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Applied Mathematics Letters 18 (2005) 827–832

Applied Mathematics Letters

www.elsevier.com/locate/aml

Existence of a global attractor for the plate equation with a critical exponent in an unbounded domain

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Received 2 August 2004; accepted 3 August 2004

Abstract

In this work, we study the asymptotic behavior of solutions for the plate equation with a critical exponent in R^n. We prove the existence of a global attractor in $W_2^2(R^n) \times L_2(R^n)$.

Keywords: Global attractor; Asymptotic compactness

1. Introduction

The subject of investigation of this work is the existence of a global attractor for the following plate equation in R^n:

$$u_{tt} + \alpha u_t + \Delta^2 u + \lambda u + f(u) = g(x)$$

where α and λ are positive constants, $g(\cdot)$ is a given function and $f(\cdot)$ is a nonlinear function satisfying some growth conditions.

The existence of a global attractor for this equation in a bounded domain, when the growth of $f(\cdot)$ is subcritical, was studied in [1]. The long-time behavior of solutions for the semilinear wave equations with interior dissipation and a critical exponent in a bounded domain was investigated in [2–4] and references therein. In bounded domains, the asymptotic compactness of the solutions, which plays an

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0893-9659/$ - see front matter
doi:10.1016/j.aml.2004.08.013

important role for the existence of a global attractor, is obtained by compactness of Sobolev embeddings. This method does not apply to unbounded domains since the embeddings are no longer compact.

The existence of a global attractor for the semilinear wave equations with critical and supercritical exponents in an unbounded domain was studied in [5,6]. In these articles, the asymptotic compactness has been established using finite speed of propagation and specific estimates for the linear wave equations in R^n, which do not seem to apply to the plate equation.

The main goal of the present work is to prove the asymptotic compactness of solutions, which, together with the results of [7], implies the existence of a global attractor.

2. Preliminaries

We consider the following Cauchy problem:

$$u_{tt} + \alpha u_t + \Delta^2 u + \lambda u + f(u) = g(x), \qquad (t, x) \in R_+ \times R^n, \tag{1}$$

$$u(0, x) = u_0(x), \qquad u_t(0, x) = u_1(x), \qquad x \in R^n, \tag{2}$$

where $\alpha > 0$, $\lambda > 0$, $g \in L_2(R^n)$ and $f(\cdot)$ satisfies the following conditions:

$$f \in C^1(R), \qquad |f'(u)| \le c(1 + |u|^p), \qquad p > 0, \qquad (n - 4)p \le 4 \tag{3}$$

$$f(u) \cdot u \ge 0 \qquad \text{for every } u \in R. \tag{4}$$

Denote the spaces $W_2^s(R^n)$ and $L_2(R^n)$ by $H^s (s \neq 0)$ and H respectively. The norms in H^s and H are denoted by $\|\cdot\|_s$ and $\|\cdot\|$ respectively. We also use the spaces $\mathcal{H}^s = H^{2+2s} \times H^{2s}$ $(s \neq 0)$ and $\mathcal{H} = H^2 \times H$. In the space $\mathcal{H}$ we introduce a linear closed operator A as follows:

$$D(A) = \mathcal{H}^1, \qquad Aw = (w_2, -\Delta^2 w_1 - \lambda w_1 - \alpha w_2), \qquad w = (w_1, w_2) \in D(A).$$

Using the substitution $\theta(t) = (u(t), u_t(t))$, we reduce problem (1) and (2) to the problem

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\theta(t) = A\theta(t) + F(\theta(t)), \qquad t \in R_+ \\ \theta(0) = \theta_0 \end{cases} \tag{5}$$

where $F(\theta(t)) = (0, -f(u(t)) + g)$, $\theta_0 = (u_0, u_1)$.

It is easy to show that A is an infinitesimal generator of C_0-semigroup e^{tA} (see [8]) and as in [1] there exist $M > 0$ and $\omega > 0$ such that for $s \in [-1, 1]$

$$\|\mathrm{e}^{tA}\|_{L(\mathcal{H}^s, \mathcal{H}^s)} \le M\mathrm{e}^{-\omega t}, \qquad \forall\, t \ge 0 \tag{6}$$

where $\mathcal{H}^0$ means $\mathcal{H}$.

