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Weak solutions to the barotropic Navier–Stokes system with slip boundary conditions in time dependent domains

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ABSTRACT

We consider the compressible (barotropic) Navier–Stokes system on time dependent domains, supplemented with slip boundary conditions. Our approach is based on penalization of the boundary behavior, viscosity, and the pressure in the weak formulation. Global-in-time weak solutions are obtained.

1. Introduction

Problems involving the motion of solid objects in fluids occur frequently in various applications of continuum fluid dynamics, where the boundary conditions on the interfaces play a crucial role. Besides the commonly used *no-slip* condition, where the velocity of the fluid coincides with that of the adjacent solid body, various slip-like conditions have been proposed to handle the situations in which the no-slip scenario fails to produce a correct description of the fluid boundary behavior, see Bulíček, Málek and Rajagopal [1], Priezjev and Troian [17] and the references therein. For *viscous* fluids, Navier proposed the boundary conditions in the form

$$[\mathbb{S}\mathbf{n}]_{\tan} + \kappa[\mathbf{u} - \mathbf{V}]_{\tan}|_{\Gamma_\tau} = 0, \quad \kappa \geqslant 0, \tag{1.1}$$

where $\mathbb{S}$ is the viscous stress tensor, κ represents a "friction" coefficient, $\mathbf{u}$ and $\mathbf{V}$ denote the fluid and solid body velocities, respectively, and Γ_τ is the position of the interface at a time τ, with the outer

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normal vector $\mathbf{n}$. If $\kappa = 0$, we obtain the *complete slip* while the asymptotic limit $\kappa \to \infty$ gives rise to the standard no-slip boundary conditions.

Besides their applications in "thin" domains occurring in nanotechnologies (see Qian, Wang and Sheng [18]), the slip boundary conditions are particularly relevant for dense viscous gases (see Coron [2]), described by means of the standard *Navier–Stokes system*:

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0, \tag{1.2}$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \varrho \mathbf{f}, \tag{1.3}$$

where ϱ is the density, $p = p(\varrho)$ the (barotropic) pressure, $\mathbf{f}$ a given external force, and $\mathbb{S}$ is determined by the standard *Newton rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\operatorname{div}_x \mathbf{u}\mathbb{I}\right) + \eta \operatorname{div}_x \mathbf{u}\mathbb{I}, \quad \mu > 0,\ \eta \geqslant 0. \tag{1.4}$$

The boundary of the domain Ω_t occupied by the fluid is described by means of a *given* velocity field $\mathbf{V}(t, \mathbf{x})$, where $t \geqslant 0$ and $\mathbf{x} \in R^3$. More specifically, assuming $\mathbf{V}$ is regular, we solve the associated system of differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t, \mathbf{x}) = \mathbf{V}\big(t, \mathbf{X}(t, \mathbf{x})\big), \quad t > 0, \qquad \mathbf{X}(0, \mathbf{x}) = \mathbf{x}, \tag{1.5}$$

and set

$$\Omega_\tau = \mathbf{X}(\tau, \Omega_0), \quad \text{where } \Omega_0 \subset R^3 \text{ is a given domain,}$$
$$\Gamma_\tau = \partial\Omega_\tau, \quad \text{and} \quad Q_\tau = \big\{(t, x)\ \big|\ t \in (0, \tau),\ x \in \Omega_\tau\big\}.$$

In addition to (1.1), we assume that the boundary Γ_τ is impermeable, meaning

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \quad \text{for any } \tau \geqslant 0. \tag{1.6}$$

Finally, the problem (1.1)–(1.6) is supplemented by the initial conditions

$$\varrho(0, \cdot) = \varrho_0, \qquad (\varrho \mathbf{u})(0, \cdot) = (\varrho \mathbf{u})_0 \quad \text{in } \Omega_0. \tag{1.7}$$

Our main goal is to show *existence* of global-in-time *weak solutions* to problem (1.1)–(1.7) for any finite energy initial data. The existence theory for the barotropic Navier–Stokes system on *fixed* spatial domains in the framework of weak solutions was developed in the seminal work by Lions [12], and later extended in [9] to a class of physically relevant pressure-density state equations. The investigation of *incompressible* fluids in time dependent domains started with a seminal paper of Ladyzhenskaja [11], see also [13–15] for more recent results in this direction.

Compressible fluid flows in time dependent domains, supplemented with the *no-slip* boundary conditions, were examined in [7] by means of Brinkman's penalization method. However, applying a penalization method to the slip boundary conditions is more delicate. Unlike for *no-slip*, where the fluid velocity coincides with the field $\mathbf{V}$ outside Ω_τ, it is only its normal component $\mathbf{u} \cdot \mathbf{n}$ that can be controlled in the case of *slip*. In particular, given the rather poor *a priori* bounds available in the class of weak solutions, we lose control over the boundary behavior of the normal stress $\mathbb{S}\mathbf{n}$ involved in Navier's condition (1.1).

A rather obvious penalty approach to slip conditions for *stationary incompressible* fluids was proposed by Stokes and Carrey [20]. In the present setting, the variational (weak) formulation of the momentum equation is supplemented by a singular forcing term

$$\frac{1}{\varepsilon}\int_0^T \int_{\Gamma_t} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\boldsymbol{\varphi}\cdot\mathbf{n}\,\mathrm{dS}_x\,\mathrm{d}t, \quad \varepsilon>0 \text{ small}, \tag{1.8}$$

penalizing the normal component of the velocity on the boundary of the fluid domain. In the time dependent geometries, the penalization can be applied in the *interior* of a fixed reference domain, however, the resulting limit system consists of *two* fluids separated by impermeable boundary and coupled through the tangential components of normal stresses. In such a way, an extra term is produced acting on the fluid by its "complementary" part outside Ω_τ. In order to eliminate these extra stresses, we use the following three level penalization scheme:

