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Asymptotic behavior of solutions of the *m*th-order nonhomogeneous difference equations

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Abstract

Asymptotic behavior of solutions of the *m*th-order difference equation of the form

(E1) $\Delta^m x_n + f(n, x_n, \ldots, \Delta^{m-1} x_n) = h_n$

and some special case (E2) of these equation are investigated.

Keywords: Difference equations; Asymptotic behavior

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The asymptotic behavior of solutions of an *n*th-order nonhomogeneous differential equation of the form $y^{(n)} + f(t, y, y^{(1)}, \ldots, y^{(n-1)}) = h(t)$ has been considered in [3]. Some asymptotic properties of the difference equation $\Delta^2 y_n + F(n, y_n, y_{n+1}) = b_n$ were given in [6]. The asymptotic behavior of solutions of the second-order difference equation of the form $\Delta(r_n \Delta x_n) + f(n, x_n, \Delta x_n) = h_n$ has been investigated in [1]. Similar problems for *m*th-order difference equation have been studied in [4].

Let $\mathbb{N} := \{n_0, n_0 + 1, \ldots\}$, where n_0 is a given nonnegative integer, $\mathbb{R}$ the set of real numbers and $\mathbb{R}_+$ the set of positive reals. For a function $f : \mathbb{N} \to \mathbb{R}$ one introduces the difference operator Δ as follows:

$$\Delta f_n = f_{n+1} - f_n, \qquad \Delta^k f_n = \Delta(\Delta^{k-1} f_n), \quad k \geqslant 1, \text{ where } f_n = f(n).$$

Moreover, $\sum_{j=k}^{t} f_j = 0$ for $t < k$ and let us denote

$$n^{(k)} = \prod_{i=0}^{k-1} (n - i) \quad \text{for } n \geqslant k,$$

$$n^{(0)} = 1.$$

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The application of the argueing method of the discrete versions of Gronwall lemma, as given in [5], allows us to perform the proofs of the following lemma.

Lemma. *Let* $u:\mathbb{N}\to\mathbb{R}_+$, $c:\mathbb{N}\to\mathbb{R}_+$, $M\in\mathbb{R}_+$, $C_0\in\mathbb{R}_+$.

If $u_n \leqslant C_0 + M\sum_{j=n_0}^{n-1} c_j(1+u_j)$

then

$$u_n \leqslant (1+C_0)\exp\left[M\sum_{j=n_0}^{n-1} c_j\right]-1$$

for all $n\in\mathbb{N}$.

1. The mth-order nonhomogeneous difference equation

(E1) $\Delta^m x_n + f(n,x_n,\Delta x_n,\ldots,\Delta^{m-1}x_n) = h_n,$

where $h:\mathbb{N}\to\mathbb{R}$, $f:\mathbb{N}\times\mathbb{R}^m\to\mathbb{R}$, $m\geqslant 2$ will be investigated subject to the following hypothese:

$$(1.1)\qquad |f(n,x_n,\Delta x_n,\ldots,\Delta^{m-1}x_n)| \leqslant \sum_{i=0}^{m-1} g_i(n)|\Delta^i x_n|^{t_i},$$

where $g_i:\mathbb{N}\to\mathbb{R}_+\cup\{0\}$, $t_i\in(0,1\rangle$, $i=0,1,\ldots,m-1$.

Theorem 1. *If condition* (1.1) *is satisfied and*

(1.2) *the series* $\sum_{j=n_0}^{\infty}|h_j|$ *is convergent,*

(1.3) *the series* $\sum_{j=n_0}^{\infty} j^{(m-i-1)t_i}g_i(j)$, $i=0,1,\ldots,m-1$ *are convergent,*

then there exist solutions x_n *of* (E1) *which possesses the asymptotic behavior*

$$(\mathrm{AB})\qquad \lim_{n\to\infty}\frac{\Delta^i x_n}{(m-1)^{(i)}n^{(m-i-1)}} = C = \mathit{const.},\quad i=0,1,\ldots,m-1.$$

