rm_2$ for some $m_1, m_2 \in A$. Suppose $m_1 \in Sm$ and $m_2 \in Sn$. Then $m_1 = um$ and $m_2 = vn$ for some $u, v \in S$. By assumption there exists an element $t \in S$ such that $true = ue$. Now

$$um = f^{-1}(ue) = f^{-1}(true) = trum,$$

i.e., $m_1 = trm_1$. From $rm_1 = rm_2$ we get $trm_1 = trm_2$. Hence $m_1 = trm_2 = trun \in Sn$. Now $m_1 = m_2$ because $Sn$ is torsion free being isomorphic to a left ideal of $S$.

3.5. **Theorem.** If all regular left $S$-acts are principally weakly flat, then for every idempotent $e \in T$ and every element $s \in S$ the product $se$ is a regular element in $S$.

**Proof.** Let $s \in S$ and $e^2 = e \in T$. If there exists $t \in S$ such that $tse = e$, then $se = setse$ and $se$ is a regular element. In the other case we have $Sse \subsetneq Se$. Let now $M$ be the regular $S$-act constructed in 3.3 for $u = 1$. By assumption $M$ must be principally weakly flat. Now $se(x, e) = (z, se)$ and $se(y, e) = (z, se)$. Hence $se(x, e) = se(y, e)$. This means that we have $se \otimes (x, e) = se \otimes (y, e)$ in the tensor product $S \otimes M$. Since $M$ is principally weakly flat we have $se \otimes (x, e) = se \otimes (y, e)$ also in $Sse \otimes M$. This means that there exists a finite sequence of pairs in $Sse \times M$ such that the first pair is $(se, (x, e))$, the last pair is $(se, (y, e))$, and every pair of the sequence can be received from the preceding pair by the transfer of an element of $S$. Let $(seu, v(x, e))$, $u, v \in S$, be the last pair of our sequence before entering $Sse \times S(y, e)$. As so far everything happened in $S(x, e) \equiv Sse$ we have $se = sueve$. To get the next pair we must have $seu = sek l$ for $k, l \in S$ and the next pair will be $(sek, lv(x, e))$. Since $(sek, lv(x, e)) \in Sse \times S(y, e)$ we have in fact $(sek, lv(x, e)) = (sek, (z, lv(e)))$. Consequently $lve \in Sse$, i.e., $lve = rse$ for some $r \in S$. Now

$$se = sueve = (sek l)ve = sek(lve) = (sek)(rse) = (se)(kr)(se)$$

which means that $se$ is regular.

3.6. **Remark.** If $S$ is (von Neumann) regular, then all left $S$-acts are principally weakly flat [9] and then, of course, all regular $S$-acts are principally weakly flat. The next theorem gives another class of monoids with this property and Example 3.8 shows that both classes are different.

3.7. **Theorem.** Let all idempotents of $S$ be central. If $se$ is a regular element for all $s \in S$ and $e^2 = e \in T$, then all regular left $S$-acts are principally weakly flat.
Proof. It suffices to show that all regular left S-acts with two generating elements are principally weakly flat. Let $A = Sm \cup Sn$ be a regular left S-act and let $a \otimes m = a \otimes n$ in the tensor product $S \otimes A$ for some $a \in S$. We have to show that $a \otimes m = a \otimes n$ in $aS \otimes A$. Note that from $a \otimes m = a \otimes n$ in $S \otimes A$ it follows that $am = an$. By Lemma 3.2 there exist isomorphisms $g : Sm \rightarrow Se$ and $h : Sn \rightarrow Sf$ such that $g(m) = e = e^2 \in T$ and $h(n) = f = f^2 \in T$, and, in particular, $em = m$ and $fn = n$. As idempotents are central we get

$$am = aem = aen = an = afn = afm.$$ 

By assumption there exist elements $u, v \in S$ such that $ae = aeuae$ and $af = afvaf$, and thus also $aeu$ and $afv$ are central. Now

$$a \otimes m = a \otimes em = ae \otimes m = aeuae \otimes m = aeu \otimes aem = aeu \otimes afn$$

$$= aeuaf \otimes n = aeu(afv)af \otimes n = afvaeu \otimes afn = afvaeu \otimes aen$$

$$= afv( aeuae) \otimes n = afvaeu \otimes n = afv \otimes aen = afv \otimes an = afv \otimes n$$

Hence, we have shown that $A$ is principally weakly flat.

