On strong ergodicity for nonhomogeneous continuous-time Markov chains

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Let $X(t)$ be a nonhomogeneous continuous-time Markov chain. Suppose that the intensity matrices of $X(t)$ and some weakly or strongly ergodic Markov chain $\tilde{X}(t)$ are close. Some sufficient conditions for weak and strong ergodicity of $X(t)$ are given and estimates of the rate of convergence are proved. Queue-length for a birth and death process in the case of asymptotically proportional intensities is considered as an example.

1. Introduction and definitions

Ergodic properties of nonhomogeneous continuous-time Markov chains have been investigated by a number of authors. One of the problems in this area is the following. Let the intensity matrix of a Markov chain $X(t)$ be close to the intensity matrix of an ergodic Markov chain $\tilde{X}(t)$ (particularly, the perturbation tends to zero as $t \to \infty$ or the perturbation is integrable). Is $X(t)$ also ergodic with the same limit regime?

The case where $\tilde{X}(t)$ is homogeneous was considered in [6], and the case of an exponential ergodic 'non-perturbed' chain was studied in [12]. Now we consider the more general case and give some estimates of the rate of convergence.

Let $X(t)$ and $\tilde{X}(t)$ be continuous-time Markov chains with the state space $S = \{0, 1, \ldots, K\}$, $K \leq \infty$. Write $p_i(t) = \Pr(X(t) = i)$ and $\mathbf{p}(t) = (p_i(t))$. Let $P(s, t) = (p_{ij}(s, t))$ and $A(t) = (a_{ij}(t))$ be transition and intensity matrices for the chain $X(t)$ respectively.

Put $\|x\| = \sum |x_i|$, $\|P\| = \sup_i \sum_j |p_{ij}|$, let $\Omega$ be the set of all stochastic vectors, and ' denote transpose.
We shall consider the Markov chains which satisfy the following condition: for any \( t \geq 0 \) and \( h > 0 \),

\[
\Pr\{X(t+h) = j/X(t) = i\} = \begin{cases} 
  a_{ij}(t)h + o(h) & \text{if } j \neq i, \\
  1 - h \sum_{k \neq i} a_{ik}(t) - o(h) & \text{if } j = i,
\end{cases}
\]

where all \( o(h) \) are uniform in \( i \), \( a_{ii}(t) = -\sum_{k \neq i} a_{ik}(t) \). Then for any \( t \geq 0 \) in \( l_1 \)-norm,

\[
x(t+h) - x(t) = B(t)x(t)h + o(h),
\]

where \( x(t) = p'(t), B(t) = A'(t) \). Hence there is the following differential equation in the space \( l_1 \):

\[
dx/dt = B(t)x.
\]

One can note that this equation is a forward Kolmogorov system for \( X(t) \). If \( U(t, s) \) is a Cauchy operator of equation (1.3) then for any \( s \geq 0, t \geq s \), \( U(t, s) = P'(s, t) \). Notice that

\[
U(t, \tau) = I + \int_\tau^t B(s) \, ds + \int_\tau^s B(s) \int_\tau^s B(s_1) \, ds_1 \, ds + \cdots
\]

and \( x(t) = U(t, s)x(s) \) for any \( s \geq 0, t \geq s \), \( x(s) \); the properties of a Cauchy operator were investigated in [1].

Denote by \( \tilde{p}_j(t), \tilde{p}(t), \tilde{P}(s, t), \tilde{A}(t) \) the respective functions for the chain \( \tilde{X}(t) \).

In the whole paper we suppose that the intensity matrices are locally integrable on \([0, \infty)\) and the forward Kolmogorov system is identified with a differential equation in the space \( l_1 \), see [1, 10]. In this case Cauchy operators for \( X(t) \) and \( \tilde{X}(t) \) exist and are unique.

**Definition 1.1.** A Markov chain \( X(t) \) is called weakly ergodic if there exists a vector function \( q(t) \in \Omega \) for every \( t \geq 0 \) such that if \( Q(t) = (q(t), q(t), \ldots)' \) then for all \( s \geq 0 \),

\[
\lim_{t \to \infty} \| P(s, t) - Q(t) \| = 0.
\]

It is noteworthy that \( X(t) \) can be proved to be weakly ergodic if the above limit is equal to zero for some \( s \geq 0 \).

