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A centerless representation of the Virasoro algebra associated with the unitary circular ensemble

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ABSTRACT

We consider the 2-dimensional Toda lattice tau functions $\tau_n(t, s; \eta, \theta)$ deforming the probabilities $\tau_n(\eta, \theta)$ that a randomly chosen matrix from the unitary group $U(n)$, for the Haar measure, has no eigenvalues within an arc (η, θ) of the unit circle. We show that these tau functions satisfy a centerless Virasoro algebra of constraints, with a boundary part in the sense of Adler, Shiota and van Moerbeke. As an application, we obtain a new derivation of a differential equation due to Tracy and Widom, satisfied by these probabilities, linking it to the Painlevé VI equation.

1. Introduction

Consider the group $U(n)$ of $n \times n$ unitary matrices, with the normalized Haar measure as a probability measure. The Weyl integral formula gives the induced density distribution on the eigenvalues of the matrices on the unit circle in the complex plane, and is given by

$$\frac{1}{n!}|\Delta_n(z)|^2 \prod_{k=1}^{n} \frac{dz_k}{2\pi i z_k}; \quad z_k = e^{i\varphi_k} \quad \text{and} \quad \Delta_n(z) = \prod_{1\le k<l\le n} (z_k - z_l).$$

Thus, for $\eta, \theta \in]-\pi, \pi[$, with $\eta \le \theta$, the probability that a randomly chosen matrix from $U(n)$ has no eigenvalues within an arc of circle $(\eta, \theta) = \{z \in S^1 | \eta < \arg(z) < \theta\}$ is given by

$$\tau_n(\eta, \theta) = \frac{1}{(2\pi)^n n!} \int_\theta^{2\pi+\eta} \cdots \int_\theta^{2\pi+\eta} \prod_{1\le k<l\le n} |e^{i\varphi_k} - e^{i\varphi_l}|^2 d\varphi_1, \ldots, d\varphi_n.$$

Obviously, this probability depends only on the length $\theta - \eta$. All of this is well known and we refer the reader to [1] for details. We shall denote by

$$R(\theta) = -\frac{1}{2}\frac{d}{d\theta} \log \tau_n(-\theta, \theta), \tag{1.1}$$

the logarithmic derivative of the probability that an arc of circle of length 2θ contains no eigenvalues of a randomly chosen unitary matrix.

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The starting motivation for our work was to understand a differential equation satisfied by the function $R(\theta)$

$$R(\theta)^2 + 2\sin\theta\cos\theta R(\theta)R'(\theta) + \sin^2\theta R'(\theta)^2$$
$$= \frac{1}{2}\left(\frac{1}{4}\sin^2\theta\frac{R''(\theta)^2}{R'(\theta)} + \sin\theta\cos\theta R''(\theta) + \left(\cos^2\theta + n^2\sin^2\theta\right)R'(\theta)\right), \tag{1.2}$$

obtained in [2], from the point of view of the Adler–Shiota–van Moerbeke [3] approach, in terms of Virasoro constraints. Introducing the 2-Toda time-dependent tau functions

$$\tau_n(t, s; \eta, \theta) = \frac{1}{n!}\int_{[\theta, 2\pi+\eta]^n} |\Delta_n(z)|^2 \prod_{k=1}^{n}\left(e^{\sum_{j=1}^{\infty}(t_j z_k^j + s_j z_k^{-j})}\frac{dz_k}{2\pi i z_k}\right), \tag{1.3}$$

with $z_k = e^{i\varphi_k}$, deforming the probabilities $\tau_n(\eta, \theta) = \tau_n(0, 0; \eta, \theta)$, we discover that they satisfy a set of Virasoro constraints indexed by *all* integers, decoupling into a boundary-part and a time-part

$$\frac{1}{i}\left(e^{ik\theta}\frac{\partial}{\partial\theta} + e^{ik\eta}\frac{\partial}{\partial\eta}\right)\tau_n(t, s; \eta, \theta) = L_k^{(n)}\tau_n(t, s; \eta, \theta), \quad k \in \mathbb{Z}, i = \sqrt{-1},$$

with the time-dependent operators $L_k^{(n)}$ providing a centerless representation of the *full* Virasoro algebra, see Section 2 (Theorem 2.2) for a precise statement and the proof of the result.

In their study of Painlevé equations satisfied (as functions of x) by integrals of Gessel's type $E_{U(n)}e^{x\,\mathrm{tr}(M+\overline{M})}$, where the expectation $E_{U(n)}$ refers to integration with respect to the Haar measure over the whole of $U(n)$, Adler and van Moerbeke [4] found the sl_2 subalgebra corresponding to $k = -1, 0, 1$, without boundary terms. The appearance of boundary terms and of a *full* centerless Virasoro algebra is to the best of our knowledge new. From this result, it is easy to obtain Eq. (1.2), using the algorithmic method of [3]. Finally, similarly to a result by the first author and Semengue [5] on the Jacobi polynomial ensemble, we show that $R(\theta)$ is the restriction to the unit circle of a function $r(z)$ defined in the complex plane, so that $\sigma(z) = -i(z-1)r(z) - n^2z/4$ satisfies a special case of the Okamoto–Jimbo–Miwa form of the Painlevé VI equation. This will be explained in Section 3 of the paper.