1. In addition to (1.8), we introduce a *variable* shear viscosity coefficient $\mu=\mu_\omega$, where μ_ω remains strictly positive in the fluid domain Q_T but vanishes in the solid domain Q_T^c as $\omega\to 0$.
2. Similarly to the existence theory developed in [9], we introduce the *artificial pressure*

$$p_\delta(\varrho)=p(\varrho)+\delta\varrho^\beta, \quad \beta\geqslant 2,\ \delta>0,$$

in the momentum equation (1.3).
3. Keeping ε, δ, $\omega>0$ fixed, we solve the modified problem in a (bounded) reference domain $B\subset R^3$ chosen in such a way that

$$\overline{\Omega}_\tau\subset B \quad \text{for any } \tau\geqslant 0.$$

To this end, we adapt the existence theory for the compressible Navier–Stokes system with variable viscosity coefficients developed in [6].
4. We take the initial density ϱ_0 vanishing outside Ω_0, and letting $\varepsilon\to 0$ for fixed δ, $\omega>0$ we obtain a "two-fluid" system, where the density vanishes in the solid part $((0,T)\times B)\setminus Q_T$ of the reference domain.
5. Letting the viscosity vanish in the solid part, we perform the limit $\omega\to 0$, where the extra stresses disappear in the limit system. The desired conclusion results from the final limit process $\delta\to 0$.

The paper is organized as follows. In Section 2, we introduce all necessary preliminary material including a weak formulation of the problem and state the main result. Section 3 is devoted to the penalized problem and to uniform bounds and existence of solutions at the starting level of approximations. In Section 4, the singular limits for $\varepsilon\to 0$, $\omega\to 0$, and $\delta\to 0$ are preformed successively. Section 5 discusses possible extensions and applications of the method.

2. Preliminaries, weak formulation, main result

In the weak formulation, it is convenient that the equation of continuity (1.2) holds in the whole physical space R^3 provided the density ϱ was extended to be zero outside the fluid domain, specifically

$$\int_{\Omega_\tau} \varrho\varphi(\tau,\cdot)\,\mathrm{d}x-\int_{\Omega_0}\varrho_0\varphi(0,\cdot)\,\mathrm{d}x=\int_0^\tau\int_{\Omega_t}(\varrho\partial_t\varphi+\varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{2.1}$$

for any $\tau\in[0,T]$ and any test function $\varphi\in C_c^\infty([0,T]\times R^3)$. Moreover, Eq. (1.2) is also satisfied in the sense of renormalized solutions introduced by DiPerna and Lions [3]:

$$\int_{\Omega_\tau} b(\varrho)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_0} b(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x$$
$$= \int_0^\tau \int_{\Omega_t} \big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\operatorname{div}_x \mathbf{u}\varphi\big)\,\mathrm{d}x\,\mathrm{d}t \tag{2.2}$$

for any $\tau \in [0,T]$, any $\varphi \in C_c^\infty([0,T]\times R^3)$, and any $b \in C^1[0,\infty)$, $b(0)=0$, $b'(r)=0$ for large r. Of course, we suppose that $\varrho \geqslant 0$ a.a. in $(0,T)\times R^3$.

Similarly, the momentum equation (1.3) is replaced by a family of integral identities

$$\int_{\Omega_\tau} \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_{\Omega_0} (\varrho\mathbf{u})_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x$$
$$= \int_0^\tau \int_{\Omega_t} \big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p(\varrho)\operatorname{div}_x\boldsymbol{\varphi} - \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\boldsymbol{\varphi} + \varrho\mathbf{f}\cdot\boldsymbol{\varphi}\big)\,\mathrm{d}x\,\mathrm{d}t \tag{2.3}$$

for any $\tau \in [0,T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T]\times R^3; R^3)$ satisfying

$$\boldsymbol{\varphi}\cdot\mathbf{n}|_{\Gamma_\tau} = 0 \quad \text{for any } \tau \in [0,T]. \tag{2.4}$$

Finally, the impermeability condition (1.6) is satisfied in the sense of traces, specifically,

$$\mathbf{u} \in L^2\big(0,T; W^{1,2}\big(R^3;R^3\big)\big) \quad \text{and} \quad (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}(\tau,\cdot)|_{\Gamma_\tau} = 0 \quad \text{for a.a. } \tau \in [0,T]. \tag{2.5}$$

At this stage, we are ready to state the main result of the present paper:

Theorem 2.1. *Let $\Omega_0 \subset R^3$ be a bounded domain of class $C^{2+\nu}$, and let $\mathbf{V} \in C^1([0,T]; C_c^3(R^3;R^3))$ be given. Assume that the pressure $p \in C[0,\infty)\cap C^1(0,\infty)$ satisfies*

$$p(0) = 0, \qquad p'(\varrho) > 0 \quad \textit{for any } \varrho > 0, \qquad \lim_{\varrho\to\infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \quad \textit{for a certain } \gamma > 3/2.$$

Let the initial data satisfy

$$\varrho_0 \in L^\gamma\big(R^3\big), \qquad \varrho_0 \geqslant 0, \qquad \varrho_0 \not\equiv 0, \qquad \varrho_0|_{R^3\setminus\Omega_0} = 0,$$
$$(\varrho\mathbf{u})_0 = 0 \quad \textit{a.a. on the set } \{\varrho_0 = 0\}, \qquad \int_{\Omega_0} \frac{1}{\varrho_0}\big|(\varrho\mathbf{u})_0\big|^2\,\mathrm{d}x < \infty.$$

Then the problem (1.1)–(1.8) admits a weak solution on any time interval $(0,T)$ in the sense specified through (2.1)–(2.5).

The rest of the paper is devoted to the proof of Theorem 2.1.