**Definition 1.2.** A Markov chain \( X(t) \) is called strongly ergodic if there exists a vector \( q \in \Omega \) such that if \( Q = (q, q, \ldots)' \) then for all \( s \geq 0 \),

\[
\lim_{t \to \infty} \| P(s, t) - Q \| = 0.
\]
Definition 1.3. A Markov chain $X(t)$ is called exponentially ergodic if there exist positive numbers $N$ and $\sigma$ such that for any $p^* \in \Omega$, $p^{**} \in \Omega$, $s, t$ $(0 \leq s \leq t)$,
\[
\|(p^*-p^{**})P(s, t)\| \leq Ne^{-\sigma(t-s)}\|p^*-p^{**}\|.
\] (1.7)

Let
\[
a(t) = \|A(t) - \bar{A}(t)\|.
\] (1.8)

Then using [6,12] we have the following:

**Proposition.** If $\bar{X}(t)$ is strong ergodic and homogeneous (or exponentially ergodic) and, in addition, $a(t) \to 0$ as $t \to \infty$, then $X(t)$ is strongly ergodic and the limit regimes for $X(t)$ and $\bar{X}(t)$ are identical. \(\square\)

**Remark 1.1.** The proposition is not correct if $\bar{X}(t)$ is assumed to be only strongly ergodic.

**Example 1.1.** Let $X(t)$, $\bar{X}(t)$ be Markov chains with two states 0, 1; and let $\lambda(t)$, $\mu(t)$, $(\bar{\lambda}(t), \bar{\mu}(t))$ be intensities of transitions $0 \to 1$ and $1 \to 0$ respectively, where $\bar{\lambda}(t) = \bar{\mu}(t) = 1/(1+t)$, $\lambda(t) = \mu(t) \equiv 0$. Then $\bar{X}(t)$ is strongly ergodic with the limit regime $q = (0.5, 0.5)$, $a(t) = 2/(1+t) \to 0$ as $t \to \infty$; and $X(t)$ is not ergodic.

On the other hand the condition of exponential ergodicity of $\bar{X}(t)$ is not necessary; for instance, if $\bar{\lambda}(t) = \bar{\mu}(t) = 1/(1+t)$, $a(t) = 1/(1+t)^2$, then the chain $X(t)$ in Example 1.1 is strongly ergodic with the limit regime $q = (0.5, 0.5)$.

The following definition seems to be applicable for the study of ergodicity of ‘perturbed’ Markov chains. Let $N > 0$ and let $\alpha(t)$ be a non-negative locally integrable function on $[0, \infty)$ with
\[
\int_0^\infty \alpha(t) \, dt = \infty.
\] (1.9)

**Definition 1.4.** A Markov chain $X(t)$ is $(N, \alpha(t))$-ergodic if for any $p^* \in \Omega$, $p^{**} \in \Omega$ and all $s, t$ $(0 \leq s \leq t)$ the following inequality is satisfied
\[
\|(p^*-p^{**})P(s, t)\| \leq N\exp\left(-\int_s^t \alpha(u) \, du\right)\|p^*-p^{**}\|.
\] (1.10)

If a Markov chain is homogeneous and $(N, \alpha(t))$-ergodic then it is exponentially strongly ergodic. In the general case $(N, \alpha(t))$-ergodicity of a Markov chain implies its weak ergodicity. One can note that the chain $\bar{X}(t)$ in Example 1.1 is $(N, \alpha(t))$-ergodic with $N = 1$, $\alpha(t) = \bar{\lambda}(t) + \bar{\mu}(t) = 2/(1+t)$. 
2. A stability theorem

We consider the following condition for a 'perturbation' of a Markov chain $\bar{X}(t)$ when $\bar{X}(t)$ is $(N, \alpha(t))$-ergodic:

for almost all $t \geq 0$, \[ a(t) \leq \omega(t) \alpha(t) \] \hspace{1cm} (2.1)

where $\omega(t) \rightarrow 0$ as $t \rightarrow \infty$.

**Theorem 2.1.** Let a Markov chain $X(t)$ be $(N, \alpha(t))$-ergodic and the perturbed chain $X(t)$ be such that condition (2.1) is fulfilled. Then for all $s, t$ ($0 \leq s \leq t$) the following inequality holds:

\[ \| P(s, t) - \bar{P}(s, t) \| \leq 12N \int_{s}^{t} \omega(\tau) \alpha(\tau) \exp \left( - \int_{\tau}^{t} \alpha(\xi) \, d\xi \right) \, d\tau. \] \hspace{1cm} (2.2)

**Proof.** Let $x(t) = p'(t), \bar{x}(t) = \bar{p}'(t), B(t) = A'(t), \bar{B}(t) = \bar{A}'(t)$. Then $U(t, s) = P'(s, t), \bar{U}(t, s) = \bar{P}'(s, t)$ are the Cauchy operators of the forward Kolmogorov systems:

\[ \frac{dx}{dt} = B(t)x, \hspace{1cm} \frac{d\bar{x}}{dt} = \bar{B}(t)\bar{x}, \] \hspace{1cm} (2.3)

respectively.

