On strong ergodicity for nonhomogeneous continuous-time Markov chains

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Let \( X(t) \) be a nonhomogeneous continuous-time Markov chain. Suppose that the intensity matrices of \( X(t) \) and some weakly or strongly ergodic Markov chain \( \hat{X}(t) \) are close. Some sufficient conditions for weak and strong ergodicity of \( X(t) \) are given and estimates of the rate of convergence are proved. Queue-length for a birth and death process in the case of asymptotically proportional intensities is considered as an example.

nonhomogeneous \* continuous-time Markov chain \* strong ergodicity \* weak ergodicity \* forward Kolmogorov system \* differential equation in the space \( l_i \)

1. Introduction and definitions

Ergodic properties of nonhomogeneous continuous-time Markov chains have been investigated by a number of authors. One of the problems in this area is the following. Let the intensity matrix of a Markov chain \( X(t) \) be close to the intensity matrix of an ergodic Markov chain \( \hat{X}(t) \) (particularly, the perturbation tends to zero as \( t \to \infty \) or the perturbation is integrable). Is \( X(t) \) also ergodic with the same limit regime?

The case where \( \hat{X}(t) \) is homogeneous was considered in [6], and the case of an exponential ergodic 'non-perturbed' chain was studied in [12]. Now we consider the more general case and give some estimates of the rate of convergence.

Let \( X(t) \) and \( \hat{X}(t) \) be continuous-time Markov chains with the state space \( S = \{0, 1, \ldots, K\} \), \( K \leq \infty \). Write \( p_i(t) = \text{Pr}(X(t) = i) \) and \( \mathbf{p}(t) = (p_i(t)) \). Let \( P(s, t) = (p_{ij}(s, t)) \) and \( A(t) = (a_{ij}(t)) \) be transition and intensity matrices for the chain \( X(t) \) respectively.

Put \( \|x\| = \sum |x_i| \), \( \|P\| = \sup \sum |p_{ij}| \), let \( \Omega \) be the set of all stochastic vectors, and ' denote transpose.

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We shall consider the Markov chains which satisfy the following condition: for any \( t \geq 0 \) and \( h > 0 \),

\[
\Pr\{X(t+h) = j/X(t) = i\} = \begin{cases} 
    a_{ij}(t)h + o(h) & \text{if } j \neq i, \\
    1 - h \sum_{k \neq i} a_{ik}(t) + o(h) & \text{if } j = i,
\end{cases}
\]  

(1.1)

where all \( o(h) \) are uniform in \( i \), \( a_{ij}(t) = -\sum_{k \neq i} a_{ik}(t) \). Then for any \( t \geq 0 \) in \( l_1 \)-norm,

\[
x(t+h) - x(t) = B(t)x(t)h + o(h),
\]

(1.2)

where \( x(t) = p'(t), B(t) = A'(t) \). Hence there is the following differential equation in the space \( l_1 \) :

\[
dx/dt = B(t)x.
\]

(1.3)

One can note that this equation is a forward Kolmogorov system for \( X(t) \). If \( U(t, s) \) is a Cauchy operator of equation (1.3) then for any \( s \geq 0, t \geq s \), \( U(t, s) = P'(s, t) \). Notice that

\[
U(t, \tau) = I + \int_\tau^t B(s) ds + \int_\tau^s B(s) \int_\tau^{s_1} B(s_1) ds_1 ds + \cdots
\]

(1.4)

and \( x(t) = U(t, s)x(s) \) for any \( s \geq 0, t \geq s, x(s) \); the properties of a Cauchy operator were investigated in [1].

Denote by \( \tilde{p}_i(t), \tilde{p}(t), \tilde{P}(s, t), \tilde{A}(t) \) the respective functions for the chain \( \tilde{X}(t) \).

In the whole paper we suppose that the intensity matrices are locally integrable on \([0, \infty)\) and the forward Kolmogorov system is identified with a differential equation in the space \( l_1 \), see [1,10]. In this case Cauchy operators for \( X(t) \) and \( \tilde{X}(t) \) exist and are unique.