3. Penalization

For the sake of simplicity, we restrict ourselves to the case $\kappa = 0$, $\eta = 0$, and $\mathbf{f} = 0$. As we shall see, the main ideas of the proof presented below require only straightforward modifications to accommodate the general case.

3.1. Penalized problem – weak formulation

Choosing $R > 0$ such that

$$\mathbf{V}|_{[0,T]\times\{|\mathbf{x}|>R\}} = 0, \quad \overline{\Omega}_0 \subset \{|\mathbf{x}| < R\} \tag{3.1}$$

we take the reference domain $B = \{|\mathbf{x}| < 2R\}$.

Next, the shear viscosity coefficient $\mu = \mu_\omega(t, \mathbf{x})$ is taken such that

$$\mu_\omega \in C_c^\infty([0,T] \times R^3), \qquad 0 < \underline{\mu_\omega} \leqslant \mu_\omega(t,x) \leqslant \mu \quad \text{in } [0,T] \times B,$$
$$\mu_\omega(\tau,\cdot)|_{\Omega_\tau} = \mu \quad \text{for any } \tau \in [0,T]. \tag{3.2}$$

Finally, we define modified initial data so that

$$\varrho_0 = \varrho_{0,\delta}, \qquad \varrho_{0,\delta} \geqslant 0, \qquad \varrho_{0,\delta} \not\equiv 0, \qquad \varrho_{0,\delta}|_{R^3\setminus\Omega_0} = 0, \qquad \int_B \left(\varrho_{0,\delta}^\gamma + \delta \varrho_{0,\delta}^\beta\right) \mathrm{d}x \leqslant c, \tag{3.3}$$

$$(\varrho\mathbf{u})_0 = (\varrho\mathbf{u})_{0,\delta}, \qquad (\varrho\mathbf{u})_{0,\delta} = 0 \quad \text{a.a. on the set } \{\varrho_{0,\delta} = 0\}, \qquad \int_{\Omega_0} \frac{1}{\varrho_{0,\delta}} \left|(\varrho\mathbf{u})_{0,\delta}\right|^2 \mathrm{d}x \leqslant c. \tag{3.4}$$

The weak formulation of the *penalized problem* reads as follows:

$$\int_B \varrho\varphi(\tau,\cdot)\,\mathrm{d}x - \int_B \varrho_0\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau \int_B (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{3.5}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T] \times R^3)$;

$$\begin{aligned}
&\int_B \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_B (\varrho\mathbf{u})_0\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x \\
&\quad = \int_0^\tau \int_B \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi} + \delta\varrho^\beta\,\mathrm{div}_x\,\boldsymbol{\varphi} \\
&\qquad - \mu_\omega\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\Big) : \nabla_x\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t + \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} \big((\mathbf{V}-\mathbf{u})\cdot\mathbf{n}\boldsymbol{\varphi}\cdot\mathbf{n}\big)\,\mathrm{dS}_x\,\mathrm{d}t
\end{aligned} \tag{3.6}$$

for any $\tau \in [0,T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0,T] \times B; R^3)$, where $\mathbf{u} \in L^2(0,T; W_0^{1,2}(B;R^3))$, meaning $\mathbf{u}$ satisfies the no-slip boundary condition

$$\mathbf{u}|_{\partial B} = 0 \quad \text{in the sense of traces.} \tag{3.7}$$

Here, ε, δ, and ω are positive parameters. The choice of the no-slip boundary condition (3.7) is not essential.

The *existence* of global-in-time solutions to the penalized problem can be shown by means of the method developed in [6] to handle the nonconstant viscosity coefficients. Indeed, for $\varepsilon > 0$ fixed, the extra penalty term in (3.6) can be treated as a “compact” perturbation. In addition, solutions can be constructed satisfying the *energy inequality*

$$\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x + \frac{1}{2}\int_0^\tau \int_B \mu_\omega \left|\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\right|^2 \mathrm{d}x\,\mathrm{d}t$$

$$+ \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} \big[(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\big]\mathbf{u}\cdot\mathbf{n}\,\mathrm{d}S_x\,\mathrm{d}t$$

$$\leqslant \int_B \left(\frac{1}{2\varrho_{0,\delta}}\left|(\varrho\mathbf{u})_{0,\delta}\right|^2 + P(\varrho_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta\right)\mathrm{d}x, \tag{3.8}$$

where

$$P(\varrho) = \varrho\int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z.$$

Note that the quantity on the right-hand side of (3.8) representing the total energy of the system is finite because of (3.3), (3.4).

In addition, since $\beta \geqslant 2$, the density is square integrable and we may use the regularization technique of DiPerna and Lions [3] to deduce the renormalized version of (3.5), namely

$$\int_B b(\varrho)\varphi(\tau,\cdot)\,\mathrm{d}x - \int_B b(\varrho_0)\varphi(0,\cdot)\,\mathrm{d}x$$

$$= \int_0^\tau \int_B \big(b(\varrho)\partial_t\varphi + b(\varrho)\mathbf{u}\cdot\nabla_x\varphi + \big(b(\varrho) - b'(\varrho)\varrho\big)\,\mathrm{div}_x\,\mathbf{u}\varphi\big)\,\mathrm{d}x\,\mathrm{d}t \tag{3.9}$$

for any φ and b as in (2.2).

