Groups with a Characteristic Cyclic Series

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Communicated by Marshall Hall, Jr.

Received June 25, 1970

1. INTRODUCTION

By a characteristic cyclic series (c.c.s.) of a group $G$ is meant a finite series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G,$$

such that each $G_i$ is characteristic in $G$ and each $G_{i+1}/G_i$ is cyclic. The purpose of this paper is to investigate groups having a c.c.s. and the close relationship between these groups and those having supersolvable automorphism groups.

In Section 2 we state some basic facts about groups with a c.c.s. and give some examples. In Section 3 we show that if $G$ has a c.c.s., then the automorphism group of $G$ is supersolvable (Theorem 1). The converse of this is true if $G$ is finite Abelian provided that $J_2 \not\subseteq J_3$ is not a direct factor of $G$ (Theorem 3). Using these theorems, it is easy to show that if $G$ is a finite Abelian $p$-group, $p$ a prime, then $G$ has a c.c.s. if and only if the automorphism group of $G$ is supersolvable and the exponent of the automorphism group of $G$ divides $p^t(p-1)$ for some $t \geq 0$ (Theorem 5). This can be extended to arbitrary finite $p$-groups if $p = 2$ or $p = 3$ (Theorem 6); we conjecture that it can be extended to all finite $p$-groups, $p$ a prime. In Section 4 we determine all finite groups $G$ such that every proper subgroup of $G$ has a U.S.

Most of the notation is standard as found in Ref. [4]. We list the exceptions below.

We write $o(G)$ for the order of a finite group $G$, and we write $\exp(G)$ for the exponent of a group $G$. If $H$ is a subset of $G$, then $\langle H \rangle$ denotes the subgroup of $G$ generated by $H$. If $G$ is a $p$-group of order $p^n$ and the nilpotency class of $G$ is $n-1$, then we say $G$ is of maximal class. We denote the derived group of $G$ by $G'$. If $G'$ and $G/G'$ are cyclic, then we say $G$ is metacyclic.

If $H$ and $K$ are groups, then $H \wr K$ is the (standard restricted) wreath product of $H$ and $K$. $J_n$ denotes the cyclic group of order $n$.

We will occasionally refer to the groups $M(p)$, $M(m, p)$, $D_m$, $Q_m$, and $S_m$. These are all defined in Ref. [4].
2. Basic Facts

It follows immediately from the definition that if $G$ has a c.c.s., then $G$ is supersolvable; however, there are supersolvable groups which do not have a c.c.s., for example $J_2 \times J_2$. This example also shows that the class of groups having a c.c.s. is not closed under direct products. But if $G$ can be written as a finite direct product $G_1 \times \cdots \times G_n$, where each $G_i$ char $G$, then $G$ has a c.c.s. if and only if each $G_i$ has a c.c.s. Hence in studying finite nilpotent groups having a c.c.s. we can restrict our attention to $p$-groups.

We also note that the class of groups with a c.c.s. is neither subgroup closed nor factor group closed, since, e.g., $J_2 \times J_4$ has a c.c.s. but $J_2 \times J_2$ does not; nor is the class closed under extensions, since $J_2 \times J_3$ does not have a c.c.s.

If $H$ char $G$, and $H$ and $G/H$ both have a c.c.s., then $G$ has a c.c.s. This follows immediately from the fact that if $H$ char $G$ and $K/H$ char $G/H$, then $K$ char $G$. One can also show that if $G$ has a c.c.s. and $H$ is a maximal characteristic Abelian subgroup of $G$, then $H$ is a maximal Abelian subgroup of $G$. (This is done in the same way that one shows that if $G$ is supersolvable and $H$ is a maximal normal Abelian subgroup of $G$, then $H$ is a maximal Abelian subgroup of $G$ [7, 7.2. 20]). Finally, we point out that for $G$ finite any two characteristic series of $G$ have equivalent refinements [7, P. 43]; also, if $G$ has a c.c.s., then, without loss of generality, we can assume each factor is either infinite cyclic or cyclic of prime order.

For the remainder of this section we will list some examples of groups with a c.c.s.