Since the nonlinear operator $F(\cdot) : \mathcal{H} \longrightarrow \mathcal{H}$ satisfies the local Lipschitz condition (thanks to (3)), using the results of [9], we find that for any $\theta_0 \in \mathcal{H}$ the problem (5) has a unique solution $\theta(\cdot) \in C([0, +\infty); \mathcal{H})$; moreover if $\theta_0 \in \mathcal{H}^1$, then $\theta(\cdot) \in C^1([0, +\infty); \mathcal{H}) \cap C([0, +\infty); \mathcal{H}^1)$. Therefore, we have the strongly continuous nonlinear semigroup $\{U(t)\}_{(t \ge 0)}$, where $\theta(t) = U(t)\theta_0$ is the solution of problem (5).

Lemma 1. *Let us assume that conditions* (3) *and* (4) *are satisfied. Then*

(i) *for all* $\theta_0 \in \mathcal{H}$

$$\sup_{t \ge 0} \|U(t)\theta_0\|_{\mathcal{H}} \le c(\|\theta_0\|_{\mathcal{H}}), \tag{7}$$

where $c(\cdot)$ is a monotone increasing function;

(ii) *if $\theta_m \to \theta$ weakly in $\mathcal{H}$, then for every $t > 0$,*

$$U(t)\theta_m \to U(t)\theta \text{ weakly in } \mathcal{H}. \tag{8}$$

Proof. (i) Multiplying (1) by u_t and integrating over $[\tau, t] \times R^n$ we obtain

$$\begin{aligned} &E(u(t), u_t(t)) + \int_{R^n} \Phi(u(t,x))\mathrm{d}x - \int_{R^n} g(x)u(t,x)\mathrm{d}x + \alpha \int_\tau^t \|u_t\|^2 \mathrm{d}s \\ &\quad = E(u(\tau), u_t(\tau)) + \int_{R^n} \Phi(u(\tau,x))\mathrm{d}x - \int_{R^n} g(x)u(\tau,x)\mathrm{d}x, \end{aligned} \tag{9}$$

where $E(u(t), u_t(t)) = \frac{1}{2}\|u_t(t)\|^2 + \frac{1}{2}\|\Delta u(t)\|^2 + \frac{\lambda}{2}\|u(t)\|^2$, $\Phi(s) = \int_0^s f(\tau)\mathrm{d}\tau$. (9), together with (3) and (4), yields (7).

(ii) Since $\theta_m \to \theta$ weakly in $\mathcal{H}$, the sequence $\{\theta_m\}$ is bounded in $\mathcal{H}$. Thus from (7) the sequence $\{U(t)\theta_m\}$ and consequently also, by the condition (3), the sequence $\{F(U(t)\theta_m)\}$ are both bounded in $L_\infty(0, T; \mathcal{H})$. From this, and the fact that $U(t)\theta_m$ is the solution of $(5)_1$, it follows that the sequence $\{\frac{\mathrm{d}}{\mathrm{d}t}U(t)\theta_m\}$ is bounded in $L_\infty(0, T; \mathcal{H}^{-1})$. Then we have a subsequence $\{m_k\}$ such that

$$\begin{cases} U(t)\theta_{m_k} \to \theta(t) & \text{weakly in } L_2(0, T; \mathcal{H}) \\ F(U(t)\theta_{m_k}) \to \chi & \text{weakly in } L_2(0, T; \mathcal{H}) \\ \dfrac{\mathrm{d}}{\mathrm{d}t}U(t)\theta_{m_k} \to \dfrac{\mathrm{d}}{\mathrm{d}t}\theta(t) & \text{weakly in } L_2(0, T; \mathcal{H}^{-1}). \end{cases} \tag{10}$$

From (10) we obtain that $\chi = F(\theta(t))$ (see for example [10, p. 12]) and $\theta(t)$ is a solution of problem (5) with $\theta_0 = \theta$. By the uniqueness of solutions, we have $\theta(t) = U(t)\theta$. This shows that any subsequence of $\{(U(t)\theta_m, \frac{\mathrm{d}}{\mathrm{d}t}U(t)\theta_m)\}$ has a weakly convergent subsequence in $L_2(0, T; \mathcal{H} \times \mathcal{H}^{-1})$ and the limit of any such subsequence is equal to $(U(t)\theta, \frac{\mathrm{d}}{\mathrm{d}t}U(t)\theta)$. Therefore the sequence $\{(U(t)\theta_m, \frac{\mathrm{d}}{\mathrm{d}t}U(t)\theta_m)\}$ weakly converges to $(U(t)\theta, \frac{\mathrm{d}}{\mathrm{d}t}U(t)\theta)$ in $L_2(0, T; \mathcal{H} \times \mathcal{H}^{-1})$ and consequently for every $t \in [0, T]$ we have $U(t)\theta_m \to U(t)\theta$ weakly in $\mathcal{H}^{-1}$. On the other hand, according to (7), for every $t \in [0, T]$ the sequence $\{U(t)\theta_m\}$ is bounded in $\mathcal{H}$. Thus we obtain (8). □