3.2. Modified energy inequality and uniform bounds

Since the vector field $\mathbf{V}$ vanishes on the boundary ∂B, it may be used as a test function in (3.6). Combining the resulting expression with the energy inequality (3.8), we obtain

$$\int_B \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + \frac{\delta}{\beta-1}\varrho^\beta\right)(\tau,\cdot)\,\mathrm{d}x + \frac{1}{2}\int_0^\tau \int_B \mu_\omega \left|\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\right|^2 \mathrm{d}x\,\mathrm{d}t$$

$$+ \frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_t} \left|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\right|^2 \mathrm{d}S_x\,\mathrm{d}t$$

$$\leqslant \int_B \left(\frac{1}{2\varrho_{0,\delta}}\left|(\varrho\mathbf{u})_{0,\delta}\right|^2 + P(\varrho_{0,\delta}) + \frac{\delta}{\beta-1}\varrho_{0,\delta}^\beta\right)\mathrm{d}x + \int_B \big((\varrho\mathbf{u}\cdot\mathbf{V})(\tau,\cdot) - (\varrho\mathbf{u})_{0,\delta}\cdot\mathbf{V}(0,\cdot)\big)\,\mathrm{d}x$$

$$+\int_0^\tau\int_B\left(\mu_\omega\left(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\operatorname{div}_x\mathbf{u}\mathbb{I}\right):\nabla_x\mathbf{V}-\varrho\mathbf{u}\cdot\partial_t\mathbf{V}-\varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\mathbf{V}\right.$$

$$\left.-\,p(\varrho)\operatorname{div}_x\mathbf{V}-\frac{\delta}{\beta-1}\varrho^\beta\operatorname{div}_x\mathbf{V}\right)\mathrm{d}x\,\mathrm{d}t. \tag{3.10}$$

Since the vector field $\mathbf{V}$ is regular and since

$$p(\varrho)\leqslant c\big(1+P(\varrho)\big)\quad\text{for all }\varrho\geqslant 0,$$

relation (3.10) gives rise to the following bounds *independent* of the parameters ε, δ, and ω:

$$\operatorname*{ess\,sup}_{t\in(0,T)}\big\|\sqrt{\varrho}\mathbf{u}(t,\cdot)\big\|_{L^2(B;R^3)}\leqslant c, \tag{3.11}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)}\int_B P(\varrho)(t,\cdot)\,\mathrm{d}x\leqslant c\quad\text{yielding}\quad\operatorname*{ess\,sup}_{t\in(0,T)}\big\|\varrho(t,\cdot)\big\|_{L^\gamma(B)}\leqslant c, \tag{3.12}$$

$$\operatorname*{ess\,sup}_{t\in(0,T)}\delta\big\|\varrho(t,\cdot)\big\|_{L^\beta(B)}^\beta\leqslant c, \tag{3.13}$$

$$\int_0^T\int_B\mu_\omega\left|\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\operatorname{div}_x\mathbf{u}\mathbb{I}\right|^2\mathrm{d}x\,\mathrm{d}t\leqslant c, \tag{3.14}$$

and

$$\int_0^T\int_{\Gamma_t}\big|(\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\big|^2\,\mathrm{dS}_x\,\mathrm{d}t\leqslant\varepsilon c. \tag{3.15}$$

Finally, we note that the total mass is conserved, meaning

$$\int_B\varrho(\tau,\cdot)\,\mathrm{d}x=\int_B\varrho_{0,\delta}\,\mathrm{d}x=\int_{\Omega_0}\varrho_{0,\delta}\,\mathrm{d}x\leqslant c\quad\text{for any }\tau\in[0,T]. \tag{3.16}$$

Thus, relations (3.11), (3.14), (3.16), combined with the generalized version of Korn's inequality (see [8, Theorem 10.17]), imply that

$$\int_0^T\big\|\mathbf{u}(t,\cdot)\big\|_{W_0^{1,2}(B;R^3)}^2\leqslant c(\omega). \tag{3.17}$$

3.3. *Pressure estimates*

Since the surfaces Γ_τ are determined *a priori*, we can use the technique based on the so-called Bogovskii operator to deduce the uniform bounds

$$\int\int_{\mathcal{K}}\big(p(\varrho)\varrho^\nu+\delta\varrho^{\beta+\nu}\big)\,\mathrm{d}x\,\mathrm{d}t\leqslant c(\mathcal{K})\quad\text{for a certain }\nu>0 \tag{3.18}$$

for any compact $\mathcal{K} \subset [0,T] \times \overline{B}$ such that

$$\mathcal{K} \cap \left(\bigcup_{\tau \in [0,T]} \left(\{\tau\} \times \Gamma_\tau \right) \right) = \emptyset,$$

see [10] for details.

Note that due to the fact that the boundaries Γ_τ change with time, *uniform estimates* like (3.18) on the whole space–time cylinder $(0,T) \times B$ seem to be a delicate matter. On the other hand, the mere *equi-integrability* of the pressure could be shown by the method based on special test functions used in [7].

4. Singular limits

In this section, we perform successively the singular limits $\varepsilon \to 0$, $\omega \to 0$, and $\delta \to 0$.

4.1. Penalization limit

Keeping the parameters δ, ω fixed, our goal is to let $\varepsilon \to 0$ in (3.5), (3.6). Let $\{\varrho_\varepsilon, \mathbf{u}_\varepsilon\}$ be the corresponding weak solution of the perturbed problem constructed in the previous section. To begin, the estimates (3.12), (3.17), combined with the equation of continuity (3.5), imply that

$$\varrho_\varepsilon \to \varrho \quad \text{in } C_{\text{weak}}\big([0,T]; L^\gamma(B)\big),$$

and

$$\mathbf{u}_\varepsilon \to \mathbf{u} \quad \text{weakly in } L^2\big(0,T; W_0^{1,2}\big(B, R^3\big)\big)$$

at least for suitable subsequences, where, as a direct consequence of (3.15),

$$(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n}(\tau, \cdot)|_{\Gamma_\tau} = 0 \quad \text{for a.a. } \tau \in [0,T]. \tag{4.1}$$

Consequently, in accordance with (3.11), (3.12) and the compact embedding $L^\gamma(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$, we obtain

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \quad \text{weakly-(*) in } L^\infty\big(0,T; L^{2\gamma/(\gamma+1)}\big(B; R^3\big)\big), \tag{4.2}$$

and, thanks to the embedding $W_0^{1,2}(B) \hookrightarrow L^6(B)$,

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \to \overline{\varrho \mathbf{u} \otimes \mathbf{u}} \quad \text{weakly in } L^2\big(0,T; L^{6\gamma/(4\gamma+3)}\big(B; R^3\big)\big),$$

where we have used the bar to denote a weak limit of a composed function.