Proof. From (1.2) it follows that there exists a constant A_0 such that

$$(1.4)\qquad \sum_{j=n_0}^{\infty}|h_j| = A_0.$$

By summation (E1) over n one yields

$$(1.5)\qquad \Delta^{m-1}x_n = \Delta^{m-1}x_{n_0} + \sum_{j=n_0}^{n-1}h_j - \sum_{j=n_0}^{n-1}f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j).$$

Hence,

$$|\Delta^{m-1}x_n| \leqslant |\Delta^{m-1}x_{n_0}| + \sum_{j=n_0}^{n-1}|h_j| + \sum_{j=n_0}^{n-1}|f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|.$$

Let us denote

$$(1.6)\qquad A_r = \frac{|\Delta^{m-r}x_{n_0}|}{n_0^{r-1}}, \quad r = 1,2,\ldots,m.$$

Using (1.4) and (1.6), one finds

$$(1.7)\qquad |\Delta^{m-1}x_n| \leqslant A_1 + A_0 + \sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|.$$

By summation (E1) over n twice one gets

$$(1.8)\qquad \Delta^{m-2}x_n = \Delta^{m-2}x_{n_0} + (n-n_0)\Delta^{m-1}x_{n_0} + \sum_{k=n_0}^{n-1}\sum_{j=n_0}^{k-1} h_j - \sum_{k=n_0}^{n-1}\sum_{j=n_0}^{k-1} f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j).$$

Dividing (1.8) by n we obtain

$$\frac{|\Delta^{m-2}x_n|}{n} \leqslant \frac{|\Delta^{m-2}x_{n_0}|}{n} + |\Delta^{m-1}x_{n_0}| + \sum_{j=n_0}^{n-1}|h_j| + \sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|.$$

Using (1.6), one finds

$$\frac{|\Delta^{m-2}x_n|}{n} \leqslant A_2 + A_1 + A_0 + \sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)| \quad \text{for all } n \geqslant n_0.$$

Similarly, summation k-times of (E1) over n and division by n^{k-1}, for all $n \geqslant n_0$ gives

$$\frac{|\Delta^{m-k}x_n|}{n^{k-1}} \leqslant A_k + A_{k-1} + \cdots + A_1 + A_0 + \sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|, \quad k = 1,2,\ldots,m.$$

Hence,

$$(1.9)\qquad \sum_{k=1}^{m} \frac{|\Delta^{m-k}x_n|}{n^{k-1}} \leqslant \sum_{k=1}^{m}\sum_{r=0}^{k} A_r + m\sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|.$$

Let us denote $A = \sum_{k=1}^{m}\sum_{r=0}^{k} A_r$ and $m-k=i$, $k=1,\ldots,m$. Then (1.9) can be rewritten in the form

$$\sum_{i=0}^{m-1} \frac{|\Delta^{i}x_n|}{n^{m-i-1}} \leqslant A + m\sum_{j=n_0}^{n-1} |f(j,x_j,\Delta x_j,\ldots,\Delta^{m-1}x_j)|.$$

By assumption (1.1) we have

$$(1.10)\qquad \sum_{i=0}^{m-1} \frac{|\Delta^{i}x_n|}{n^{m-i-1}} \leqslant A + m\sum_{j=n_0}^{n-1}\sum_{i=0}^{m-1} g_i(j)\, j^{(m-i-1)t_i} \left|\frac{\Delta^{i}x_j}{j^{m-i-1}}\right|^{t_i}.$$

It is evident that the inequality

$$(1.11)\qquad a_i^w \leqslant 1 + a_1 + a_2 + \cdots + a_m$$

holds for any $a_i \in \mathbb{R}_+$ and $w \in (0,1\rangle$.