Put $p_{0} = 1 - \sum_{i \geq 1} p_{i}$; then from the forward Kolmogorov systems for the chains $X(t)$ and $\bar{X}(t)$ we have the equations

\[ \frac{dz}{dt} = B_{1}(t)z + f(t) \] \hspace{1cm} (2.4)

and

\[ \frac{d\bar{z}}{dt} = \bar{B}_{1}(t)\bar{z} + \bar{f}(t), \] \hspace{1cm} (2.5)

respectively. Here $z = (p_{1}, p_{2}, \ldots)', f = (a_{01}, a_{02}, \ldots)'$, $B_{i} = (b_{ij}), i, j \geq 1$, $b_{ij} = a_{ij} - a_{0i}$, and $\bar{z}, \bar{f}, \bar{B}_{1}$ are the corresponding functions for the chain $\bar{X}(t)$.

The equation (2.4) can be rewritten in the following form:

\[ \frac{d\bar{z}}{dt} = \bar{B}_{1}(t)\bar{z} + \bar{f}(t) + (B_{1}(t) - \bar{B}_{1}(t))z + (f(t) - \bar{f}(t)) \]

\[ = \bar{B}_{1}(t)z + \bar{f}(t) + \bar{B}_{1}(t)z + \bar{f}(t), \] \hspace{1cm} (2.6)

where $\bar{B}_{1}(t) = B_{1}(t) - \bar{B}_{1}(t)$, $\bar{f}(t) = f(t) - \bar{f}(t)$.

Let $V(t, s)$ be a Cauchy operator for the differential equation

\[ \frac{dy}{dt} = \bar{B}_{1}(t)y. \] \hspace{1cm} (2.7)

Then known formulae (see, for example, [1]) with $\bar{z}(s) = z(s)$ imply

\[ \bar{z}(t) = V(t, s)\bar{z}(s) + \int_{s}^{t} V(t, \tau)\bar{f}(\tau) \, d\tau, \] \hspace{1cm} (2.8)
\[ z(t) = V(t, s)z(s) + \int_{s}^{t} V(t, \tau)\dot{z}(\tau) \, d\tau + \int_{s}^{t} V(t, \tau)(\dot{B}_1(\tau)z(\tau) + \ddot{f}(\tau)) \, d\tau \]

\[ = \ddot{z}(t) + \int_{s}^{t} V(t, \tau)(\dot{B}_1(\tau)z(\tau) + \ddot{f}(\tau)) \, d\tau. \]  

Therefore

\[ \|z(t) - \ddot{z}(t)\| \leq \int_{s}^{t} \|V(t, \tau)\|(\|\dot{B}_1(\tau)\|\|z(\tau)\| + \|\ddot{f}(\tau)\|) \, d\tau. \]  

We have

\[ \|\dot{B}_1(t)\| \leq 2\alpha(t) \leq 2\omega(t)\alpha(t), \]  

\[ \|\ddot{f}(t)\| \leq \alpha(t) \leq \omega(t)\alpha(t), \]  

\[ \|z(t)\| \leq \|x(t)\| = \|p(t)\| = 1. \]  

Now one can estimate \(\|V(t, \tau)\|\). Notice that

\[ x = \left(1 - \|z\| \right) z. \]  

It implies the inequality

\[ \|z^* - z^{**}\| \leq \|p^* - p^{**}\| \leq 2\|z^* - z^{**}\|. \]  

Then one has from (2.13) and (1.10) the following estimate: For any \(s, t \ (0 \leq s \leq t)\) and any \(z^*(s) = z^*, z^{**}(s) = z^{**}\),

\[ \|V(t, s)(z^* - z^{**})\| = \|z^*(t) - z^{**}(t)\| \leq \|p^*(t) - p^{**}(t)\| \leq \|p^* - p^{**}\| \leq 2N \exp\left(-\int_{s}^{t} \alpha(\tau) \, d\tau\right) \|z^* - z^{**}\|. \]  

Hence for all \(t, s \ (0 \leq s \leq t)\),

\[ \|V(t, s)\| \leq 2N \exp\left(-\int_{s}^{t} \alpha(\tau) \, d\tau\right). \]  