Finally, it follows from the momentum equation (3.6) that

$$\varrho_\varepsilon \mathbf{u}_\varepsilon \to \varrho \mathbf{u} \quad \text{in } C_{\text{weak}}\big([T_1, T_2]; L^{2\gamma/(\gamma+1)}\big(O; R^3\big)\big)$$

for any space–time cylinder

$$(T_1, T_2) \times O \subset [0,T] \times B, \qquad [T_1, T_2] \times \overline{O} \cap \bigcup_{\tau \in [0,T]} \big(\{\tau\} \times \Gamma_\tau\big) = \emptyset.$$

Seeing that $L^{2\gamma/(\gamma+1)}(B) \hookrightarrow\hookrightarrow W^{-1,2}(B)$ we conclude, exactly as in (4.2), that

$$\overline{\varrho\mathbf{u}\otimes\mathbf{u}} = \varrho\mathbf{u}\otimes\mathbf{u} \quad \text{a.a. in } (0,T)\times B.$$

Passing to the limit in (3.5) we obtain

$$\int_B \varrho\varphi(\tau,\cdot)\,\mathrm{d}x - \int_B \varrho_{0,\delta}\varphi(0,\cdot)\,\mathrm{d}x = \int_0^\tau\int_B (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{4.3}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^\infty([0,T]\times R^3)$.

The limit in the momentum equation (3.6) is more delicate. Since we have at hand only the *local estimates* (3.18) on the pressure, we have to restrict ourselves to the class of test functions

$$\begin{gathered}\boldsymbol{\varphi} \in C^1\big([0,T]; W_0^{1,\infty}\big(B;R^3\big)\big), \qquad \mathrm{supp}\big[\mathrm{div}_x\,\boldsymbol{\varphi}(\tau,\cdot)\big]\cap\Gamma_\tau = \emptyset,\\ \boldsymbol{\varphi}\cdot\mathbf{n}|_{\Gamma_t} = 0 \quad \text{for all } \tau\in[0,T].\end{gathered} \tag{4.4}$$

Passing to the limit in (3.6), we obtain

$$\begin{aligned}&\int_B \varrho\mathbf{u}\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_B (\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x\\ &\quad= \int_0^\tau\int_B \bigg(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\boldsymbol{\varphi} + \overline{p(\varrho)}\,\mathrm{div}_x\,\boldsymbol{\varphi} + \delta\overline{\varrho^\beta}\,\mathrm{div}_x\,\boldsymbol{\varphi}\\ &\qquad - \mu_\omega\bigg(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\bigg):\nabla_x\boldsymbol{\varphi}\bigg)\,\mathrm{d}x\,\mathrm{d}t\end{aligned} \tag{4.5}$$

for any test function $\boldsymbol{\varphi}$ as in (4.4). Note that the requirement of smoothness of $\boldsymbol{\varphi}$ postulated in (3.6) can be easily relaxed by means of a density argument.

Finally, we show pointwise (a.a.) convergence of the sequence $\{\varrho_\varepsilon\}_{\varepsilon>0}$. To this end, we adopt the method developed in [6] to accommodate the variable viscosity coefficient μ_ω. The crucial observation is the effective viscous pressure identity that can be established exactly as in [6]:

$$\overline{p_\delta(\varrho)T_k(\varrho)} - \overline{p_\delta(\varrho)}\,\overline{T_k(\varrho)} = \frac{4}{3}\mu_\omega\big(\overline{T_k(\varrho)\,\mathrm{div}_x\,\mathbf{u}} - \overline{T_k(\varrho)}\,\mathrm{div}_x\,\mathbf{u}\big) \tag{4.6}$$

where we have denoted

$$p_\delta(\varrho) = p(\varrho) + \delta\varrho^\beta, \qquad T_k(\varrho) = \min\{\varrho, k\}.$$

Similarly to the pressure estimates (3.18), identity (4.6) holds only on compact sets $\mathcal{K}\subset[0,T]\times B$ satisfying

$$\mathcal{K}\cap\bigg(\bigcup_{\tau\in[0,T]}\big(\{\tau\}\times\Gamma_\tau\big)\bigg) = \emptyset.$$

We recall that this step leans on the satisfaction of the renormalized equation (3.9) for both ϱ_ε and the limit ϱ that can be shown by the regularization procedure of DiPerna and Lions [3].

Following [6], we introduce the *oscillations defect measure*

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho](\mathcal{K}) = \sup_{k \geqslant 0}\left(\limsup_{\varepsilon \to 0} \int_{\mathcal{K}} \left|T_k(\varrho_\varepsilon) - T_k(\varrho)\right|^q \, \mathrm{d}x\, \mathrm{d}t\right),$$

and use (4.6) to conclude that

$$\mathbf{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho](\mathcal{K}) \leqslant c(\omega) < \infty, \tag{4.7}$$

where the constant c is *independent* of $\mathcal{K}$. Thus

$$\mathbf{osc}_{\gamma+1}[\varrho_\varepsilon \to \varrho]\big([0,T] \times B\big) \leqslant c(\omega), \tag{4.8}$$

which implies, by virtue of the procedure developed in [5], the desired conclusion

$$\varrho_\varepsilon \to \varrho \quad \text{a.a. in } (0,T) \times B. \tag{4.9}$$