**Definition 1.1.** A Markov chain \( X(t) \) is called weakly ergodic if there exists a vector function \( q(t) \in \Omega \) for every \( t \geq 0 \) such that if \( Q(t) = (q(t), q(t), \ldots)' \) then for all \( s \geq 0 \),

\[
\lim_{t \to \infty} \|P(s, t) - Q(t)\| = 0.
\]

(1.5)

It is noteworthy that \( X(t) \) can be proved to be weakly ergodic if the above limit is equal to zero for some \( s \geq 0 \).

**Definition 1.2.** A Markov chain \( X(t) \) is called strongly ergodic if there exists a vector \( q \in \Omega \) such that if \( Q = (q, q, \ldots)' \) then for all \( s \geq 0 \),

\[
\lim_{t \to \infty} \|P(s, t) - Q\| = 0.
\]

(1.6)
Definition 1.3. A Markov chain $X(t)$ is called exponentially ergodic if there exist positive numbers $N$ and $\sigma$ such that for any $p^{*} \in \Omega, p^{**} \in \Omega, s, t \ (0 \leq s \leq t)$,
\[
\| (p^{*} - p^{**}) P(s, t) \| \leq N e^{-\sigma(t-s)} \| p^{*} - p^{**} \| .
\]  
(1.7)

Let
\[
a(t) = \| A(t) - \bar{A}(t) \| .
\]  
(1.8)
Then using [6,12] we have the following:

**Proposition.** If $\tilde{X}(t)$ is strong ergodic and homogeneous (or exponentially ergodic) and, in addition, $a(t) \to 0$ as $t \to \infty$, then $X(t)$ is strongly ergodic and the limit regimes for $X(t)$ and $\tilde{X}(t)$ are identical.

Remark 1.1. The proposition is not correct if $\tilde{X}(t)$ is assumed to be only strongly ergodic.

Example 1.1. Let $X(t), \tilde{X}(t)$ be Markov chains with two states 0, 1; and let $\lambda(t), \mu(t), (\bar{\lambda}(t), \bar{\mu}(t))$ be intensities of transitions $0 \to 1$ and $1 \to 0$ respectively, where $\bar{\lambda}(t) = \bar{\mu}(t) = 1/(1+t), \lambda(t) = \mu(t) \equiv 0$. Then $\tilde{X}(t)$ is strongly ergodic with the limit regime $q = (0.5, 0.5)$, $a(t) = 2/(1+t) \to 0$ as $t \to \infty$; and $X(t)$ is not ergodic.

On the other hand the condition of exponential ergodicity of $\tilde{X}(t)$ is not necessary; for instance, if $\bar{\lambda}(t) = \bar{\mu}(t) = 1/(1+t), a(t) = 1/(1+t)^2$, then the chain $X(t)$ in Example 1.1 is strongly ergodic with the limit regime $q = (0.5, 0.5)$.

The following definition seems to be applicable for the study of ergodicity of ‘perturbed’ Markov chains. Let $N > 0$ and let $\alpha(t)$ be a non-negative locally integrable function on $[0, \infty)$ with
\[
\int_{0}^{\infty} \alpha(t) \ dt = \infty .
\]  
(1.9)

Definition 1.4. A Markov chain $X(t)$ is $(N, \alpha(t))$-ergodic if for any $p^{*} \in \Omega, p^{**} \in \Omega$ and all $s, t \ (0 \leq s \leq t)$ the following inequality is satisfied
\[
\| (p^{*} - p^{**}) P(s, t) \| \leq N \exp \left( - \int_{s}^{t} \alpha(u) \ du \right) \| p^{*} - p^{**} \| .
\]  
(1.10)

If a Markov chain is homogeneous and $(N, \alpha(t))$-ergodic then it is exponentially strongly ergodic. In the general case $(N, \alpha(t))$-ergodicity of a Markov chain implies its weak ergodicity. One can note that the chain $\tilde{X}(t)$ in Example 1.1 is $(N, \alpha(t))$-ergodic with $N = 1, \alpha(t) = \bar{\lambda}(t) + \bar{\mu}(t) = 2/(1+t)$.
2. A stability theorem

We consider the following condition for a 'perturbation' of a Markov chain $\bar{X}(t)$ when $\bar{X}(t)$ is $(N, \alpha(t))$-ergodic:

for almost all $t > 0$, \[ a(t) \leq \omega(t) \alpha(t), \] (2.1)

where $\omega(t) \to 0$ as $t \to \infty$.