2. A centerless representation of the Virasoro algebra

The proof of the Virasoro constraints satisfied by the integral (1.3) is a non-trivial adaptation of the self-similarity argument exploited in the case of the Gaussian ensembles, based on the invariance of the integrals with respect to translations, see [6] and references therein. Here, we replace translations by appropriate rotations. More precisely, setting

$$dI_n(t, s, z) = |\Delta_n(z)|^2 \prod_{\alpha=1}^{n}\left(e^{\sum_{j=1}^{\infty}(t_j z_\alpha^j + s_j z_\alpha^{-j})}\frac{dz_\alpha}{2\pi i z_\alpha}\right), \tag{2.1}$$

with $z_\alpha = e^{i\varphi_\alpha}$ and $|\Delta_n(z)|^2 = \prod_{1\le\alpha<\beta\le n}|z_\alpha - z_\beta|^2$, we have the fundamental next proposition.

Proposition 2.1. *The following variational formulas hold*

$$\frac{d}{d\varepsilon}dI_n\left(z_\alpha \mapsto z_\alpha e^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = \left(L_k^{(n)} - L_{-k}^{(n)}\right)dI_n, \tag{2.2}$$

$$\frac{d}{d\varepsilon}dI_n\left(z_\alpha \mapsto z_\alpha e^{i\varepsilon(z_\alpha^k + z_\alpha^{-k})}\right)\Big|_{\varepsilon=0} = i\left(L_k^{(n)} + L_{-k}^{(n)}\right)dI_n, \tag{2.3}$$

for all $k \ge 0$, with

$$L_k^{(n)} = \sum_{j=1}^{k-1}\frac{\partial^2}{\partial t_j\partial t_{k-j}} + n\frac{\partial}{\partial t_k} + \sum_{j=1}^{\infty}jt_j\frac{\partial}{\partial t_{j+k}} - \sum_{j=k+1}^{\infty}js_j\frac{\partial}{\partial s_{j-k}} - \sum_{j=1}^{k-1}js_j\frac{\partial}{\partial t_{k-j}} - nks_k, \quad k \ge 1, \tag{2.4}$$

$$L_0^{(n)} = \sum_{j=1}^{\infty}jt_j\frac{\partial}{\partial t_j} - \sum_{j=1}^{\infty}js_j\frac{\partial}{\partial s_j}, \tag{2.5}$$

$$L_{-k}^{(n)} = -\sum_{j=1}^{k-1}\frac{\partial^2}{\partial s_j\partial s_{k-j}} - n\frac{\partial}{\partial s_k} - \sum_{j=1}^{\infty}js_j\frac{\partial}{\partial s_{j+k}} + \sum_{j=k+1}^{\infty}jt_j\frac{\partial}{\partial t_{j-k}} + \sum_{j=1}^{k-1}jt_j\frac{\partial}{\partial s_{k-j}} + nkt_k, \quad k \ge 1. \tag{2.6}$$

Proof. We shall only give the proof of (2.2), the proof of (2.3) is similar. Upon setting

$$E=\prod_{\alpha=1}^{n}\mathrm{e}^{\sum_{j=1}^{\infty}(t_jz_\alpha^j+s_jz_\alpha^{-j})},$$

the following four relations hold, for $k\geq 0$,

$$\begin{aligned}&\left(\frac{\partial}{\partial t_k}+n\delta_{k,0}\right)E=\left(\sum_{\alpha=1}^{n}z_\alpha^k\right)E\\&\left(\frac{\partial}{\partial s_k}+n\delta_{k,0}\right)E=\left(\sum_{\alpha=1}^{n}z_\alpha^{-k}\right)E,\end{aligned}\tag{2.7}$$

$$\begin{aligned}&\left(\frac{1}{2}\sum_{\substack{i+j=k\\i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}-\frac{n}{2}\delta_{k,0}\right)E=\left(\sum_{\substack{1\leq\alpha<\beta\leq n\\i+j=k\\i,j>0}}z_\alpha^iz_\beta^j+\frac{k-1}{2}\sum_{\alpha=1}^{n}z_\alpha^k\right)E\\&\left(\frac{1}{2}\sum_{\substack{i+j=k\\i,j>0}}\frac{\partial^2}{\partial s_i\partial s_j}-\frac{n}{2}\delta_{k,0}\right)E=\left(\sum_{\substack{1\leq\alpha<\beta\leq n\\i+j=k\\i,j>0}}z_\alpha^{-i}z_\beta^{-j}+\frac{k-1}{2}\sum_{\alpha=1}^{n}z_\alpha^{-k}\right)E.\end{aligned}\tag{2.8}$$

We split the computation into four contributions, corresponding to various factors in (2.1).

Contribution 1: For $k>0$, we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\left|\Delta_n\left(z\mathrm{e}^{\varepsilon(z^k-z^{-k})}\right)\right|^2\Big|_{\varepsilon=0}&=|\Delta_n(z)|^2\sum_{1\leq\alpha<\beta\leq n}\frac{(z_\alpha+z_\beta)(z_\alpha^k-z_\beta^k-(z_\alpha^{-k}-z_\beta^{-k}))}{z_\alpha-z_\beta}\\&=|\Delta_n(z)|^2\sum_{1\leq\alpha<\beta\leq n}(z_\alpha+z_\beta)\left(\sum_{i=0}^{k-1}z_\alpha^iz_\beta^{k-1-i}+\sum_{i=0}^{k-1}z_\alpha^{-i-1}z_\beta^{i-k}\right)\\&=|\Delta_n(z)|^2E^{-1}\left[2\sum_{\substack{1\leq\alpha<\beta\leq n\\i+j=k\\i,j>0}}(z_\alpha^iz_\beta^j+z_\alpha^{-i}z_\beta^{-j})+(n-1)\sum_{\alpha=1}^{n}(z_\alpha^k+z_\alpha^{-k})\right]E.\end{aligned}$$