In accordance with (4.9), the momentum equation (4.5) reads

$$\begin{aligned} &\int_B \varrho\mathbf{u} \cdot \boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x - \int_B (\varrho\mathbf{u})_{0,\delta} \cdot \boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x \\ &\quad = \int_0^\tau \int_B \Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi} + \varrho[\mathbf{u}\otimes\mathbf{u}] : \nabla_x\boldsymbol{\varphi} + p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi} + \delta\varrho^\beta\,\mathrm{div}_x\,\boldsymbol{\varphi} \\ &\qquad - \mu_\omega\Big(\nabla_x\mathbf{u} + \nabla_x^t\mathbf{u} - \frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\Big) : \nabla_x\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t \end{aligned} \tag{4.10}$$

for any test function $\boldsymbol{\varphi}$ as in (4.4). In addition, as already observed, the limit solution $\{\varrho, \mathbf{u}\}$ satisfies also the renormalized equation (3.9).

4.1.1. Fundamental lemma

Our next goal is to use the specific choice of the initial data $\varrho_{0,\delta}$ to get rid of the density-dependent terms in (4.10) supported by the "solid" part $((0,T) \times B) \setminus Q_T$. To this end, we show the following result, rather obvious for regular solutions but a bit more delicate in the weak framework, that may be of independent interest.

Lemma 4.1. *Let $\varrho \in L^\infty(0,T;L^2(B))$, $\varrho \geqslant 0$, $\mathbf{u} \in L^2(0,T;W_0^{1,2}(B;R^3))$ be a weak solution of the equation of continuity, specifically,*

$$\int_B \big(\varrho(\tau,\cdot)\varphi(\tau,\cdot) - \varrho_0\varphi(0,\cdot)\big)\,\mathrm{d}x = \int_0^\tau \int_B (\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi)\,\mathrm{d}x\,\mathrm{d}t \tag{4.11}$$

for any $\tau \in [0,T]$ and any test function $\varphi \in C_c^1([0,T] \times R^3)$.

In addition, assume that

$$(\mathbf{u} - \mathbf{V})(\tau,\cdot) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \quad \textit{for a.a. } \tau \in (0,T), \tag{4.12}$$

and that

$$\varrho_0 \in L^2(R^3), \qquad \varrho_0 \geqslant 0, \qquad \varrho_0|_{B\setminus\Omega_0} = 0.$$

Then

$$\varrho(\tau,\cdot)|_{B\setminus\Omega_\tau} = 0 \quad \textit{for any } \tau \in [0,T].$$

Proof. We use the level set approach to describe the interface Γ_τ, see Osher and Fedkiw [16]. To this end, we introduce a function $d = d(t,x)$ defined as the unique solution of the transport equation

$$\partial_t d + \nabla_x d(t,x)\cdot \mathbf{V} = 0, \quad t > 0,\ x \in R^3,$$

with the initial data

$$d(0,x) = d_0(x) \in C^\infty(R^3), \qquad d_0(x) = \begin{cases} > 0 & \text{for } x \in B\setminus\Omega_0, \\ < 0 & \text{for } x \in \Omega_0 \cup (R^3\setminus\overline{B}), \end{cases} \qquad \nabla_x d_0 \neq 0 \quad \text{on } \Gamma_0.$$

Note that the interface Γ_τ can be identified with a component of the level set $\{d(\tau,\cdot) = 0\}$, while the sets $B\setminus\Omega_\tau$ correspond to $\{d(\tau,\cdot) > 0\}$. Finally,

$$\begin{gathered} \nabla_x d(\tau,x) = \lambda(\tau,x)\mathbf{n}(x) \quad \text{for any } x \in \Gamma_\tau, \\ \lambda(\tau,x) \geqslant 0 \quad \text{for } \tau \in [0,T]. \end{gathered} \tag{4.13}$$

For a given $\xi > 0$, we take

$$\varphi = \left[\min\left\{\frac{1}{\xi}d; 1\right\}\right]^+$$

as a (Lipschitz) test function in the variational formulation (4.11) to obtain

$$\int_{B\setminus\Omega_\tau} \varrho\varphi(\tau,\cdot)\,\mathrm{d}x = \frac{1}{\xi}\int_0^\tau \int_{\{0\leqslant d(t,x)<\xi\}} (\varrho\partial_t d + \varrho\mathbf{u}\cdot\nabla_x d)\,\mathrm{d}x\,\mathrm{d}t. \tag{4.14}$$

Now, we have

$$\varrho\partial_t d + \varrho\mathbf{u}\cdot\nabla_x d = \varrho(\partial_t d + \mathbf{u}\cdot\nabla_x d) = \varrho(\mathbf{V}-\mathbf{u})\cdot\nabla_x d$$

where, by virtue of hypothesis (4.12) and relation (4.13),

$$(\mathbf{V}-\mathbf{u})\cdot\nabla_x d \in W_0^{1,2}(B\setminus\Omega_t) \quad \text{for a.a. } t \in (0,\tau). \tag{4.15}$$

Introducing

$$\delta(t,x) = \mathrm{dist}_{R^3}\big[x, \partial(B\setminus\Omega_t)\big] \quad \text{for } t \in [0,\tau],\ x \in B\setminus\Omega_t,$$

we deduce from (4.15) and Hardy's inequality that

$$\frac{1}{\delta}(\mathbf{V}-\mathbf{u})\cdot\nabla_x d \in L^2\big([0,\tau]\times B\setminus Q_\tau\big).$$