Theorem 2.1. Let a Markov chain $X(t)$ be $(N, \alpha(t))$-ergodic and the perturbed chain $X(t)$ be such that condition (2.1) is fulfilled. Then for all $s, t \ (0 < s < t)$ the following inequality holds:

\[ \| P(s, t) - \bar{P}(s, t) \| \leq 12N \int_x^t \omega(\tau) \alpha(\tau) \exp \left( - \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau. \] (2.2)

Proof. Let $x(t) = p'(t), \bar{x}(t) = \bar{p}'(t), B(t) = A'(t), \bar{B}(t) = \bar{A}'(t)$. Then $U(t, s) = P'(s, t), \bar{U}(t, s) = \bar{P}'(s, t)$ are the Cauchy operators of the forward Kolmogorov systems:

\[ \frac{dx}{dt} = B(t)x, \quad \frac{\bar{dx}}{dt} = \bar{B}(t)\bar{x}, \] (2.3)

respectively.

Put $p_0 = 1 - \sum_{i \geq 1} p_i$ then from the forward Kolmogorov systems for the chains $X(t)$ and $\bar{X}(t)$ we have the equations

\[ \frac{dz}{dt} = B_1(t)z + f(t) \] (2.4)

and

\[ \frac{d\bar{z}}{dt} = \bar{B}_1(t)\bar{z} + \bar{f}(t), \] (2.5)

respectively. Here $z = (p_1, p_2, \ldots)'$, $f = (a_{01}, a_{02}, \ldots)'$, $B_1 = (b_{ij})$, $i, j \geq 1$, $b_{ij} - a_{ij} = a_{ji} - a_{0i}$, and $\bar{z}, \bar{f}, \bar{B}_1$ are the corresponding functions for the chain $\bar{X}(t)$.

The equation (2.4) can be rewritten in the following form:

\[ \frac{dz}{dt} = \bar{B}_1(t)z + \bar{f}(t) + (B_1(t) - \bar{B}_1(t))z + (f(t) - \bar{f}(t)) = \bar{B}_1(t)z + \bar{f}(t) \] (2.6)

where $\bar{B}_1(t) = B_1(t) - \bar{B}_1(t), \bar{f}(t) = f(t) - \bar{f}(t)$.

Let $V(t, s)$ be a Cauchy operator for the differential equation

\[ \frac{dy}{dt} = \bar{B}_1(t)y. \] (2.7)

Then known formulae (see, for example, [1]) with $\bar{z}(s) = z(s)$ imply

\[ \bar{z}(t) = V(t, s)\bar{z}(s) + \int_s^t V(t, \tau)\bar{f}(\tau) \, d\tau, \] (2.8)
\[
z(t) = V(t, s)\tilde{z}(s) + \int_s^t V(t, \tau)\tilde{f}(\tau)\,d\tau + \int_s^t V(t, \tau) (\tilde{B}_1(\tau)z(\tau) + \tilde{f}(\tau))\,d\tau
\]

\[
= \tilde{z}(t) + \int_s^t V(t, \tau) (\tilde{B}_1(\tau)z(\tau) + \tilde{f}(\tau))\,d\tau.
\]

Therefore

\[
\|z(t) - \tilde{z}(t)\| \leq \int_s^t \|V(t, \tau)\| (\|\tilde{B}_1(\tau)\| \|z(\tau)\| + \|\tilde{f}(\tau)\|)\,d\tau.
\]

We have

\[
\|\tilde{B}_1(t)\| \leq 2a(t) \leq 2\omega(t)\alpha(t),
\]

\[
\|\tilde{f}(t)\| \leq a(t) \leq \omega(t)\alpha(t),
\]

\[
\|z(t)\| \leq \|x(t)\| = \|p(t)\| = 1.
\]

Now one can estimate \(\|V(t, \tau)\|\). Notice that

\[
x = \left(1 - \frac{\|z\|}{z}\right).
\]

It implies the inequality

\[
\|z^* - z^{**}\| \leq \|p^* - p^{**}\| \leq 2\|z^* - z^{**}\|.
\]