Lemma 2. *Let us assume that the conditions (3) and (4) are satisfied and B is a bounded subset of $\mathcal{H}$. Then for any $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon, B)$ and $r_0 = r_0(\varepsilon, B)$ such that for every $t \geq t_0$, $r \geq r_0$ and every $\theta \in B$ we have*

$$\frac{1}{t}\int_0^t \|U(s)\theta\|^2_{W_2^2(R^n \setminus B(0,r)) \times L_2(R^n \setminus B(0,r))} \mathrm{d}s \leq \varepsilon \tag{11}$$

where $B(0,r) = \{x \in R^n / |x| \leq r\}$.

Proof. Using the notation $\eta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\theta(t)$, from (5) we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\eta(t) = A\eta(t) + F_1(t), \qquad \eta(0) = \eta_0,$$

where $F_1(t) = (0, -f'(u)u_t)$ and $\eta_0 = (u_1, -\alpha u_1 - \Delta^2 u_0 - \lambda u_0 - f(u_0) + g)$. From (3) and (7) we have

$$\|F_1(t)\|_{\mathcal{H}^{-1}} \leq c_1\|u_t(t)\|, \qquad \forall\, t \geq 0. \tag{12}$$

Taking into account (6) and (12) in

$$\eta(t) = e^{tA}\eta_0 + \int_0^t e^{(t-s)A}(F_1(s))ds$$

for every $t \geq 0$ we obtain

$$\|\eta\|_{\mathcal{H}^{-1}} \leq Me^{-\omega t}\|\eta_0\|_{\mathcal{H}^{-1}} + Mc_1\int_0^t e^{-\omega(t-s)}\|u_t(s)\|ds,$$

which yields

$$\int_0^t \|\eta(s)\|^2_{\mathcal{H}^{-1}}ds \leq \frac{M^2}{\omega}\|\eta_0\|^2_{\mathcal{H}^{-1}} + c_2\int_0^t\left(\int_0^s e^{-\omega(s-\tau)}\|u_t(\tau)\|d\tau\right)^2 ds. \tag{13}$$

On the other hand,

$$\begin{aligned}\int_0^t\left(\int_0^s e^{-\omega(s-\tau)}\|u_t(\tau)\|d\tau\right)^2 ds &\leq \int_0^t\left(\int_0^s e^{-\omega(s-\tau)}d\tau\right)\left(\int_0^s e^{-\omega(s-\tau)}\|u_t(\tau)\|^2 d\tau\right)ds\\ &\leq \frac{1}{\omega}\int_0^t\int_0^s e^{-\omega(s-\tau)}\|u_t(\tau)\|^2 d\tau ds = \frac{1}{\omega}\int_0^t e^{\omega\tau}\|u_t(\tau)\|^2\left(\int_\tau^t e^{-\omega s}ds\right)d\tau\\ &\leq \frac{1}{\omega^2}\int_0^t\|u_t(\tau)\|^2 d\tau,\end{aligned} \tag{14}$$

and thus from (9), (13) and (14) we have

$$\int_0^t (\|u_t(\tau)\|^2 + \|u_{tt}(\tau)\|^2_{-2})d\tau \leq c_3, \qquad \forall\, t \geq 0. \tag{15}$$

Let $\varphi(\cdot) \in C^\infty(R^n)$ be such that

$$0 \leq \varphi(x) \leq 1 \qquad \text{and} \qquad \varphi(x) = \begin{cases}1, & |x| \geq 2\\ 0, & |x| \leq 1.\end{cases}$$

Multiplying (1) by $\varphi(\frac{x}{r})u(t,x)$, integrating over $[0,t] \times R^n$ and taking into account (4), (7) and (15), we obtain

$$\int_0^t\left(\|\Delta u\|^2_{L_2(R^n\setminus B(0,2r))} + \|u\|^2_{L_2(R^n\setminus B(0,2r))}\right)ds \leq c_4\left(1 + \frac{t}{r} + t\|g\|^2_{L_2(R^n\setminus B(0,r))}\right),$$

which, together with (15), yields (11). □

Lemma 3. *Assume that the conditions* (3) *and* (4) *are satisfied, and B is a bounded subset of $\mathcal{H}$. If $\{\theta_m\}$ is a sequence in B, weakly converging to θ in $\mathcal{H}$, then for any $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon, B)$ such that whenever $T \geq T_0$*

$$\limsup_{m\longrightarrow\infty}\|U(T)\theta_m - U(T)\theta\|_{\mathcal{H}} \leq \varepsilon \tag{16}$$

holds.