3.8. Example. Let $S = \{1\} \cup \{x^n | n \in \mathbb{N}\} \cup \{e, 0\}$ where $\{x^n\}$ is the free semigroup generated by $x$, $\{e, 0\}$ is the two-element semi-lattice, $0$ is the zero of $S$, and $x^n e = ex^n = e$ for all $n \in \mathbb{N}$. Note that the largest regular left ideal of $S$ consists of $e$ and $0$. Obviously, $S$ is not von Neumann regular. At the same time $S$ satisfies the condition of Theorem 3.7.

3.9. Theorem. The following conditions on $S$ are equivalent.

(i) All regular left $S$-acts are projective.

(ii) All regular left $S$-acts are strongly flat.

(iii) All regular left $S$-acts are completely reducible.

(iv) Every idempotent of $T$ generates a minimal left ideal.

Proof. (i) $\Rightarrow$ (ii). Follows from Proposition 1.4.

(ii) $\Rightarrow$ (iv). Let $e^2 = e \in T$. Then $Se$ is regular by Proposition 1.6. Suppose $I$ is a left ideal of $S$ such that $I \subseteq Se$, $I \neq Se$. Let $M$ be the regular $S$-act constructed in 3.3 for $u = 1$. By assumption $M$ must be strongly flat. By Proposition 5.5 of [21], $M$ must then be a coproduct of cyclic $S$-acts which is impossible because $S(x, e) \cap S(y, e) = \{(z, se)\}$. Hence, $Se$ must be a minimal left ideal.

(iii) $\Rightarrow$ (iv). Take the regular left $S$-act $M$ constructed in 3.3 for $u = 1$. Again, by assumption and by definition of complete reducibility, $M$ must be a coproduct of simple and, in particular, cyclic left $S$-acts which was shown to be impossible, and thus $Se$ is a minimal left ideal.

(iv) $\Rightarrow$ (i). Let $A$ be an arbitrary regular left $S$-act. For any $a \in A$ the cyclic subact
Sa is, by Lemma 3.2, isomorphic to some left ideal Se, e ∈ T. By assumption, all such ideals Se are simple. Hence, M is a coproduct of simple subacts each of which is isomorphic to a left ideal generated by an idempotent. By Lemma 1.2, A is projective.

(iv) ⇒ (iii). In the proof of the previous implication we actually have shown that assuming (iv) an arbitrary regular left S-act A is completely reducible.

The following theorem sharpens Theorem 5 of [22] as taking into account Theorem 2.3 exactly for groups the classes of regular and of free acts coincide.

3.10. Theorem. The following conditions on S are equivalent.

(i) All regular left S-acts are free.
(ii) All regular left S-acts are projective generators in S-act.
(iii) S is a group.

Proof. (i) ⇒ (ii). Follows from Proposition 1.4.

(ii) ⇒ (iii). Consider Se, e ∈ T. Then, in particular, Se is strongly flat by (ii) and thus, by Theorem 3.9, simple. Since Se is a generator in S-Act there exists an epimorphism f: Se → S by Lemma 1.1. Let f(xe) = 1, x ∈ S. Define g: S → Se as follows: g(s) = sxe for all s ∈ S. Then gf = 1 and f must be a monomorphism. Hence S ≅ Se and thus S is simple, i.e., S is a group.

(iii) ⇒ (i). Let A be an arbitrary regular act over a group S. Then, A is a coproduct of cyclic left S-acts Ai, i ∈ I, by [20]. Each Ai, i ∈ I, must be regular by Proposition 1.6. Hence Ai, i ∈ I, is projective by Proposition 1.5. Using Lemma 1.2 it follows that A ≅ S, i ∈ I. Hence, A is free.

3.11. Theorem. All regular left S-acts are divisible if and only if all left ideals Se, e^2 = e ∈ T, are divisible.

Proof. Necessity is obvious because Se is regular by Proposition 1.6.

Sufficiency. Let A be an arbitrary regular S-act and let a ∈ A. Then Sa is isomorphic to Se, e ∈ T, by Lemma 3.2. Since Se is divisible dSe = Se follows for any right cancellable d ∈ S and thus dSa = Sa. But then

\[ dA = d \left( \bigcup_{a ∈ A} Sa \right) = \bigcup_{a ∈ A} dSa = \bigcup_{a ∈ A} Sa = A \]

which shows that A is divisible.