Using the four relations (2.7) and (2.8), we obtain

$$\frac{\partial}{\partial\varepsilon}\left|\Delta_n\left(z\mathrm{e}^{\varepsilon(z^k-z^{-k})}\right)\right|^2\Big|_{\varepsilon=0}=2|\Delta_n(z)|^2E^{-1}\left[\frac{1}{2}\sum_{\substack{i+j=k\\i,j>0}}\frac{\partial^2}{\partial t_i\partial t_j}+\frac{1}{2}\sum_{\substack{i+j=k\\i,j>0}}\frac{\partial^2}{\partial s_i\partial s_j}+\frac{n-k}{2}\frac{\partial}{\partial t_k}+\frac{n-k}{2}\frac{\partial}{\partial s_k}\right]E,\tag{2.9}$$

which is also trivially satisfied for $k=0$.

Contribution 2: For $k\geq 0$, using the relations (2.7), we have

$$\begin{aligned}\frac{\partial}{\partial\varepsilon}\prod_{\alpha=1}^{n}\mathrm{d}\left(z_\alpha\mathrm{e}^{\varepsilon(z_\alpha^k-z_\alpha^{-k})}\right)\Big|_{\varepsilon=0}&=E^{-1}\sum_{\alpha=1}^{n}\left((k+1)z_\alpha^k+(k-1)z_\alpha^{-k}\right)E\prod_{\alpha=1}^{n}\mathrm{d}z_\alpha\\&=E^{-1}\left[(k+1)\frac{\partial}{\partial t_k}+(k-1)\frac{\partial}{\partial s_k}\right]E\prod_{\alpha=1}^{n}\mathrm{d}z_\alpha.\end{aligned}\tag{2.10}$$

Contribution 3: For $k\geq 0$, using the relations (2.7), we have

$$\frac{\partial}{\partial\varepsilon}\prod_{\alpha=1}^{n}\mathrm{e}^{\sum_{j=1}^{\infty}\left(t_j\left(z_\alpha\mathrm{e}^{\varepsilon(z_\alpha^k-z_\alpha^{-k})}\right)^j+s_j\left(z_\alpha\mathrm{e}^{\varepsilon(z_\alpha^k-z_\alpha^{-k})}\right)^{-j}\right)}\Bigg|_{\varepsilon=0}=\sum_{\alpha=1}^{n}\left[\sum_{j=1}^{\infty}jt_jz_\alpha^j(z_\alpha^k-z_\alpha^{-k})-\sum_{j=1}^{\infty}js_jz_\alpha^{-j}(z_\alpha^k-z_\alpha^{-k})\right]E$$

$$=\left[\sum_{j=1}^{\infty}jt_j\sum_{\alpha=1}^{n}z_\alpha^{j+k}-\sum_{j=1}^{k-1}jt_j\sum_{\alpha=1}^{n}z_\alpha^{j-k}-\sum_{j=k}^{\infty}jt_j\sum_{\alpha=1}^{n}z_\alpha^{j-k}-\sum_{j=1}^{k-1}js_j\sum_{\alpha=1}^{n}z_\alpha^{k-j}-\sum_{j=k}^{\infty}js_j\sum_{\alpha=1}^{n}z_\alpha^{k-j}+\sum_{j=1}^{\infty}js_j\sum_{\alpha=1}^{n}z_\alpha^{-k-j}\right]E$$

$$= \left[\sum_{j=1}^{\infty} jt_j \frac{\partial}{\partial t_{k+j}} - \sum_{j=1}^{k-1} jt_j \frac{\partial}{\partial s_{k-j}} - \sum_{j=k+1}^{\infty} jt_j \frac{\partial}{\partial t_{j-k}} - nkt_k - \sum_{j=1}^{k-1} js_j \frac{\partial}{\partial t_{k-j}} - \sum_{j=k+1}^{\infty} js_j \frac{\partial}{\partial s_{j-k}} - nks_k + \sum_{j=1}^{\infty} js_j \frac{\partial}{\partial s_{k+j}}\right] E. \tag{2.11}$$

Contribution 4: For $k \geq 0$, using the relations (2.7), we have

$$\frac{\partial}{\partial \varepsilon} \prod_{\alpha=1}^{n} \frac{1}{2\pi \mathrm{i} z_\alpha \mathrm{e}^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}}\bigg|_{\varepsilon=0} = E^{-1}\left[-\sum_{\alpha=1}^{n} z_\alpha^k + \sum_{\alpha=1}^{n} z_\alpha^{-k}\right] E \prod_{\alpha=1}^{n} \frac{1}{2\pi \mathrm{i} z_\alpha}$$
$$= E^{-1}\left[-\frac{\partial}{\partial t_k} + \frac{\partial}{\partial s_k}\right] E \prod_{\alpha=1}^{n} \frac{1}{2\pi \mathrm{i} z_\alpha}. \tag{2.12}$$

Adding up (2.9)–(2.12) gives (2.2). This concludes the proof of Proposition 2.1. □

We are now able to state our main result.