Then one has from (2.13) and (1.10) the following estimate: For any \(s, t \ (0 \leq s \leq t)\) and any \(z^*(s) = z^*, \ z^{**}(s) = z^{**}\),

\[
\|V(t, s)(z^* - z^{**})\| = \|z^*(t) - z^{**}(t)\| \leq \|p^*(t) - p^{**}(t)\|
\]

\[
= \|(p^* - p^{**})P(s, t)\| \leq N \exp\left(-\int_s^t \alpha(\tau)\,d\tau\right) \|p^* - p^{**}\|
\]

\[
\leq 2N \exp\left(-\int_s^t \alpha(\tau)\,d\tau\right) \|z^* - z^{**}\|.
\]

Hence for all \(t, s \ (0 \leq s \leq t)\),

\[
\|V(t, s)\| \leq 2N \exp\left(-\int_s^t \alpha(\tau)\,d\tau\right).
\]

Now using (2.15) and (2.11) we have from the estimate (2.10) the following inequality:
\[ \| \mathbf{z}(t) - \mathbf{z}(t) \| \leq 6N \int_\tau^t \omega(\tau) \alpha(\tau) \exp \left( - \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau. \]  

(2.16)

Inequalities (2.13) and (2.16) imply (2.2). \( \square \)

**Remark 2.1.** One can prove that estimate (2.2) has the exact order.

### 3. Ergodic properties

**Theorem 3.1.** Let \( \tilde{X}(t) \) be \((N, \alpha(t))\)-ergodic with limit regime \( Q(t) \) and let a perturbed chain \( X(t) \) is such that (2.1) is fulfilled. Then \( X(t) \) is weakly ergodic and

\[
\| P(s, t) - Q(t) \| \leq 2N \exp \left( - \int_\tau^t \alpha(\tau) \, d\tau \right)
\times \left\{ 1 + 6 \int_\tau^t \omega(\tau) \alpha(\tau) \exp \left( \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau \right\}.
\]  

(3.1)

**Proof.** Using inequality (2.15) and Theorem 2.1 one has the following estimate:

\[
\| P(s, t) - Q(t) \|
\leq \| P(s, t) - \bar{P}(s, t) \| + \| \bar{P}(s, t) - Q(t) \|
\leq 12N \int_\tau^t \omega(\tau) \alpha(\tau) \exp \left( - \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau + 2N \exp \left( - \int_\tau^t \alpha(\tau) \, d\tau \right)
\times \left\{ 1 + 6 \int_\tau^t \omega(\tau) \alpha(\tau) \exp \left( \int_\tau^t \alpha(\xi) \, d\xi \right) \, d\tau \right\}. \quad \square
\]  

(3.2)

**Remark 3.1.** The conditions of weak ergodicity of nonhomogeneous continuous-time Markov chains were obtained in [5, 10]. Particularly, if the conditions of Theorem 1 in [10] are fulfilled then the respective chain is \((N, \alpha(t))\)-ergodic.
Corollary 3.1. If under the assumptions of Theorem 3.1 the Markov chain \( \tilde{X}(t) \) is strongly ergodic with limit regime \( Q \) (for instance there exists a constant vector \( q \in \Omega \) such that \( qA(t) = 0 \) for almost all \( t \geq 0 \)) then \( X(t) \) is also strongly ergodic with the same limit regime. \( \Box \)

Similar statements for the case of homogeneous \( \tilde{X}(t) \) were considered in [61].

Corollary 3.2. Let \( \tilde{X}(t) \) be a strongly ergodic homogeneous chain \((\tilde{A}(t) = \tilde{A})\) with limit regime \( Q \), and let the intensity matrix \( A(t) \) of the chain \( X(t) \) be asymptotically constant \((a(t) \leq \alpha \omega(t)) \). Then \( X(t) \) is strongly ergodic and

\[
\| P(s, t) - Q \| \leq 2Ne^{-\alpha(t-s)} \left\{ 1 + 6\alpha \int_s^t \omega(\tau)e^{\alpha(\tau-s)} d\tau \right\}. \tag{3.3}
\]

One can obtain a more explicit estimate of the rate of convergence using the following statement.