If $G \cong \prod_{i=1}^{n} J_{p^e_i}$, $m_i \geq m_{i+1}$, $1 \leq i \leq n - 1$, then, using some results by Burnside [2, Section 82], one can show that $G$ has a c.c.s. if and only if $m_i > m_{i+1}$, $1 \leq i \leq n - 1$. Since a finite Abelian group $G$ can be written as the direct product of its Sylow $p$-subgroups, each of which is characteristic in $G$, this completely determines all finite Abelian groups with a c.c.s. We can also show that, for any positive integer $n$, all groups of order $n$ have a c.c.s. if and only if $n$ is square-free (use [5, Corollary 9.4.1]). Using a lemma of Blackburn's [1, Lemma 2.5], it is easy to show that if $G$ is a $p$-group of order $p^n$, $m \geq 4$, and $G$ is of maximal class, then $G$ has a c.c.s. From this it immediately follows that if $G \cong D_m$, $Q_m$, or $S_m$, $m \geq 4$, then $G$ has a c.c.s., since, in each case, $G$ is a 2-group of maximal class [4, Theorem 5.4.3]. We can also show that $M(m, p)$ has a c.c.s. (use [4, Theorem 5.4.3]).

We have also determined which groups of order $p^3$ and $p^4$ have a c.c.s. We state our results below but omit the proofs.

**Proposition 1.** Let $G$ be a nonabelian $p$-group of order $p^3$, $p$ a prime. Then $G$ has a c.c.s. if and only if $G \cong D_5$ or $G \cong M(3, p)$, $p$ an odd prime.
PROPOSITION 2. Let $G$ be a group of order $p^k$, $p$ a prime, such that $c_1(G) = 2$. Then $G$ has a c.c.s. unless $G$ is isomorphic to one of the following groups:

1) $Q_8 \times J_2$,
2) $M(p) \times J_p$, $p \neq 2$,
3) $(M(p) \times J_p)/H$ if $p \neq 2$ or $(D_3 \times J_p)/H$ if $p = 2$, where $H = gp(xy)$ with $x$ an element of order $p$ in the center of $M(p)$ or $D_3$ and $y$ an element of order $p$ in $J_p$.

We have attempted to determine all finite wreath products having a c.c.s., but have not yet been able to do so. (Finite supersolvable wreath products have been completely determined [3]). We have some results which we state below without proof.

PROPOSITION 3.

1) Let $W = A \wr B$. If $W$ has a c.c.s., then $A$ and $B$ each have c.c.s.
2) Let $W = A \wr B$, where $A$ is finite and $B \cong J_{p^m}$, $p$ an odd prime. If $W$ has a c.c.s., then $A$ is a $p$-group.
3) Let $p$ be a prime. Then $J_{p^m} \wr J_p$ has a c.c.s.
4) Let $A$ be a finite Abelian group with a c.c.s., and let $B \cong J_2$. Then $A \wr B$ has a c.c.s.
5) Let $A$ be a finite Abelian 3-group with a c.c.s., and let $B \cong J_3$. Then $A \wr B$ has a c.c.s.

3. AUTOMORPHISM GROUPS OF GROUPS WITH A C.C.S.

If $G$ is a group and $H$ char $G$, then \{$g \in \text{aut}(G)$ | $g^{-1}(g_0) \in H$ for each $g \in G$\} is a normal subgroup of $\text{aut}(G)$. Conversely, if $B$ is a normal subgroup of $\text{aut}(G)$, then \{ $g \in G$ | $g_\alpha = g$ for each $\alpha \in B$\} is a characteristic subgroup of $G$. Hence it is natural to ask the following questions:

1) If $G$ has a c.c.s., what can we say about the structure of $\text{aut}(G)$?
2) What conditions can we impose on $\text{aut}(G)$ so that $G$ will have a c.c.s.?

Before attempting to answer these questions we need the following definition. Let

\[ 1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G \]  

be a series of subgroups of $G$. Let $B = \{ \alpha \in \text{aut}(G) | g^{-1}(g_\alpha) \in G_{i-1} \text{ for each } g \in G_i, 1 \leq i \leq n \}$. Then $B$ is called the stability group of Eq. (1). It is easy to show that if each $G_i$ char $G$ then $B$ is a normal subgroup of $\text{aut}(G)$.
**Theorem 1.** Let $G$ be a group. If $G$ has a c.c.s., then $\text{aut}(G)$ is supersolvable.