Theorem 2.2. (i) *The tau functions*[1] $\tau_n(t, s; \eta, \theta)$, $n \geq 1$, *defined in* (1.3), *satisfy*

$$\mathcal{B}_k(\eta, \theta)\tau_n(t, s; \eta, \theta) = L_k^{(n)} \tau_n(t, s; \eta, \theta), \quad k \in \mathbb{Z}, \tag{2.13}$$

with $L_k^{(n)}$, $k \in \mathbb{Z}$, *defined as in* (2.4)–(2.6), *and*

$$\mathcal{B}_k(\eta, \theta) = \frac{1}{\mathrm{i}}\left(\mathrm{e}^{\mathrm{i}k\theta} \frac{\partial}{\partial \theta} + \mathrm{e}^{\mathrm{i}k\eta} \frac{\partial}{\partial \eta}\right); \quad \mathrm{i} = \sqrt{-1}. \tag{2.14}$$

(ii) *The operators* $L_k^{(n)}$, $k \in \mathbb{Z}$, *satisfy the commutation relations of the centerless Virasoro algebra, that is*

$$\left[L_k^{(n)}, L_l^{(n)}\right] = (k - l) L_{k+l}^{(n)}, \quad k, l \in \mathbb{Z}. \tag{2.15}$$

Proof. (i) Denoting $z_\alpha = \mathrm{e}^{\mathrm{i}\varphi_\alpha}$, the change of variable $z_\alpha \mapsto z_\alpha \mathrm{e}^{\varepsilon(z_\alpha^k - z_\alpha^{-k})}$ in the integral (1.3) gives the following transformation on the angle $\varphi_\alpha \mapsto \varphi_\alpha + 2\varepsilon \sin(k\varphi_\alpha)$, inducing a change in the limits of integration given by the inverse map

$$\varphi_\alpha \mapsto \varphi_\alpha - 2\varepsilon \sin(k\varphi_\alpha) + O(\varepsilon^2), \tag{2.16}$$

for ε small enough. Making the change of variable in the integral (1.3), with the corresponding change in the limits of integration, leaves it invariant. Thus, by differentiating the result with respect to ε and evaluating it at $\varepsilon = 0$, using the chain rule together with (2.2) and (2.16), we obtain

$$0 = \left(-2\sin(k\theta)\frac{\partial}{\partial \theta} - 2\sin(k\eta)\frac{\partial}{\partial \eta} + L_k^{(n)} - L_{-k}^{(n)}\right) \tau_n(t, s; \eta, \theta). \tag{2.17}$$

Similarly, the change of variable $z_\alpha \mapsto z_\alpha \mathrm{e}^{\mathrm{i}\varepsilon(z_\alpha^k + z_\alpha^{-k})}$ corresponds to the transformation $\varphi_\alpha \mapsto \varphi_\alpha + 2\varepsilon \cos(k\varphi_\alpha)$, with inverse

$$\varphi_\alpha \mapsto \varphi_\alpha - 2\varepsilon \cos(k\varphi_\alpha) + O(\varepsilon^2),$$

which, using (2.3), leads to

$$0 = \left(-\frac{2}{\mathrm{i}}\cos(k\theta)\frac{\partial}{\partial \theta} - \frac{2}{\mathrm{i}}\cos(k\eta)\frac{\partial}{\partial \eta} + L_k^{(n)} + L_{-k}^{(n)}\right) \tau_n(t, s; \eta, \theta). \tag{2.18}$$

Adding and subtracting (2.17) and (2.18) gives the constraints (2.13), with $\mathcal{B}_k(\eta, \theta)$ defined as in (2.14).
(ii) Consider the complex Lie algebra $\mathcal{A}$ given by the direct sum of two commuting copies of the Heisenberg algebra with bases $\{\hbar_a, a_j | j \in \mathbb{Z}\}$ and $\{\hbar_b, b_j | j \in \mathbb{Z}\}$ and defining commutation relations

$$\begin{aligned} &[\hbar_a, a_j] = 0, \quad [a_j, a_k] = j\delta_{j,-k}\,\hbar_a, \\ &[\hbar_b, b_j] = 0, \quad [b_j, b_k] = j\delta_{j,-k}\,\hbar_b, \\ &[\hbar_a, \hbar_b] = 0, \quad [a_j, b_k] = 0, \quad [\hbar_a, b_j] = 0, \quad [\hbar_b, a_j] = 0, \end{aligned} \tag{2.19}$$

[1] See the beginning of Section 3, for a justification of the terminology.

with $j, k \in \mathbb{Z}$. Let $\mathcal{B}$ be the space of formal power series in the variables $t_1, t_2, \ldots$ and $s_1, s_2, \ldots$, and consider the following representation of $\mathcal{A}$ in $\mathcal{B}$:

$$\begin{aligned} &a_j = \frac{\partial}{\partial t_j}, \qquad a_{-j} = jt_j, \qquad b_j = \frac{\partial}{\partial s_j}, \qquad b_{-j} = js_j, \\ &a_0 = b_0 = \mu, \qquad \hbar_a = \hbar_b = 1, \end{aligned} \tag{2.20}$$

for $j > 0$, and $\mu \in \mathbb{C}$. Define the operators

$$A_k^{(n)} = \frac{1}{2}\sum_{j\in\mathbb{Z}} : a_{-j}a_{j+k} :, \qquad B_k^{(n)} = \frac{1}{2}\sum_{j\in\mathbb{Z}} : b_{-j}b_{j+k} :,$$

where $k \in \mathbb{Z}$, a_j, b_j are as in (2.20) with $\mu = n$, and where the colons indicate normal ordering, defined by

$$: a_j a_k := \begin{cases} a_j a_k & \text{if } j \le k, \\ a_k a_j & \text{if } j > k, \end{cases}$$

and a similar definition for $: b_j b_k :$, obtained by changing the a's in b's in the former. Using these notations, we can rewrite (2.4)–(2.6) as follows

$$\begin{aligned} L_k^{(n)} &= A_k^{(n)} - B_{-k}^{(n)} + \frac{1}{2}\sum_{j=1}^{k-1}(a_j - b_{-j})(a_{k-j} - b_{j-k}), \quad k \ge 1 \\ L_0^{(n)} &= A_0^{(n)} - B_0^{(n)}, \\ L_{-k}^{(n)} &= A_{-k}^{(n)} - B_k^{(n)} - \frac{1}{2}\sum_{j=1}^{k-1}(a_{-j} - b_j)(a_{j-k} - b_{k-j}), \quad k \ge 1. \end{aligned}$$