If \( \alpha(t) \) is a non-negative locally integrable function on \( [0, \infty) \) and condition (1.9) is satisfied, then there exists a non-decreasing function \( \varphi(t) \) such that \( \varphi(t) \to \infty \) as \( t \to \infty \) and

\[
\int_{\varphi(t)}^t \alpha(u) du \to \infty \quad \text{as} \quad t \to \infty. \tag{3.4}
\]

For example one can suppose \( \varphi(t) = n \) for \( t \in (t_n, t_{n+1}) \) where \( \int_{t_n}^{t_n+1} \alpha(u) du = n \).

Let \( \omega(t) \) be a non-negative locally integrable function and \( \omega(t) \to 0 \) as \( t \to \infty \). Then the function

\[
\psi(t) = \text{ess sup}_{\tau \in [n, n+1]} |\omega(\tau)| \quad \text{if} \quad t \in [n, n+1), \quad n \in \mathbb{N}, \tag{3.5}
\]
decreases to zero as \( t \to \infty \) and \( |\omega(t)| \leq \psi(t) \) for any \( t \geq 0 \).

Theorem 3.2. Let \( \alpha(t), \omega(t) \) be non-negative and such that condition (1.9) is fulfilled, with \( \omega(t) \to 0 \) as \( t \to \infty \). Then the following estimate is true: for any \( s \geq 0, \varphi(t) \geq s, \)

\[
\int_s^t \omega(\tau) \alpha(\tau) \exp \left( - \int_\tau^t \alpha(\xi) d\xi \right) d\tau 
\leq \psi(\varphi(t)) + \psi(s) \exp \left\{ - \int_{\varphi(t)}^t \alpha(\tau) d\tau \right\} \tag{3.6}
\]

and therefore the left-hand side of inequality (3.6) tends to zero as \( t \to \infty \).

Proof. One has for any \( s \geq 0, \) and \( \varphi(t) \geq s, \)
Now we consider the condition
\[ a(t) \, dt < \infty \] instead of (2.1). In this case the condition of \((N, \alpha(t) )\)-ergodicity is unnecessary.

**Theorem 3.3.** Let condition (3.8) be fulfilled and let the Markov chain \(\tilde{X}(t)\) be weakly ergodic. Then the Markov chain \(X(t)\) is also weakly ergodic and their limit regimes are identical.

**Proof.** From the forward Kolmogorov systems (2.3) for \(X(t)\) and \(\tilde{X}(t)\) one has
\[ d\tilde{x}/dt = B(t)\tilde{x} + (\tilde{B}(t) - B(t))\tilde{x}, \] and if \(\tilde{x}(s) = x(s)\) then
\[
\tilde{x}(t) = U(t, s)x(s) + \int_s^t U(t, \tau)(\tilde{B}(\tau) - B(\tau))\tilde{x}(\tau) \, d\tau \\
= x(t) + \int_s^t U(t, \tau)(\tilde{B}(\tau) - B(\tau))\tilde{x}(\tau) \, d\tau. \]
For a sequence \( (x(t)) \) in a Banach space \( X \), and a measurable function \( f : [0, T] \to X \), we have:
\[
\| x(t) - x(s) \| \leq \int_s^t \| f(\tau) \| \, d\tau.
\]

And hence
\[
\| P(s, t) - Q(t) \| \leq \int_s^t \| f(\tau) \| \, d\tau.
\]

Now let \( X(t) \) be weakly ergodic with a limit regime \( Q(t) \). Let \( s > 0 \) be fixed. There exists \( s \) such that \( \int_0^1 a(\tau) \, d\tau < \frac{1}{2} \varepsilon \). One can choose \( t_0 \) such that \( \| P(s, t) - Q(t) \| < \frac{1}{2} \varepsilon \) for \( t > t_0 \). Then using (3.12) one has
\[
\| P(s, t) - Q(t) \| \leq \| P(s, t) - \bar{P}(s, t) \| + \| \bar{P}(s, t) - Q(t) \|
\leq \int_s^t \| f(\tau) \| \, d\tau + \frac{1}{2} \varepsilon < \varepsilon.
\]

Then
\[
\| P(0, t) - Q(t) \| \leq \| P(0, s) \| \| P(s, t) - Q(t) \|
\leq \| P(s, t) - Q(t) \| < \varepsilon.
\]

Inequality (3.14) implies weak ergodicity of \( X(t) \) and the equality of the limit regimes because \( \varepsilon > 0 \) is arbitrary.