**Proof.** Assume $G$ is a group with a c.c.s. Then we have a series

$$1 = G_0 \subseteq G_1 \subseteq \cdots \subseteq G_n = G \quad (1)$$

of $G$ such that each $G_i$ char $G$ and each factor is either infinite cyclic or cyclic of prime order.

Let $A = \text{aut}(G)$, and let $B$ be the stability group of Eq. (1). Then $B$ is a normal subgroup of $A$, and we will first show $A/B$ is supersolvable.

Let $B_i = \{ \alpha \in A \mid g^{-1}(\alpha g) \in G_{i-1} \text{ for each } g \in G_i \}$, $1 \leq i \leq n$, and let $C_i = \bigcap_{j=1}^{i-1} B_j$, $1 \leq j \leq n$; let $C_0 = A$. Each $C_i$ is a normal subgroup of $A$, and $C_n = B$. Thus we have an invariant series

$$B = C_n \subseteq C_{n-1} \subseteq \cdots \subseteq C_0 = A$$

of $A$. For each $i$, $\alpha \in C_i-1$ induces an automorphism on $G_i/G_{i-1}$. Hence we have a homomorphism $\beta : C_i \to \text{aut}(G_i/G_{i-1})$. Ker $\beta = C_i$, so $C_i/C_i \subseteq \text{aut}(G_i/G_{i-1})$, so $C_i/C_i$ is cyclic. Hence $A/B$ is supersolvable.

To show $A$ is supersolvable, it suffices to prove the following: Let $1 \leq k \leq n - 1$, and let $C$ be the stability group of $1 \subseteq G_k \subseteq G_{k+1} \subseteq \cdots \subseteq G_n = G$. Let $D$ be the stability group of $1 \subseteq G_{k+1} \subseteq G_{k+2} \subseteq \cdots \subseteq G_n = G$. Then there exist subgroups $C_0 \subseteq C_1 \subseteq \cdots \subseteq C_k$, such that each $C_i A$, $C_0 = D$, $C_k = C$, and $C_i/C_{i-1}$ is cyclic, $1 \leq i \leq k$.

To prove this, let $D_i$ be the stability group of $1 \subseteq G_i \subseteq G_{i+1} \subseteq \cdots \subseteq G_n = G$, $0 \leq i \leq k$. Let $C_i \subseteq D_i \cap D_i$, $0 \leq i \leq k$. Then $C_0 = D$, $C_k = C$, each $C_i A$, and $C_i \subseteq C_i$, $1 \leq i \leq k$, so we only need to show that $C_i/C_{i-1}$ is cyclic, $1 \leq i \leq k$.

Fix $i$, and consider $C_i/C_{i-1}$. Let $\alpha C_{i-1}$ be a nontrivial element of $C_i/C_{i-1}$. Then $\alpha \in D_i \cap D_i$, $\alpha \notin D_{i-1}$. If we let $G_i = \text{gp}(x_i)$ and $G_i = \text{gp}(\{G_i, x_i\})$, $2 \leq j \leq n$, then we have $x_i \alpha = x_i$, $0 \leq j \leq k$, and $x_{k+1} \alpha = x_{k+1}(x_i)g$, $g \in G_{i+1}$, $a$ an integer, $a \neq 0$. Choose $\alpha$ so that $|a|$ is minimal.

Let $\beta \in C_i$. Then $x_i \beta - x_i$, $0 \leq j \leq k$, and $x_{k+1} \beta = x_{k+1}(x_i)^{b}h$, $h \in G_{i+1}$, $b$ an integer. Let $r = (a, b)$. There exist integers $s, t$ such that $sa + tb = r$. Hence $x_{k+1} \beta^{t} x_i^{s} = x_{k+1}(x_i)^{q}g'$, $g' \in G_{i+1}$. By the minimality of $|a|$, $|r| = |\alpha|$, so $a | b$. Hence $b = ca$ for some integer $c$. Then $x_{k+1} \beta^{c} x_i^{s} = x_{k+1} h', h' \in G_{i+1}$.