As shown in [7] (see Lecture 2) the operators $A_k^{(n)}$, $k \in \mathbb{Z}$, provide a representation of the Virasoro algebra in $\mathcal{B}$ with central charge $c = 1$, that is

$$[A_k^{(n)}, A_l^{(n)}] = (k-l)A_{k+l}^{(n)} + \delta_{k,-l}\frac{k^3-k}{12}, \tag{2.21}$$

for $k, l \in \mathbb{Z}$. Similarly, the operators $B_k^{(n)}$ satisfy the commutation relations

$$[B_k^{(n)}, B_l^{(n)}] = (k-l)B_{k+l}^{(n)} + \delta_{k,-l}\frac{k^3-k}{12}, \tag{2.22}$$

for $k, l \in \mathbb{Z}$. Furthermore we have for $k, l \in \mathbb{Z}$

$$\begin{aligned} &[a_k, A_l^{(n)}] = ka_{k+l}, \qquad [b_k, B_l^{(n)}] = kb_{k+l}, \\ &[a_k, B_l^{(n)}] = 0, \qquad [b_k, A_l^{(n)}] = 0. \end{aligned} \tag{2.23}$$

Let us now establish the commutation relations (2.15). We give the proof for $k, l \ge 0$, the other cases being similar. As $[A_i^{(n)}, B_j^{(n)}] = 0$, $i, j \in \mathbb{Z}$, we have using (2.19), (2.21), (2.22) and (2.23)

$$\begin{aligned} [L_k^{(n)}, L_l^{(n)}] = {} & (k-l)\left(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\right) - \frac{1}{2}\sum_{j=1}^{l-1} j(a_{j+k} - b_{-j-k})(a_{l-j} - b_{j-l}) - \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\ & + \frac{1}{2}\sum_{j=1}^{k-1} j(a_{j+l} - b_{-j-l})(a_{k-j} - b_{j-k}) + \frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}). \end{aligned}$$

Relabeling the indices in the sums, we have

$$\begin{aligned} [L_k^{(n)}, L_l^{(n)}] = {} & (k-l)\left(A_{k+l}^{(n)} - B_{-k-l}^{(n)}\right) - \frac{1}{2}\sum_{j=k+1}^{k+l-1}(j-k)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \\ & - \frac{1}{2}\sum_{j=1}^{l-1}(l-j)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) + \frac{1}{2}\sum_{j=l+1}^{k+l-1}(j-l)(a_j - b_{-j})(a_{k+l-j} - b_{j-k-l}) \end{aligned}$$

$$+\frac{1}{2}\sum_{j=1}^{k-1}(k-j)(a_j-b_{-j})(a_{k+l-j}-b_{j-k-l})$$
$$=(k-l)L_{k+l}^{(n)}.$$

This concludes the proof of Theorem 2.2. □

3. The unitary circular ensemble and the Painlevé VI equation

It is well known, see for instance [1], that the integral $\tau_n(t, s; \eta, \theta)$ defined in (1.3) can be represented as a Toeplitz determinant

$$\tau_n(t, s; \eta, \theta) = \det\left(\mu_{k-l}(t, s; \eta, \theta)\right)_{0\le k,l\le n-1}, \tag{3.1}$$

with

$$\mu_k(t, s; \eta, \theta) = \int_\theta^{2\pi+\eta} z^k\left(e^{\sum_{j=1}^{\infty}(t_jz^j+s_jz^{-j})}\frac{dz}{2\pi i z}\right); \quad z = e^{i\varphi}, k \in \mathbb{Z}.$$

A nice consequence of this representation is that $\tau_n(t, s; \eta, \theta)$ is a tau function of a reduction of the 2-Toda lattice hierarchy, that was called the Toeplitz hierarchy in [4]. Therefore, as with any 2-Toda tau function (see [8]), it satisfies the KP equation in the $t = (t_1, t_2, \ldots)$ (or $s = (s_1, s_2, \ldots)$) variables separately

$$\left(\frac{\partial^4}{\partial t_1^4}+3\frac{\partial^2}{\partial t_2^2}-4\frac{\partial^2}{\partial t_1\partial t_3}\right)\log\tau_n+6\left(\frac{\partial^2}{\partial t_1^2}\log\tau_n\right)^2=0. \tag{3.2}$$

As announced in the introduction, in this section, using the method of [3], we establish the following result.

Theorem 3.1. *The Virasoro constraints (2.13), combined with the KP equation (3.2) in the t variables (or the KP equation in the s variables), imply that the function $R(\theta)$ defined in (1.1) satisfies (1.2).*

Proof. Remembering the definition of $L_0^{(n)}$ in (2.5), the Virasoro constraint in (2.13) for $k = 0$, evaluated along the locus $t = s = 0$, gives

$$\left.\frac{\partial\log\tau_n(t, s; \eta, \theta)}{\partial\theta}\right|_{t=s=0} = -\left.\frac{\partial\log\tau_n(t, s; \eta, \theta)}{\partial\eta}\right|_{t=s=0}, \tag{3.3}$$

which is a reformulation of the fact that the gap probability $\tau_n(0, 0; \eta, \theta)$ only depends on the length $\theta-\eta$.