An element p ∈ S is said to be q-cancellable, q ∈ S, if sp = tp, s, t ∈ S, always implies sq = tq.

3.12. Theorem. All regular left S-acts are principally weakly injective if and only if the largest regular left ideal T ⊂ S is (von Neumann) regular and whenever
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Proof. Necessity. Let \( t \in T \). Then \( St \subseteq T \) and \( St \) is regular by Proposition 1.6. Hence, \( St \) is principally weakly injective. Then there exists a homomorphism \( g : S \to St \) such that \( gi = 1_{St} \) where \( i : St \to S \) is the inclusion. Hence \( st = g(1) \) for some \( s \in S \). Now \( tst = tg(1) = g(t) = t \). Hence, \( t \) is regular.

Let now \( p \in S \setminus T \) be \( e \)-cancellable for \( e^2 = e \in T \). Then by setting \( f(p) = e \) we get a homomorphism \( f \) from \( Sp \) into \( Se \). Since \( Se \) is regular it is principally weakly injective and there exists \( g : S \to Se \) such that \( f = gi \) where \( i : Sp \to S \) is the inclusion. Now

\[ e = f(p) = g(p) = pg(1) \in pS. \]

Sufficiency. Let \( A \) be an arbitrary regular left \( S \)-act, \( p \in S \), and let \( f : Sp \to A \) be a homomorphism. Denote \( f(p) = a \). By Lemma 3.2 there exists an isomorphism \( h : Sa \to Se \) where \( e^2 = e \in T \) and \( f(a) = e \). Since \( Se \) is regular it is principally weakly injective. Hence, there exists a homomorphism \( k : S \to Se \) such that \( hf = ki \) where \( i : Sp \to S \) is the inclusion. Now consider again \( f : Sp \to A \) and \( g = h^{-1}k \). Then \( g(p) = h^{-1}k(p) = h^{-1}ki(p) = h^{-1}hf(p) = f(p) \), i.e., \( f = gi \). Hence, \( A \) is principally weakly injective.

3.13. Example. Let \( S = P \cup T \), where \( P \) is a monoid with only one idempotent, \( T \) a von Neumann regular semigroup and \( pt = tp = p \) for all \( p \in T \), \( t \in T \). Then \( T \) is the largest regular left ideal of \( S \). This monoid \( S \) fulfills the conditions of Theorem 3.4, taking the required element \( t \) equal to 1, for example, and consequently all regular left \( S \)-acts are torsion-free. The monoid \( S \) also fulfills the conditions of Theorem 3.12 (and thus a forteriori those of Theorem 3.11), as every \( p \in P \) is \( e \)-cancellable for any \( e \in T \) and \( e = pe \in pS \). Consequently all regular left \( S \)-acts are principally weakly flat. Requiring \( T \) to be completely simple in the above \( S \) we get a monoid which fulfills (iv) of Theorem 3.9 and thus all regular left \( S \)-acts are projective and completely reducible. If we require \( T \) to be commutative we get another monoid fulfilling the conditions of Theorem 3.7 and thus all regular left \( S \)-acts are principally weakly flat.

So far it remains an open problem to characterize monoids over which all regular left \( S \)-acts are weakly injective or injective.

For the two strongest concepts on the line of injectivity the answer to the question of this section is ‘never’ as the following theorem shows:

3.14. Theorem. If there exist regular left \( S \)-acts, then there exist regular left \( S \)-acts which are not cofree, and there exist regular left \( S \)-acts which are not injective cogenerators in \( S \)-Act.

Proof. If there exist regular left \( S \)-acts, then by Lemma 3.2 there exists a regular left ideal \( Se \), \( e^2 = e \in T \). If all regular \( S \)-acts were cofree, then \( Se = X^S \) for some set
S. If $|X| \geq 2$, then $|X^S| > |S| \geq |Se|$. Hence $|X| = 1$. But then $|X^S| = 1$. Hence, $Se$ is one element which means that $e$ is a right zero. But then $0 \equiv 0$ where $0$ is the one element left $S$-act is regular by Proposition 1.6. Hence $0 \equiv 0$ is cofree which again is impossible.

If all regular left $S$-acts were injective cogenerators, then $Se$ must be injective. Hence $Se$ contains a right zero $0$. This implies that the one element left $S$-act $0$ is regular. By assumption $0$ is an injective cogenerator; but this contradicts Proposition 2 of [18] from which it follows that every cogenerator must contain a two-element subact.

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References

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