Hence $\beta^{c} x_i^{s} \in C_{i+1}$. Therefore $\beta C_{i+1} = (x C_{i+1})^{s}$, so $C_i/C_{i+1}$ is cyclic. Hence $\text{aut}(G)$ is supersolvable.

The converse of Theorem 1 is not true, since, for example, $J_2 \times J_2$ has supersolvable automorphism group but does not have a c.c.s. We will show that the converse is true if $G$ is a finite Abelian group such that $J_2 \times J_2$ is not a direct factor of $G$. We need the following lemma.

**Lemma 2.** Let $G \cong J_p^m \times J_p^n$, where $p$ is a prime, $n \geq 1$ if $p \neq 2$, and $n > 1$ if $p = 2$. Then $\text{aut}(G)$ is not supersolvable.
Proof. Let $G = \langle a \rangle \times \langle b \rangle$, where $o(a) = o(b) = p^s$. Assume $\text{aut}(G)$ is supersolvable. If $p \neq 2$, then $p$ is the largest prime divisor of $\text{aut}(G)$ [4, Theorem 5.4.15]; hence $\text{aut}(G)$ has a normal Sylow $p$-subgroup $P$. There exist automorphisms $\alpha, \beta \in \text{aut}(G)$ such that $\alpha a = ab$, $\beta a = a$ and $\alpha b = b$, $\beta b = ab$. This is a contradiction, so for $p \neq 2$, $\text{aut}(G)$ is not supersolvable. Now assume $p = 2$. $\text{Aut}(G)$ is supersolvable, so it has a normal subgroup $H$ such that $(o(H), 2) = 1$ and $\text{aut}(G)/H$ is a 2-group. There exist automorphisms $\alpha, \beta \in \text{aut}(G)$ such that $\alpha a = a$, $\beta a = ab$, $\alpha b = b$, $\beta b = ab$. Then $\alpha, \beta \in H$, but $\alpha \beta \notin H$. This is a contradiction; so $\text{aut}(G)$ is not supersolvable.

**Theorem 3.** Let $G$ be a finite Abelian group such that $J_2 \times J_2$ is not a direct factor of $G$. Then $G$ has a c.c.s. if and only if $\text{aut}(G)$ is supersolvable.

Proof. If $G$ has a c.c.s., then $\text{aut}(G)$ is supersolvable by Theorem 1. Now assume $\text{aut}(G)$ is supersolvable. Without loss of generality, we may assume $G$ is a $p$-group for some prime $p$. Then $G \cong \prod_{i=1}^{n} J_{p^i}$, $m_i \geq m_{i+1}$, $1 < i < n - 1$. $\prod_{i=1}^{n} (\text{aut}(J_{p^i}))/\cong \text{aut}(G)$; so, by Lemma 2, $m_i > m_{i+1}$, $1 < i < n - 1$. Using the characterization of finite Abelian groups with a c.c.s. given in Section 2, we see that $G$ has a c.c.s.

It follows from Theorem 3 that if $G$ is a finite Abelian group such that $J_2 \times J_2$ is not a direct factor of $G$, then $G$ has a supersolvable holomorph if and only if $\text{aut}(G)$ is supersolvable.

By placing more restrictions on the automorphism group of $G$ we can get rid of the "exceptions" in the above theorem. We need the following lemma.

**Lemma 4.** Let $G$ be a finite $p$-group, $p$ a prime. If $G$ has a c.c.s., then $\exp(\text{aut}(G)) = p^t(p - 1)$ for some $t \geq 0$. 

Proof. If $G$ has a c.c.s., we can find a series

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G$$

(1)

of $G$ with each $G_i \text{ char } G$ and each factor cyclic of order $p$. Let $\alpha \in \text{aut}(G)$. Then $\alpha^{p-1}$ is in the stability group of Eq. (1). Therefore $\alpha^{p-1}$ has order a power of $p$ [4, p. 214]. Hence $\exp(\text{aut}(G)) = p^t(p - 1)$ for some $t \geq 0$.