Define the operator $\mathcal{D} = \frac{\partial}{\partial\theta}-\frac{\partial}{\partial\eta}$ and put for a fixed n

$$\begin{aligned} f(t, s; \eta, \theta) &= \log\tau_n(t, s; \eta, \theta),\\ g(\eta, \theta) &= -\frac{1}{2}\mathcal{D}\log\tau_n(t, s; \eta, \theta)\Big|_{t=s=0}. \end{aligned} \tag{3.4}$$

Notice that for $k \ge 0$

$$\mathcal{D}^k\log\tau_n(t, s; \eta, \theta)\Big|_{\substack{t=s=0\\ \eta=-\theta}} = \frac{d^k}{d\theta^k}\log\tau_n(t, s; -\theta, \theta)\Big|_{t=s=0}.$$

Clearly, from the definition of $R(\theta)$ in (1.1), we have

$$R(\theta) = g(-\theta, \theta) = -\frac{1}{2}\frac{d}{d\theta}\log\tau_n(t, s; -\theta, \theta)\Big|_{t=s=0}.$$

Remembering the definition of $L_k^{(n)}$ in (2.4), the constraints in (2.13) for $k = 1, 2$, evaluated at $s = (s_1, s_2, s_3, \ldots) = (0, 0, 0, \ldots)$, can be written

$$\mathcal{B}_1(\eta, \theta)f|_{s=0} = \sum_{j\ge1}jt_j\left.\frac{\partial f}{\partial t_{j+1}}\right|_{s=0}+n\left.\frac{\partial f}{\partial t_1}\right|_{s=0}, \tag{3.5}$$

$$\mathcal{B}_2(\eta, \theta)f|_{s=0} = \sum_{j\ge1}jt_j\left.\frac{\partial f}{\partial t_{j+2}}\right|_{s=0}+\left.\frac{\partial^2 f}{\partial t_1^2}\right|_{s=0}+\left.\left(\frac{\partial f}{\partial t_1}\right)^2\right|_{s=0}+n\left.\frac{\partial f}{\partial t_2}\right|_{s=0}. \tag{3.6}$$

Using (3.3) and the definition of $g(\eta,\theta)$ (3.4), the constraint (3.5) evaluated along the locus $t=s=0$ gives

$$\left.\frac{\partial f}{\partial t_1}\right|_{t=s=0} = \frac{1}{\mathrm{i}n}(\mathrm{e}^{\mathrm{i}\eta}-\mathrm{e}^{\mathrm{i}\theta})g(\eta,\theta). \tag{3.7}$$

Consequently, along the locus $\eta=-\theta$, we have

$$\left.\frac{\partial f}{\partial t_1}\right|_{\substack{t=s=0\\ \eta=-\theta}} = -\frac{2}{n}\sin(\theta)R(\theta).$$

We then proceed by induction. We call

$$\frac{\partial^n f}{\partial t_{j_1}\partial t_{j_2}\cdots\partial t_{j_n}},$$

a t derivative of weighted degree $|j| = j_1+j_2+\cdots+j_n$. Then, for $k\geq 1$, we compute the system formed by

$$\begin{cases}\text{all } t\text{-derivatives of weighted degree } k \text{ of (3.5)},\\ \text{all } t\text{-derivatives of weighted degree } k-1 \text{ of (3.6)},\end{cases} \tag{3.8}$$

evaluated at $t=s=0$. For instance, for $k=1$, (3.8) reduces to

$$\mathcal{B}_1(\eta,\theta)\left(\left.\frac{\partial f}{\partial t_1}\right|_{t=s=0}\right) = \left.\frac{\partial f}{\partial t_2}\right|_{t=s=0} + n\left.\frac{\partial^2 f}{\partial t_1^2}\right|_{t=s=0},$$

$$\mathcal{B}_2(\eta,\theta)f|_{t=s=0} = \left.\frac{\partial^2 f}{\partial t_1^2}\right|_{t=s=0} + n\left.\frac{\partial f}{\partial t_2}\right|_{t=s=0} + \left(\left.\frac{\partial f}{\partial t_1}\right|_{t=s=0}\right)^2.$$

After substitution of (3.7), this system of equations can be solved for $\left.\frac{\partial^2 f}{\partial t_1^2}\right|_{t=s=0}$ and $\left.\frac{\partial f}{\partial t_2}\right|_{t=s=0}$ in terms of $\eta,\theta,g(\eta,\theta)$ and $\mathcal{D}g(\eta,\theta)$, whenever $n\neq 1$. Consequently, on the locus $\eta=-\theta$, the partials $\left.\frac{\partial^2 f}{\partial t_1^2}\right|_{\substack{t=s=0\\ \eta=-\theta}}$ and $\left.\frac{\partial f}{\partial t_2}\right|_{\substack{t=s=0\\ \eta=-\theta}}$ can be expressed in terms of θ, $R(\theta)$ and $R'(\theta)$.