Finally, since $\mathbf{V}$ is regular, we have

$$\frac{\delta(t,x)}{\xi}\leqslant c \quad \text{whenever } 0\leqslant d(t,x)<\xi;$$

whence, letting $\xi\to 0$ in (4.14), we obtain the desired conclusion

$$\int\limits_{B\setminus\Omega_\tau}\varrho(\tau,\cdot)\,\mathrm{d}x=0,$$

where we have used the fact that $\varrho\in L^\infty(0,T;L^2(B))$. □

Thus, by virtue of Lemma 4.1, the momentum equation (4.10) reduces to

$$\begin{aligned}&\int\limits_{\Omega_\tau}\varrho\mathbf{u}\cdot\boldsymbol{\varphi}(\tau,\cdot)\,\mathrm{d}x-\int\limits_{\Omega_0}(\varrho\mathbf{u})_{0,\delta}\cdot\boldsymbol{\varphi}(0,\cdot)\,\mathrm{d}x\\&=\int_0^\tau\int\limits_{\Omega_t}\Big(\varrho\mathbf{u}\cdot\partial_t\boldsymbol{\varphi}+\varrho[\mathbf{u}\otimes\mathbf{u}]:\nabla_x\boldsymbol{\varphi}+p(\varrho)\,\mathrm{div}_x\,\boldsymbol{\varphi}+\delta\varrho^\beta\,\mathrm{div}_x\,\boldsymbol{\varphi}\\&\quad-\mu\Big(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\Big):\nabla_x\boldsymbol{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_0^\tau\int\limits_{B\setminus\Omega_t}\mu_\omega\Big(\nabla_x\mathbf{u}+\nabla_x^t\mathbf{u}-\frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}\mathbb{I}\Big):\nabla_x\boldsymbol{\varphi}\,\mathrm{d}x\,\mathrm{d}t\end{aligned}\tag{4.16}$$

for any test function $\boldsymbol{\varphi}$ as in (4.4). We remark that it was exactly this step when we needed the extra pressure term $\delta\varrho^\beta$ ensuring the density ϱ to be square integrable.

4.2. Vanishing viscosity limit

In order to get rid of the last integral in (4.16), we take the viscosity coefficient

$$\mu_\omega=\begin{cases}\mu=\mathrm{const}>0 & \text{in } Q_T,\\ \mu_\omega\to 0 & \text{a.a. in } ((0,T)\times B)\setminus Q_T.\end{cases}$$

Denoting $\{\varrho_\omega,\mathbf{u}_\omega\}$ the corresponding solution constructed in the previous section, we may use (3.14) to deduce that

$$\int_0^T\int\limits_{\Omega_t}\Big|\nabla_x\mathbf{u}_\omega+\nabla_x^t\mathbf{u}_\omega-\frac{2}{3}\,\mathrm{div}_x\,\mathbf{u}_\omega\mathbb{I}\Big|^2\,\mathrm{d}x\,\mathrm{d}t<c,\tag{4.17}$$

while

$$\int_0^T \int_{B\setminus\Omega_t} \mu_\omega \left| \nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right|^2 \mathrm{d}x\, \mathrm{d}t \leqslant c,$$

where the latter estimates yield

$$\int_0^\tau \int_{B\setminus\Omega_t} \mu_\omega \left(\nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t$$
$$= \int_0^\tau \int_{B\setminus\Omega_t} \sqrt{\mu_\omega}\sqrt{\mu_\omega} \left(\nabla_x \mathbf{u}_\omega + \nabla_x^t \mathbf{u}_\omega - \frac{2}{3} \operatorname{div}_x \mathbf{u}_\omega \mathbb{I} \right) : \nabla_x \boldsymbol{\varphi}\, \mathrm{d}x\, \mathrm{d}t \to 0 \quad \text{as } \omega \to 0$$

for any fixed $\boldsymbol{\varphi}$.

On the other hand, as we know from Lemma 4.1 that the density ϱ_ω is supported by the “fluid” region Q_T, we can still use (3.11), (4.17), together with Korn’s inequality to obtain

$$\int_0^T \int_{\Omega_t} |\nabla_x \mathbf{u}_\omega|^2 \, \mathrm{d}x\, \mathrm{d}t \leqslant c.$$

Repeating step by step the arguments of the preceding section, we let $\omega \to 0$ to obtain the momentum equation in the form

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot)\, \mathrm{d}x - \int_{\Omega_0} (\varrho \mathbf{u})_{0,\delta} \boldsymbol{\varphi}(0, \cdot)\, \mathrm{d}x = \int_0^\tau \int_{\Omega_t} \big(\varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho [\mathbf{u} \otimes \mathbf{u}] : \nabla_x \boldsymbol{\varphi} + p(\varrho) \operatorname{div}_x \boldsymbol{\varphi}$$
$$+ \delta \varrho^\beta \operatorname{div}_x \boldsymbol{\varphi} - \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \boldsymbol{\varphi} \big)\, \mathrm{d}x\, \mathrm{d}t \tag{4.18}$$

for any test function $\boldsymbol{\varphi}$ as in (4.4). Note that compactness of the density is now necessary only in the “fluid” part Q_T so a possible loss of regularity of $\mathbf{u}_\omega$ outside Q_T is irrelevant.

4.3. *Vanishing artificial pressure*

The final step is standard, we let $\delta \to 0$ in (4.18) to get rid of the artificial pressure term $\delta \varrho^\beta$ and to adjust the initial conditions, see [5, Chapter 6]. However, the momentum equation identity (4.18) holds only for the class of functions specified in (4.4). The last step of the proof of Theorem 2.1 is therefore to show that the class of admissible test functions can be extended by density arguments. This will be shown in the following part.