Using Theorem 3 and Lemma 4 we have the following result:

**Theorem 5.** Let $G$ be a finite Abelian $p$-group. Then $G$ has a c.c.s. if and only if $\text{aut}(G)$ is supersolvable and $\exp(\text{aut}(G)) = p^t(p - 1)$ for some $t \geq 0$.

This theorem is true for arbitrary finite $p$-groups if $p = 2$ or $p = 3$, as the next theorem shows.
Theorem 6. Let $G$ be a finite $p$-group, where $p = 2$ or $p = 3$. Then $G$ has a c.c.s. if and only if $\text{aut}(G)$ is supersolvable and $\exp(\text{aut}(G)) \mid p^t(p - 1)$ for some $t \geq 0$.

Proof. By Theorem 1 and Lemma 4, if $G$ has a c.c.s., then $\text{aut}(G)$ is supersolvable and $\exp(\text{aut}(G)) \mid p^t(p - 1)$ for some $t \geq 0$.

Now assume $\text{aut}(G)$ is supersolvable and $\exp(\text{aut}(G)) \mid p^t(p - 1)$ for some $t \geq 0$. Let $H$ and $K$ be characteristic subgroups of $G$ such that $K$ is a minimal characteristic subgroup of $G$ containing $H$. Then $K/H$ is an elementary Abelian $p$-group and hence may be considered as a vector space over the field of $p$ elements. Let $V = K/H$ and let $F$ be the field of $p$ elements. Let $A = \text{aut}(G)$. Every element of $A$ induces an automorphism of $K/H$, so we have a representation $\alpha : A \to GL(V, F)$. If $B = \ker \alpha$, then $\alpha$ induces a faithful representation $\alpha^* : A/B \to GL(V, F)$. By the minimality of $K$, $\alpha^*$ is irreducible. Hence $A/B$ has no nontrivial normal $p$-subgroups [4, Theorem 3.1.3]. Since $A$ is supersolvable and $p$ is the largest prime divisor of $A$ we have that $\exp(A/B) \mid (p - 1)$. For $p = 2$, this means $A = B$, so by the minimality of $K$, $\sigma(K/H) = 2$. If $p = 3$, then $\exp(A/B) = 2$, so $A/B$ is Abelian. Hence $A/B$ is, in fact, cyclic [4, Theorem 3.2.3]. Therefore $\alpha^*$ is linear [4, Theorem 3.2.4], so $\sigma(K/H) = 3$. Thus we have shown that $K/H$ is cyclic of prime order. Hence $G$ has a c.c.s.

Corollary 7. A finite $2$-group $G$ has a c.c.s. if and only if $\text{aut}(G)$ is a $2$-group.

We conjecture that Theorem 6 can be extended to all finite $p$-groups. By Theorem 1 and Lemma 4 we know that if a $p$-group $G$ is a counterexample to this conjecture then $G$ does not have a c.c.s., but $G$ does have supersolvable automorphism group and $\exp(\text{aut}(G)) \mid p^t(p - 1)$ for some $t \geq 0$. We have shown that certain groups cannot be counterexamples to the above conjecture. In practically every case we have done this by showing that $\text{aut}(G)$ is not supersolvable. In fact, the only examples we now have of groups which do not have a c.c.s. but do have a supersolvable automorphism group are groups with $J_2 \times J_2$ as a direct factor. This leads us to pose the following open question. (Compare the fact that a finite group is supersolvable if its automorphism group is supersolvable [5, p. 164].)

Question. If $G$ is finite, $J_2 \times J_2$ is not a direct factor of $G$, and $\text{aut}(G)$ is supersolvable, does it follow that $G$ has a c.c.s.?

4. Minimal Groups without C.C.S.

The main difficulty encountered in the study of groups with a c.c.s. is that the class of these groups is neither subgroup closed nor factor group
closed, so that there is no natural way to induct. The following theorem shows how far the class is from being subgroup closed.

**Theorem 1.** Let $G$ be a finite group. Every subgroup of $G$ has a c.c.s. if and only if every Sylow subgroup of $G$ is cyclic.