Proof. Let $\theta_m = (u_{0m}, u_{1m})$; then $U(t)\theta_m = (u^{(m)}(t), u_t^{(m)}(t))$, where $u^m(t,\cdot)$ is the solution of Eq. (1) subject to the conditions $u^m(0,x) = u_{0m}(x)$ and $u_t^m(0,x) = u_{1m}(x)$. Multiplying (1) by $(u_t + \frac{\alpha}{2}u)$, integrating over $[0, T] \times R^n$ and taking into account (3) and (7), we obtain that for every $T \geq 0$

$$\left| \int_0^T \left[E(u(t), u_t(t)) + \int_{R^n} f(u(t,x))u(t,x)\mathrm{d}x - \int_{R^n} g(x)u(t,x)\mathrm{d}x \right] \mathrm{d}t \right| \leq c_5. \tag{17}$$

Similarly to the case for (17), since B is bounded in $\mathcal{H}$ and $\theta_m \in B$, for every $T \geq 0$,

$$\left| \int_0^T \left[E(u^{(m)}(t), u_t^{(m)}(t)) + \int_{R^n} f(u^{(m)}(t))u^{(m)}(t)\mathrm{d}x - \int_{R^n} g(x)u^{(m)}(t)\mathrm{d}x \right] \mathrm{d}t \right| \leq c_6 \tag{18}$$

holds. From (9) and (17),

$$\begin{aligned} &E(u(T), u_t(T)) + \int_{R^n} \Phi(u(T,x))\mathrm{d}x - \int_{R^n} g(x)u(T,x)\mathrm{d}x + \frac{\alpha}{T}\int_0^T \int_t^T \|u_t\|^2 \mathrm{d}s\mathrm{d}t \\ &\geq \frac{1}{T}\int_0^T \int_{R^n} [\Phi(u(t,x)) - f(u(t,x))u(t,x)]\mathrm{d}x\mathrm{d}t - \frac{c_5}{T}. \end{aligned} \tag{19}$$

In a similar way, from (9) and (18) we find

$$\begin{aligned} &E(u^{(m)}(T), u_t^{(m)}(T)) + \int_{R^n} \Phi(u^{(m)}(T,x))\mathrm{d}x - \int_{R^n} g(x)u^{(m)}(T,x)\mathrm{d}x \\ &\quad + \frac{\alpha}{T}\int_0^T \int_t^T \|u_t^{(m)}\|^2 \mathrm{d}s\mathrm{d}t \\ &\leq \frac{1}{T}\int_0^T \int_{R^n} [\Phi(u^{(m)}(t,x)) - f(u^{(m)}(t,x))u^{(m)}(t,x)]\mathrm{d}x\mathrm{d}t + \frac{c_6}{T}. \end{aligned} \tag{20}$$

By (3) and (8) and compact embedding theorems, we have

$$\begin{cases} \lim\limits_{m\to\infty} \int_{B(0,r)} \Phi(u^{(m)}(T,x))\mathrm{d}x = \int_{B(0,r)} \Phi(u(T,x))\mathrm{d}x \\ \lim\limits_{m\to\infty} \frac{1}{T}\int_0^T \int_{B(0,r)} \Phi(u^{(m)}(t,x))\mathrm{d}x = \frac{1}{T}\int_0^T \int_{B(0,r)} \Phi(u(t,x))\mathrm{d}x \\ \lim\limits_{m\to\infty} \frac{1}{T}\int_0^T \int_{B(0,r)} f(u^{(m)}(t,x))u^{(m)}(t,x)\mathrm{d}x = \frac{1}{T}\int_0^T \int_{B(0,r)} f(u(t,x))u(t,x)\mathrm{d}x \end{cases} \tag{21}$$

for every $T > 0$ and $r > 0$. Since $\Phi(\cdot) \geq 0$, $(21)_1$ yields

$$\liminf_{m\longrightarrow\infty} \int_{R^n} \Phi(u^{(m)}(T,x))\mathrm{d}x \geq \int_{R^n} \Phi(u(T,x))\mathrm{d}x, \qquad \text{for } \forall\, T \geq 0. \tag{22}$$