Now using (2.15) and (2.11) we have from the estimate (2.10) the following inequality:
\[ \|z(t) - \bar{z}(t)\| \leq 6N \int_{s}^{t} \omega(\tau)\alpha(\tau)\exp\left(-\int_{s}^{\tau} \alpha(\xi)\,d\xi\right)\,d\tau. \]  
\tag{2.16}

Inequalities (2.13) and (2.16) imply (2.2). \( \square \)

**Remark 2.1.** One can prove that estimate (2.2) has the exact order.

3. **Ergodic properties**

**Theorem 3.1.** Let \( \tilde{X}(t) \) be \((N, \alpha(t))\)-ergodic with limit regime \( Q(t) \) and let a perturbed chain \( X(t) \) is such that (2.1) is fulfilled. Then \( X(t) \) is weakly ergodic and

\[ \|P(s, t) - Q(t)\| \leq 2N \exp\left(-\int_{s}^{t} \alpha(\tau)\,d\tau\right) \times \left\{ 1 + 6 \int_{s}^{t} \omega(\tau)\alpha(\tau)\exp\left(\int_{s}^{\tau} \alpha(\xi)\,d\xi\right)\,d\tau \right\}. \]  
\tag{3.1}

**Proof.** Using inequality (2.15) and Theorem 2.1 one has the following estimate:

\[ \|P(s, t) - Q(t)\| \leq \|P(s, t) - \tilde{P}(s, t)\| + \|\tilde{P}(s, t) - Q(t)\| \leq 12N \int_{s}^{t} \omega(\tau)\alpha(\tau)\exp\left(-\int_{s}^{\tau} \alpha(\xi)\,d\xi\right)\,d\tau + 2N \exp\left(-\int_{s}^{t} \alpha(\tau)\,d\tau\right) \] 
\[ \times \left\{ 1 + 6 \int_{s}^{t} \omega(\tau)\alpha(\tau)\exp\left(\int_{s}^{\tau} \alpha(\xi)\,d\xi\right)\,d\tau \right\}. \]  
\tag{3.2}

**Remark 3.1.** The conditions of weak ergodicity of nonhomogeneous continuous-time Markov chains were obtained in [5,10]. Particularly, if the conditions of Theorem 1 in [10] are fulfilled then the respective chain is \((N, \alpha(t))\)-ergodic.
Corollary 3.1. If under the assumptions of Theorem 3.1 the Markov chain \( \tilde{X}(t) \) is strongly ergodic with limit regime \( Q \) (for instance there exists a constant vector \( q \in \Omega \) such that \( qA(t) = 0 \) for almost all \( t \geq 0 \)) then \( X(t) \) is also strongly ergodic with the same limit regime. □

Similar statements for the case of homogeneous \( \tilde{X}(t) \) were considered in [61].

Corollary 3.2. Let \( \tilde{X}(t) \) be a strongly ergodic homogeneous chain (\( \tilde{A}(t) = \tilde{A} \)) with limit regime \( Q \), and let the intensity matrix \( A(t) \) of the chain \( X(t) \) be asymptotically constant (\( a(t) \leq \alpha \omega(t) \)). Then \( X(t) \) is strongly ergodic and

\[
\|P(s, t) - Q\| \leq 2N e^{-\alpha(t-s)} \left\{ 1 + 6\alpha \int_s^t \omega(\tau) e^{\alpha(\tau-s)} \, d\tau \right\}.
\]  

One can obtain a more explicit estimate of the rate of convergence using the following statement.

If \( a(t) \) is a non-negative locally integrable function on \([0, \infty)\) and condition (1.9) is satisfied, then there exists a non-decreasing function \( \varphi(t) \) such that \( \varphi(t) \to \infty \) as \( t \to \infty \) and

\[
\int_{\varphi(t)}^t a(u) \, du \to \infty \quad \text{as} \quad t \to \infty.
\]  

For example one can suppose \( \varphi(t) = n \) for \( t \in (t_n, t_{n+1}) \) where \( \int_{t_n}^{t_{n+1}} a(u) \, du = n \).

Let \( \omega(t) \) be a non-negative locally integrable function and \( \omega(t) \to 0 \) as \( t \to \infty \). Then the function

\[
\psi(t) = \text{ess sup}_{\tau \in [n, n+1]} |\omega(\tau)| \text{ if } t \in [n, n+1), \, n \in \mathbb{N},
\]

decreases to zero as \( t \to \infty \) and \( |\omega(t)| \leq \psi(t) \) for any \( t \geq 0 \).