**Proof.** Assume every subgroup of $G$ has a c.c.s. Let $p$ be a prime divisor of $o(G)$, and let $S$ be a Sylow $p$-subgroup of $G$. Since $J_p \times J_p$ does not have a c.c.s., every Abelian subgroup of $S$ must be cyclic. Hence either $S$ is cyclic or $p = 2$ and $S \cong Q_m, m \geq 3$ [4, Theorem 5.4.10]. But if $S \cong Q_m, m \geq 3$, then there exists a subgroup $H \subseteq S$ such that $H \cong Q_3$. But $Q_3$ does not have a c.c.s.; so $S$ must be cyclic.

Conversely, if every Sylow subgroup of $G$ is cyclic, then for any subgroup $H \subseteq G$, every Sylow subgroup of $H$ is cyclic. Then $H$ is metacyclic [7, Theorem 12.6.17], and $H$ has a c.c.s.

All groups having cyclic Sylow subgroups are known [7, Theorem 12.6.17]; so the above theorem completely determines all groups $G$ such that every subgroup of $G$ has a c.c.s.

The following theorem characterizes the minimal finite groups not having a c.c.s.

**Theorem 2.** Let $G$ be a finite group. If $G$ does not have a c.c.s. but every proper subgroup of $G$ does have a c.c.s., then $G \cong J_p \times J_p$, $p$ a prime, or $G \cong Q_3$.

**Proof.** We will first show $G$ is supersolvable. Suppose $G$ is not supersolvable. Every proper subgroup of $G$ is supersolvable, so for some prime divisor $p$ of $o(G)$, there exists a Sylow $p$-subgroup $S$ of $G$ such that $S \not\lhd G$, $S/\Phi(S)$ is a minimal normal subgroup of $G/\Phi(S)$, and $S/\Phi(S)$ is not cyclic [6, p. 721]. But $S$ is properly contained in $G$, so $S$ has a c.c.s. Hence there exists a subgroup $T$ char $S$ such that $\Phi(S) \subseteq T \subseteq S$ and $o(T/\Phi(S)) = p$. But then $T/\Phi(S) \not\lhd G/\Phi(S)$, and this contradicts the minimality of $S/\Phi(S)$. Hence $G$ is supersolvable.

We can now show $G$ must be a $p$-group for some prime $p$. Assume $o(G) = p_1^{n_1} \cdots p_k^{n_k}, k \geq 2$, each $p_i$ a prime and $p_i > p_{i+1}, 1 \leq i \leq k - 1$. $G$ is supersolvable; so we have a Sylow tower

$$1 = S(0) \subseteq S(1) \subseteq \cdots \subseteq S(k) = G,$$

where $S(i)/S(i - 1)$ is the unique normal Sylow $p_i$-subgroup of $G/S(i - 1)$ [6, Theorem VI.9.1]. $S(k - 1)$ has a complement $T$ in $G$; $S(k - 1)$ and $T$ are both properly contained in $G$ and hence have a c.c.s. $G/S(k - 1) \cong T$; so
\[ G/S(k - 1) \text{ has a c.c.s. Thus, since } S(k - 1) \text{ char } G, \text{ we can refine the series} \]
\[ 1 \subseteq S(k - 1) \subseteq G \]
to a c.c.s. of \( G \). This is a contradiction, since \( G \) does not have a c.c.s. Thus \( G \)
is a \( p \)-group for some prime \( p \).

If \( G \) is Abelian, then there exists a subgroup \( H \subseteq G \) such that \( H \cong \mathbb{F}_p \times \mathbb{F}_p \).
\( H \) does not have a c.c.s., so we must have \( G = H \). Hence \( G \cong \mathbb{F}_p \times \mathbb{F}_p \).

Assume \( G \) is non-Abelian. Every Abelian subgroup of \( G \) is cyclic, so \( G \cong Q_m, \ m \geqslant 3 \) [4, Theorem 5.4.10]. But for \( m > 3 \), \( Q_m \) has a c.c.s.; so we must have \( G \cong Q_3 \).

**Remark.** We can show that if \( G \) is an infinite nilpotent group with torsion subgroup \( T \), then every subgroup of \( G \) is a C-group if and only if \( T \) is cyclic of finite order and \( G/T \) is an infinite cyclic group. We can also show that Theorem 2 holds for any finitely generated nilpotent group.

**References**