On the other hand, by (3) and (11), for any $\varepsilon > 0$ there exist $t_0 = t_0(\varepsilon, B)$ and $r_0 = r_0(\varepsilon, B)$ such that for every $T \geq t_0$, $r \geq r_0$,

$$\frac{1}{T}\int_0^T \int_{R^n\setminus B(0,r)} [\Phi(u^{(m)}(t,x)) + f(u(t,x))u(t,x)]\mathrm{d}x\mathrm{d}t \leq \frac{\varepsilon}{2}. \tag{23}$$

Taking into account (8) and (19), $(21)_2$, $(21)_3$, (22) and (23) in (20) and passing to the limit we get

$$\limsup_{m\longrightarrow\infty} E(u^{(m)}(T), u_t^{(m)}(T)) \leq E(u(T), u_t(T)) + \frac{c_5 + c_6}{T} + \frac{\varepsilon}{2}$$

which, together with (8), gives (20). □

3. Asymptotic compactness and the global attractor

In this section, we shall show the existence of the global attractor. To this end, we first prove the asymptotic compactness of $U(t)$ in $\mathcal{H}$.

Theorem 1. *Assume that conditions* (3) *and* (4) *hold. Then for any bounded subset B of $\mathcal{H}$, the set $\{U(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $\mathcal{H}$, where $t_m \to \infty$ and $\{\theta_m\}_{m=1}^{\infty} \subset B$.*

Proof. Since B is bounded, by Lemma 1 we have $\sup_{t\geq 0} \sup_{\theta\in B} \|U(t)\theta\|_{\mathcal{H}} < \infty$. Therefore there exists a bounded subset B_0 of $\mathcal{H}$ such that $U(t)\theta \in B_0$, for every $t \geq 0$ and $\theta \in B$. Thus $\{U(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence $b_k := U(t_{m_k})\theta_{m_k}$ weakly converging in $\mathcal{H}$ to an a. From Lemma 3 we know that, if $\{\varphi_\nu\}_{\nu=1}^{\infty} \subset B_0$ and $\varphi_\nu \to \varphi$ weakly in $\mathcal{H}$, then for any $\varepsilon > 0$ there exists a $T_0 = T_0(\varepsilon, B_0)$ such that

$$\limsup_{\nu\longrightarrow\infty} \|U(T_0)\varphi_\nu - U(T_0)\varphi\|_{\mathcal{H}} \leq \varepsilon. \tag{24}$$

For $t_{m_k} \geq T_0$, since $U(t_{m_k} - T_0)\theta_{m_k} \in B_0$, there is a subsequence $\{k_\nu\}$ such that $\{U(t_{m_{k_\nu}} - T_0)\theta_{m_{k_\nu}}\}$ weakly converges to some φ in $\mathcal{H}$. Then by Lemma 1, the sequence $b_{k_\nu} := \{U(T_0)U(t_{m_{k_\nu}} - T_0)\theta_{m_{k_\nu}}\}$ weakly converges to $U(T_0)\varphi$ in $\mathcal{H}$. Hence from the uniqueness of the limit we get $a = U(T_0)\varphi$. Taking $\varphi_\nu = U(t_{m_{k_\nu}} - T_0)\theta_{m_{k_\nu}}$ in (24) we obtain $\limsup_{\nu\longrightarrow\infty} \|b_{k_\nu} - a\|_{\mathcal{H}} \leq \varepsilon$ and consequently $\liminf_{k\longrightarrow\infty} \|b_k - a\|_{\mathcal{H}} = 0$. In other words, the sequence $\{U(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence strongly convergent in $\mathcal{H}$. It can be seen in a similar way that every subsequence of $\{U(t_m)\theta_m\}_{m=1}^{\infty}$ has a subsequence strongly convergent in $\mathcal{H}$. Thus the set $\{U(t_m)\theta_m\}_{m=1}^{\infty}$ is relatively compact in $\mathcal{H}$. □

Since the problem (1) and (2) admits a "good" Lyapunov function $L(u, u_t) := E(u(t), u_t(t)) + \int_{R^n} \Phi(u(t,x))\mathrm{d}x - \int_{R^n} g(x)u(t,x)\mathrm{d}x$ and since by (4) the set of stationary solutions is bounded in H^2 (even in H^4), using the results of [7] we can formulate our main result.

Theorem 2. *Assume that* (3) *and* (4) *hold. Then problem* (1) *and* (2) *has a global attractor in $\mathcal{H}$, which is invariant and compact.*

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