4.3.1. *Extending the class of test functions*

Consider a test function $\boldsymbol{\varphi} \in C_c^\infty([0,T] \times R^3; R^3)$ such that

$$\boldsymbol{\varphi}(\tau, \cdot) \cdot \mathbf{n}|_{\Gamma_\tau} = 0 \quad \text{for any } \tau. \tag{4.19}$$

Our goal is to show the existence of an approximating sequence of functions $\boldsymbol{\varphi}_n$ belonging to the class specified in (4.4) and such that

$$\|\boldsymbol{\varphi}_n\|_{W^{1,\infty}((0,T)\times B;R^3)} \leqslant c, \qquad \boldsymbol{\varphi}_n \to \boldsymbol{\varphi}, \qquad \partial_t \boldsymbol{\varphi}_n \to \partial_t \boldsymbol{\varphi}, \quad \text{and}$$
$$\nabla_x \boldsymbol{\varphi}_n \to \nabla_x \boldsymbol{\varphi} \quad \text{a.a. in } Q_T. \tag{4.20}$$

Combining (4.20) with Lebesgue dominated convergence theorem we may infer that $\boldsymbol{\varphi}$ belongs to the class of admissible test functions for (2.3).

In other words, we have to find a suitable *solenoidal extension* of the tangent vector field $\boldsymbol{\varphi}|_{\Gamma_\tau}$ inside Ω_τ. Since Γ_τ is regular, there is an open neighborhood $\mathcal{U}_\tau$ of Γ_τ such that each point $\mathbf{x} \in \mathcal{U}_\tau$ admits a *single* closest point $\mathbf{b}_\tau(\mathbf{x}) \in \Gamma_\tau$. We set

$$\mathbf{h}(\tau, \mathbf{x}) = \boldsymbol{\varphi}\big(\tau, \mathbf{b}_\tau(\mathbf{x})\big) \quad \text{for all } \mathbf{x} \in \mathcal{U}_\tau.$$

Finally, we define

$$\mathbf{w}(\tau, \mathbf{x}) = \mathbf{h}(\tau, \mathbf{x}) + \mathbf{g}(\tau, \mathbf{x}),$$

where

$$\mathbf{g}(\tau, \mathbf{x}) = 0 \quad \text{whenever } \mathbf{x} \in \Gamma_\tau,$$

and, taking the local coordinate system at $\mathbf{x}$ so that e_3 coincides with $\mathbf{x} - \mathbf{b}_\tau(\mathbf{x})$, we set

$$\mathbf{g}(\tau, \mathbf{x}) = \big[0, 0, g^3(\tau, \mathbf{x})\big], \qquad \partial_{x_3} g^3(\tau, \mathbf{x}) = -\partial_{x_1} h^1(\tau, \mathbf{x}) - \partial_{x_2} h^2(\tau, \mathbf{x}).$$

We check that

$$\mathrm{div}_x \mathbf{w}(\tau, \cdot) = 0 \quad \text{in } \mathcal{U}_\tau, \qquad \mathbf{w}(\tau, \cdot)|_{\Gamma_\tau} = \boldsymbol{\varphi}(\tau, \cdot)|_{\Gamma_\tau}.$$

Furthermore, extending $\mathbf{w}(\tau, \cdot)$ inside Ω_τ, we may use smoothness of $\boldsymbol{\varphi}$ and Γ_τ to conclude that

$$\mathbf{w} \in W^{1,\infty}(Q_T).$$

As a matter of fact, a (smooth) extension of $\boldsymbol{\varphi}$, *solenoidal* in the whole domain Q_T, was constructed by Shifrin [19, Theorem 4].

Writing

$$\boldsymbol{\varphi} = (\boldsymbol{\varphi} - \mathbf{w}) + \mathbf{w},$$

we check that the field $\mathbf{w}$ belongs to the class (4.4), while

$$(\boldsymbol{\varphi} - \mathbf{w})(\tau, \cdot)|_{\partial\Omega_\tau} = 0 \quad \text{for any } \tau \geqslant 0.$$

Thus, finally, it is a matter of routine to construct a sequence $\mathbf{a}_n$ such that

$$\mathbf{a}_n \in C_c^\infty\big([0, T] \times B; R^3\big), \qquad \mathrm{supp}\big[\mathbf{a}_n(\tau, \cdot)\big] \subset \Omega_\tau \quad \text{for any } \tau \in [0, T],$$

in particular $\mathbf{a}_n$ belongs to the class (4.4), and

$$\|\mathbf{a}_n\|_{W^{1,\infty}((0,T)\times B;R^3)} \leqslant c, \qquad \mathbf{a}_n \to (\boldsymbol{\varphi} - \mathbf{w}), \qquad \partial_t \mathbf{a}_n \to \partial_t(\boldsymbol{\varphi} - \mathbf{w}), \quad \text{and}$$
$$\nabla_x \mathbf{a}_n \to \nabla_x(\boldsymbol{\varphi} - \mathbf{w}) \quad \text{a.a. in } Q_T.$$

Clearly, the sequence

$$\boldsymbol{\varphi}_n = \mathbf{a}_n + \mathbf{w}$$

complies with (4.20).

We have completed the proof of Theorem 2.1.

5. Discussion

The assumption on monotonicity of the pressure is not necessary, the same result can be obtained for a non-monotone pressure adopting the method developed in [4].

As already pointed out, the technicalities of Section 4.3.1 could be avoided by means of the construction of special test functions used in [7]. However, it would be interesting to show that the pressure is bounded in some $L^q(Q_T)$, with $q > 1$, meaning that the estimate (3.18) holds in Q_T.

As pointed out in the introduction, the general Navier slip conditions (1.1) are obtained introducing another boundary integral in the weak formulation, namely

$$\int_0^T \int_{\Gamma_t} \kappa(\mathbf{u} - \mathbf{V}) \cdot \boldsymbol{\varphi} \, \mathrm{dS}_x \, \mathrm{d}t.$$

Taking $\kappa = \kappa(\mathbf{x})$ as a singular parameter, we can deduce results for mixed type no-slip – (partial) slip boundary conditions prescribed on various components of Γ_t.

Last but not least, the method can be extended to unbounded (exterior) domains with prescribed boundary conditions “at infinity”.

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