For general $k\geq 1$, suppose all the t-derivatives of f of weighted degree k, evaluated at $t=s=0$, have been expressed in terms of η,θ and $g(\eta,\theta),\ldots,\mathcal{D}^{k-1}g(\eta,\theta)$, whenever $n\neq 1,\ldots,k-1$. Then (3.8) is a system of linear equations where the unknowns are all the t-derivatives of f of weighted degree $k+1$, evaluated at $t=s=0$, and the coefficients can be expressed in terms of η,θ and $g(\eta,\theta),\ldots,\mathcal{D}^{k-1}g(\eta,\theta)$. This is a system of $p(k)+p(k-1)$ linear equations in $p(k+1)$ unknowns, where $p(k)$ is the number of partitions of the natural number k. As $p(k+1)\leq p(k)+p(k-1)$, this system can be solved and all the t-derivatives of f of weighted degree $k+1$, evaluated at $t=s=0$ can be expressed in terms of η,θ, and $g(\eta,\theta),\ldots,\mathcal{D}^k g(\eta,\theta)$, whenever $n\neq k$. Consequently, on the locus $\eta=-\theta$, the t-derivatives of f of weighted degree $k+1$, evaluated at $t=s=0$ and on the locus $\eta=-\theta$, can be expressed in terms of θ, $R(\theta),R'(\theta),\ldots,R^{(k)}(\theta)$.

Since the KP equation (3.2) contains t-derivatives of f of weighted degree less or equal to 4, by performing the above scheme up to $k=3$, we can express all these derivatives, evaluated at $t=s=0$ and $\eta=-\theta$, in terms of θ, $R(\theta)$ and its first three derivatives, whenever $n\geq 4$. This gives us a third order differential equation for $R(\theta)$:

$$0 = 4R(\theta)^2 - 2\left(n^2+(1-n^2)\cos 2\theta\right)R'(\theta) + 8\sin 2\theta R(\theta)R'(\theta) - 2\sin 2\theta R''(\theta) + \sin^2\theta\left(12R'(\theta)^2 - R'''(\theta)\right).$$

Multiplying the left-hand and the right-hand side of this equation with $\frac{1}{4}\sin\theta\left(2\cos\theta R'(\theta)+\sin\theta R''(\theta)\right)$, we obtain

$$0 = \frac{\mathrm{d}}{\mathrm{d}\theta}\left(\sin^2\theta R'(\theta)W(\theta)\right), \tag{3.9}$$

with

$$\begin{aligned} W(\theta) &= R(\theta)^2 + 2\sin\theta\cos\theta R(\theta)R'(\theta) + \sin^2\theta R'(\theta)^2 \\ &\quad - \frac{1}{2}\left(\frac{1}{4}\sin^2\theta\frac{R''(\theta)^2}{R'(\theta)} + \sin\theta\cos\theta R''(\theta) + \left(\cos^2\theta + n^2\sin^2\theta\right)R'(\theta)\right).\end{aligned}$$

Eq. (3.9) implies that $W(\theta)=0$, which is the Eq. (1.2), obtained in [2]. This concludes the proof of Theorem 3.1. □

Remark 3.2. In the above proof, we had to assume that $n\geq 4$, where n is the size of the random unitary matrices. For $n=1,2,3$, the function $R(\theta)$ also satisfies (1.2), as can be shown by direct computation, using the representation (3.1) of the probability $\tau_n(\eta,\theta)$ as a Toeplitz determinant. It would be interesting to relate the proof with the original derivation in [2]. For the Gaussian ensembles, the relation between the two methods has been studied in [9].

Finally, similarly to the case of the Jacobi polynomial ensemble (see [5]), we observe that $R(\theta)$ in (1.1) is linked to the Painlevé VI equation. Precisely, we show that it is the restriction to the unit circle of a solution of (a special case of) the Painlevé VI equation, defined for $z \in \mathbb{C}$.

Corollary 3.3. *Put* $R(\theta) = r(\mathrm{e}^{-2\mathrm{i}\theta})$. *Then, the function*

$$\sigma(z) = -\mathrm{i}(z-1)r(z) - \frac{n^2}{4}z$$

satisfies the Okamoto–Jimbo–Miwa form of the Painlevé VI equation

$$[z(z-1)\sigma'']^2 + 4z(z-1)(\sigma')^3 + 4\sigma'\sigma^2 + 4(1-2z)\sigma(\sigma')^2 - c_1(\sigma')^2 + [2(1-2z)c_4 - c_2]\sigma' + 4c_4\sigma - c_3 = 0, \tag{3.10}$$

with

$$c_1 = n^2, \qquad c_2 = \frac{3n^4}{8}, \qquad c_3 = \frac{n^6}{16}, \qquad c_4 = -\frac{n^4}{16}. \tag{3.11}$$

Proof. From (1.2), by a straightforward computation, putting $R(\theta) = r(\mathrm{e}^{-2\mathrm{i}\theta})$, we obtain that $r(z)$ satisfies

$$[z(z-1)r'']^2 + 4z^2(z-1)r'r'' - 4\mathrm{i}z(z-1)^2(r')^3 - 4\mathrm{i}(z^2-1)r(r')^2 + [4z^2 - n^2(z-1)^2](r')^2 - 4\mathrm{i}r^2r' = 0. \tag{3.12}$$

Substituting in (3.12)

$$r(z) = \mathrm{i}\frac{\sigma(z) + xz}{z-1}$$

for some constant x, and annihilating the coefficient of σ^2, one finds that $x = n^2/4$. With this choice of x, the new function $\sigma(z)$ satisfies the Painlevé VI equation (3.10) if we pick c_1, c_2, c_3 and c_4 as in (3.11), which establishes Corollary 3.3. □

4. Discussion of the results and some further directions

Our starting motivation was to understand a differential equation (1.2) discovered in [2], satisfied by the logarithmic derivative of the gap probability that an arc of circle of length 2θ contains no eigenvalues of a randomly chosen unitary $n \times n$ matrix, from the point of view of the algebraic approach initiated in [3]. The main surprise is that the 2-dimensional Toda tau functions (1.3) deforming these gap probabilities, satisfy a centerless *full* Virasoro algebra of constraints. The result stands in contrast with the corresponding integrals for the Gaussian or the orthogonal polynomial ensembles, which roughly satisfy only "half of" a Virasoro type algebra of constraints, see [3,5,6,9].