Theorem 3.2. Let \( a(t), \omega(t) \) be non-negative and such that condition (1.9) is fulfilled, with \( \omega(t) \to 0 \) as \( t \to \infty \). Then the following estimate is true: for any \( s \geq 0, \, \varphi(t) \geq s \),

\[
\int_s^t \omega(\tau) a(\tau) \exp \left\{ -\int_s^t a(\xi) \, d\xi \right\} \, d\tau \leq \psi(\varphi(t)) + \psi(s) \exp \left\{ -\int_{\varphi(t)}^t a(\tau) \, d\tau \right\}
\]  

and therefore the left-hand side of inequality (3.6) tends to zero as \( t \to \infty \).

Proof. One has for any \( s \geq 0, \) and \( \varphi(t) \geq s \).
\[
\int_s^t \omega(\tau) \alpha(\tau) \exp \left( - \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau \\
\leq \exp \left( - \int_0^t \alpha(\xi) \, d\xi \right) \int_s^t \psi(\tau) \alpha(\tau) \exp \left( \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau \\
\leq \exp \left( - \int_0^t \alpha(\xi) \, d\xi \right) \left\{ \psi(s) \int_s^t \alpha(\tau) \exp \left( \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau \right\} \\
+ \psi(\varphi(t)) \int_{\varphi(t)}^t \alpha(\tau) \exp \left( \int_0^\tau \alpha(\xi) \, d\xi \right) \, d\tau \right\} \\
\leq \exp \left( - \int_0^t \alpha(\xi) \, d\xi \right) \left\{ \psi(s) \exp \left( \int_0^t \alpha(\tau) \, d\tau \right) + \psi(\varphi(t)) \int_0^t \alpha(\tau) \, d\tau \right\} \\
= \psi(\varphi(t)) + \psi(s) \exp \left\{ - \int_{\varphi(t)}^t \alpha(\tau) \, d\tau \right\}. \quad \square (3.7)
\]

Now we consider the condition
\[
\int_0^\infty a(t) \, dt < \infty \quad (3.8)
\]
instead of (2.1). In this case the condition of \((N, \alpha(t))\)-ergodicity is unnecessary.

**Theorem 3.3.** Let condition (3.8) be fulfilled and let the Markov chain \(\tilde{X}(t)\) be weakly ergodic. Then the Markov chain \(X(t)\) is also weakly ergodic and their limit regimes are identical.

**Proof.** From the forward Kolmogorov systems (2.3) for \(X(t)\) and \(\tilde{X}(t)\) one has
\[
d\tilde{x}/dt = B(t)\tilde{x} + (\tilde{B}(t) - B(t))\tilde{x}, \quad (3.9)
\]
and if \(\tilde{x}(s) = x(s)\) then
\[
\tilde{x}(t) = U(t, s)x(s) + \int_s^t U(t, \tau)(\tilde{B}(\tau) - B(\tau))\tilde{x}(\tau) \, d\tau \\
= x(t) + \int_s^t U(t, \tau)(\tilde{B}(\tau) - B(\tau))\tilde{x}(\tau) \, d\tau. \quad (3.10)
\]
Next
\[
\| \tilde{x}(t) - x(t) \| \leq \int_s^t \| U(t, \tau) \| \| \tilde{B}(\tau) - B(\tau) \| \| \tilde{x}(\tau) \| \, d\tau
\]
\[
\leq \int_s^t \| B(\tau) - B(\tau) \| \, d\tau = \int_s^t a(\tau) \, d\tau ,
\]  
(3.11)
and hence
\[
\| P(s, t) - \bar{P}(s, t) \| \leq \int_s^t a(\tau) \, d\tau .
\]  
(3.12)

Now let \( \tilde{x}(t) \) be weakly ergodic with a limit regime \( Q(t) \). Let \( \varepsilon > 0 \) be fixed. There exists \( s \) such that \( \int_s^\infty a(\tau) \, d\tau < \frac{1}{2}\varepsilon \). One can choose \( t_0 \) such that \( \| P(s, t) - Q(t) \| < \frac{1}{2}\varepsilon \) for \( t \geq t_0 \). Then using (3.12) one has
\[
\| P(s, t) - Q(t) \| \leq \| P(s, t) - \bar{P}(s, t) \| + \| \bar{P}(s, t) - Q(t) \|
\]
\[
\leq \int_s^t a(\tau) \, d\tau + \frac{1}{2}\varepsilon < \varepsilon .
\]  
(3.13)