**Corollary 3.3.** If under the assumptions of Theorem 3.3 the Markov chain \( \bar{X}(t) \) is strongly ergodic with limit regime \( Q \), then \( X(t) \) is also strongly ergodic with the same limit regime. \( \square \)

### 4. Queue-length process for \( M(t)/M(t)/N/0 \) queue

Let \( X(t) \) be a queue-length process for an \( M(t)/M(t)/N/0 \) queue (there are \( N \) servers and no waiting rooms, see, for example, [8]). It is a birth and death process with the state space \( S = \{ 0, 1, \ldots, N \} \) and intensities \( \lambda_{n-1}(t) = \lambda(t) \), \( \mu_n(t) = n \mu(t) \), \( n = 1, \ldots, N \).

There have been a number of investigations of this process in some particular cases, see [2, 4, 7].

Let the intensities be asymptotically proportional:
\[
\lambda(t) / \mu(t) \to \phi \quad \text{as } t \to \infty.
\]
Then one can write
\[ h(t) = F(t) \left[ 1 + \frac{w(t)}{1 + w(t)} \right] \]
where \( w(t) \to 0 \) as \( t \to \infty \).

(4.2)

Here \( \omega(t) \) can be supposed to be positive for any \( t \geq 0 \) and to be decreasing. Also assume that
\[ \int_0^\infty \mu(t) \, dt = \infty. \]  

(4.3)

It is noteworthy that if condition (4.3) is not fulfilled then the process is not weakly ergodic.

Let \( X(t) \) be a birth and death process with proportional intensities
\[ \lambda_n(t) = \phi \mu(t), \quad \mu_n(t) = n \mu(t), \quad n = 1, \ldots, N. \]

(4.4)

Then condition (4.3) implies strong ergodicity of \( X(t) \) by Theorem 2 of [10], see also [9]; let \( q \) be the steady-state distribution of \( X(t) \). The process \( X(t) \) with intensities \( \lambda(t) \) and \( \mu(t) \) is a perturbation of \( X(t) \) in the sense of this paper. Hence we can prove ergodicity of \( X(t) \) with limit regime \( q \) and can estimate the rate of convergence. Let \( p(t) \) be the probability distribution for \( X(t) \).

**Theorem 4.1.** Let the intensities of \( X(t) \) be asymptotically proportional and conditions (4.2), (4.3) be satisfied. Then \( X(t) \) is strongly ergodic with the limit regime \( q \). Moreover for any \( s > 0, t > 0 \) and any \( p(s) = p \) the following estimate holds:
\[ \| p(t) - q \| \leq 8N \exp \left( -\int_t^\infty \mu(\tau) \, d\tau \right) \]
\[ \times \left( 1 + 6\phi \int_t^\infty \omega(\tau) \mu(\tau) \exp \left( \int_\infty^\tau \mu(\xi) \, d\xi \right) \, d\tau \right). \]  

(4.5)

**Proof.** Let \( X'(t) \) be the homogeneous birth and death process with state space \( S \), probability distribution \( p'(t) \) and intensities
\[ \lambda'_{n-1} = \phi, \quad \mu'_n = n, \quad n = 1, \ldots, N. \]

(4.6)

Such a process was considered in [11], Example 2, estimate (14). One has
\[ \| p'(t) - q \| \leq 8N \exp(-t). \]  

(4.7)

Then by Theorem 2 of [10],
\[ \| p(t) - q \| \leq 8N \exp \left( \int_t^\infty \mu(\tau) \, d\tau \right). \]  

(4.8)
and $\tilde{X}(t)$ is $(\mathcal{N}, \alpha(t))$-ergodic with $\mathcal{N} = 4N$, $\alpha(t) = \mu(t)$. Instead of (2.1) one has $\alpha(t) = \phi \mu(t) \omega(t)$ and our claim follows from Theorem 3.1. \qed

**Remark 4.1.** Ergodicity of this process was proved by D. Gnedenko [4] without any estimates. The results of [3] show that the condition of asymptotic proportionality of intensities is close to a necessary condition for ergodicity.

**References**