As mentioned at the beginning of Section 3, the integrals (1.3) can be expressed as Toeplitz determinants, see (3.1). As such, they are very special instances of tau functions for the so-called Toeplitz lattices [4], that is

$$\tau_n(t,s) = \det\left(\mu_{k-l}(t,s)\right)_{0\le k,l\le n-1}, \tag{4.1}$$

where

$$\mu_k(t,s) = \int_{S^1} z^k \mathrm{e}^{\sum_{j=1}^{\infty}(t_j z^j + s_j z^{-j})} w(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}z}, \quad k \in \mathbb{Z}, \tag{4.2}$$

and $w(z)$ is some (complex-valued) weight function defined on the unit circle S^1, such that the trigonometric moments

$$\mu_k = \mu_k(0,0) = \int_{S^1} z^k w(z)\frac{\mathrm{d}z}{2\pi\mathrm{i}z}, \quad k \in \mathbb{Z},$$

satisfy $\det\left(\mu_{k-l}\right)_{0\le k,l\le n-1} \neq 0$, $\forall n \geq 1$. In the special case (3.1) that we consider in this paper, $w(z) = \chi_{(\eta,\theta)^c}(z)$ is the characteristic function of the complement of the arc of circle $(\eta,\theta) = \{z \in S^1 | \eta < \arg(z) < \theta\}$.

As it immediately follows from (4.2), at the level of the trigonometric moments, the Toeplitz hierarchy is given by the simple equations

$$T_j\mu_k \equiv \frac{\partial \mu_k}{\partial t_j} = \mu_{k+j}, \qquad T_{-j}\mu_k \equiv \frac{\partial \mu_k}{\partial s_j} = \mu_{k-j}, \quad \forall j \geq 1.$$

Obviously $[T_i, T_j] = 0, \forall i,j \in \mathbb{Z}$, if we define $T_0\mu_k = \mu_k$. Following an idea introduced in [5] in the context of the 1-dimensional Toda lattices, we define the following vector fields on the trigonometric moments

$$V_j\mu_k = (k+j)\mu_{k+j}, \quad \forall j \in \mathbb{Z}. \tag{4.3}$$

These vector fields trivially satisfy the commutation relations

$$[V_i, V_j] = (j-i)V_{i+j} \tag{4.4}$$

$$[V_i, T_j] = jT_{i+j}, \quad \forall i, j \in \mathbb{Z}, \tag{4.5}$$

from which it follows that

$$[[V_i, T_j], T_j] = j[T_{i+j}, T_j] = 0, \quad \forall i, j \in \mathbb{Z}. \tag{4.6}$$

Eqs. (4.4)–(4.6) mean that the vector fields $V_j, j \in \mathbb{Z}$, form a Virasoro algebra of master symmetries, in the sense of Fuchssteiner [10], for the Toeplitz hierarchy.

The tau functions (4.1) admit the following expansion

$$\tau_n(t,s) = \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}} S_{i_{n-1}-(n-1),\ldots,i_0}(t) S_{j_{n-1}-(n-1),\ldots,j_0}(s),$$

where

$$p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}} = \det\left(\mu_{i_k - j_l}(0,0)\right)_{0 \le k,l \le n-1}, \tag{4.7}$$

are the so-called Plücker coordinates, and $S_{i_1,\ldots,i_k}(t)$ denote the Schur polynomials

$$S_{i_1,\ldots,i_k}(t) = \det\left(S_{i_r+s-r}(t)\right)_{1 \le r,s \le k},$$

with $S_n(t)$ the so-called elementary Schur polynomials defined by the generating function

$$\exp\left(\sum_{k=1}^{\infty} t_k x^k\right) = \sum_{n \in \mathbb{Z}} S_n(t_1, t_2, \ldots) x^n.$$

In a forthcoming publication, we shall establish the next result:

Theorem 4.1. *For all $k \in \mathbb{Z}$, we have*

$$L_k^{(n)} \tau_n(t,s) = \sum_{\substack{0 \le i_0 < \cdots < i_{n-1} \\ 0 \le j_0 < \cdots < j_{n-1}}} V_k\left(p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}}\right) S_{i_{n-1}-(n-1),\ldots,i_0}(t) S_{j_{n-1}-(n-1),\ldots,j_0}(s),$$

with $L_k^{(n)}$, $k \in \mathbb{Z}$, defined as in (2.4)–(2.6), and $V_k\left(p_{\substack{i_0,\ldots,i_{n-1} \\ j_0,\ldots,j_{n-1}}}\right)$ the Lie derivative of the Plücker coordinates (4.7) in the direction of the master symmetries V_k of the Toeplitz hierarchy, as defined in (4.3).

Thus the operators $L_k^{(n)}$, $k \in \mathbb{Z}$, precisely describe the master symmetries of the Toeplitz hierarchy on the tau functions of this hierarchy. Since master symmetries are usually connected with a bi-Hamiltonian structure in the sense of Magri [11] (see [12,13] for an overview), it suggests investigating the relation with the recursion operator for this hierarchy that was found in [14].

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