Then
\[
\| P(0, t) - Q(t) \| \leq \| P(0, s) \| \| P(s, t) - Q(t) \|
\]
\[
\leq \| P(s, t) - Q(t) \| < \varepsilon .
\]  
(3.14)
Inequality (3.14) implies weak ergodicity of \( X(t) \) and the equality of the limit regimes because \( \varepsilon > 0 \) is arbitrary. \( \square \)

**Corollary 3.3.** If under the assumptions of Theorem 3.3 the Markov chain \( \tilde{x}(t) \) is strongly ergodic with limit regime \( Q \) then \( X(t) \) is also strongly ergodic with the same limit regime. \( \square \)

### 4. Queue-length process for M(t)/M(t)/N/0 queue

Let \( X(t) \) be a queue-length process for a M(t)/M(t)/N/0 queue (there are \( N \) servers and no waiting rooms, see, for example, [8]). It is a birth and death process with the state space \( S = \{0, 1, \ldots, N\} \) and intensities \( \lambda_{n-1}(t) = \lambda(t) \), \( \mu_n(t) = n\mu(t) \), \( n = 1, \ldots, N \).

There have been a number of investigations of this process in some particular cases, see [2,4,7].

Let the intensities be asymptotically proportional:
\[
\lambda(t) / \mu(t) \to \phi \quad \text{as } t \to \infty .
\]  
(4.1)
Then one can write

$$h(t) = \frac{F(t)}{1 + \omega(t)}$$

(4.2)

where \( \omega(t) \to 0 \) as \( t \to \infty \).

Here \( \omega(t) \) can be supposed to be positive for any \( t \geq 0 \) and to be decreasing. Also assume that

$$\int_0^\infty \mu(t) \, dt = \infty.$$  

(4.3)

It is noteworthy that if condition (4.3) is not fulfilled then the process is not weakly ergodic.

Let \( \tilde{X}(t) \) be a birth and death process with proportional intensities

$$\tilde{\lambda}_{n-1}(t) = \phi \mu(t), \quad \tilde{\mu}_n(t) = n \mu(t), \quad n = 1, \ldots, N.$$  

(4.4)

Then condition (4.3) implies strong ergodicity of \( \tilde{X}(t) \) by Theorem 2 of [10], see also [9]; let \( q \) be the steady-state distribution of \( \tilde{X}(t) \). The process \( X(t) \) with intensities \( \lambda(t) \) and \( \mu(t) \) is a perturbation of \( \tilde{X}(t) \) in the sense of this paper. Hence we can prove ergodicity of \( X(t) \) with limit regime \( q \) and can estimate the rate of convergence. Let \( p(t) \) be the probability distribution for \( X(t) \).

**Theorem 4.1.** Let the intensities of \( X(t) \) be asymptotically proportional and conditions (4.2), (4.3) be satisfied. Then \( X(t) \) is strongly ergodic with the limit regime \( q \). Moreover for any \( s > 0, t > 0 \) and any \( p(s) = p \) the following estimate holds:

$$\|p(t) - q\| \leq 8N \exp\left(-\int_s^t \mu(\tau) \, d\tau\right) \times \left(1 + 6\phi \int_s^t \omega(\tau) \mu(\tau) \exp\left(\int_s^\tau \mu(\xi) \, d\xi\right) \, d\tau\right).$$

(4.5)

**Proof.** Let \( X'(t) \) be the homogeneous birth and death process with state space \( S \), probability distribution \( p'(t) \) and intensities

$$\lambda'_{n-1} = \phi, \quad \mu'_n = n, \quad n = 1, \ldots, N.$$  

(4.6)

Such a process was considered in [11], Example 2, estimate (14). One has

$$\|p'(t) - q\| \leq 8N \exp(-t).$$

(4.7)

Then by Theorem 2 of [10],

$$\|\tilde{p}(t) - q\| \leq 8N \exp\left(\int_s^t \mu(\tau) \, d\tau\right).$$

(4.8)
and $\bar{X}(t)$ is $(\mathcal{N}, \alpha(t))$-ergodic with $\mathcal{N} = 4N$, $\alpha(t) = \mu(t)$. Instead of (2.1) one has $\alpha(t) = \phi \mu(t) \omega(t)$ and our claim follows from Theorem 3.1. □

**Remark 4.1.** Ergodicity of this process was proved by D. Gnedenko [4] without any estimates. The results of [3] show that the condition of asymptotic proportionality of intensities is close to a necessary condition for ergodicity.

**References**