Using (1.11) to (1.10), one gets

$$(1.12)\qquad \sum_{i=0}^{m-1} \frac{|\Delta^i x_n|}{n^{m-i-1}} \leqslant A + m \sum_{j=n_0}^{n-1} \sum_{i=0}^{m-1} j^{(m-i-1)t_i} g_i(j) \left[1 + \sum_{i=0}^{m-1} \frac{|\Delta^i x_j|}{j^{m-i-1}}\right].$$

Let us denote

$$u_n = \sum_{i=0}^{m-1} \frac{|\Delta^i x_n|}{n^{m-i-1}}.$$

Then (1.12) takes the form

$$u_n \leqslant A + m \sum_{j=n_0}^{n-1} \sum_{i=0}^{m-1} j^{(m-i-1)t_i} g_i(j)[1 + u_j].$$

Using Lemma with $C_0 = A$, $M = m$ and $c_j = \sum_{i=0}^{m-1} j^{(m-i-1)t_i} g_i(j)$ we have

$$u_n \leqslant (1 + A) \exp\left[m \sum_{j=n_0}^{n-1} \sum_{i=0}^{m-1} j^{(m-i-1)t_i} g_i(j)\right] - 1.$$

If n approaches infinity then

$$u_n \leqslant (1 + A) \exp\left[m \sum_{i=0}^{m-1} \left(\sum_{j=n_0}^{\infty} j^{(m-i-1)t_i} g_i(j)\right)\right].$$

By virtue of (1.3) there exists a constant L such that

$$\sum_{i=0}^{m-1} \frac{|\Delta^i x_n|}{n^{m-i-1}} \leqslant L.$$

Hence and from (1.1) it follows the inequality

$$(1.13)\qquad \sum_{j=n_0}^{n-1} |f(j, x_j, \Delta x_j, \ldots, \Delta^{m-1} x_j)| \leqslant \sum_{i=1}^{m-1} L^{t_i} \sum_{j=n_0}^{n-1} j^{(m-i-1)t_i} g_i(j).$$

In face of (1.3) the right-hand side of (1.13) is bounded if $n \to \infty$. Hence, the series

$$\sum_{j=n_0}^{\infty} |f(j, x_j, \Delta x_j, \ldots, \Delta^{m-1} x_j)| \quad \text{is convergent.}$$

So all terms on the right-hand side of (1.5) are bounded. There exists the finite limit

$$\lim_{n\to\infty} \frac{\Delta^{m-1} x_n}{(m-1)!} = C = \text{const.}$$

Using the Stolz theorem [2] m-times one can find the thesis of the theorem

$$\lim_{n\to\infty}\frac{x_n}{n^{(m-1)}}=\lim_{n\to\infty}\frac{\Delta x_n}{(m-1)n^{(m-2)}}=\lim_{n\to\infty}\frac{\Delta^2 x_n}{(m-1)^{(2)}n^{(m-3)}}=\cdots=$$

$$=\lim_{n\to\infty}\frac{\Delta^{m-2}x_n}{(m-1)^{(m-2)}n^{(1)}}=\lim_{n\to\infty}\frac{\Delta^{m-1}x_n}{(m-1)!}=C. \qquad \square$$

2. In the next theorem, a special case of equation (E1) will be considered. The second-order non-homogeneous difference equation of the form

(E2) $\Delta^2 x_n + g_0(n)x_n^{t_0} + g_1(n)[\Delta x_n]^{t_1} = h_n,$

where $g_i:\mathbb{N}\to\mathbb{R}_+\cup\{0\}$, $h:\mathbb{N}\to\mathbb{R}$, $t_i\in(0,1)$, $i=0,1$ will be investigated subject to the following set of hypotheses:

(2.1) if $g_i(n)$is not identically zero, then $\lim_{n\to\infty} n^{p_i} g_i(n)=c_i>0$ for some p_i, $c_i=\text{const.}$, $i=0,1$;

(2.2) $t_i=u_i/w_i$, where u_i, w_i are odd integers, $i=0,1$.

Theorem 2. *Assume* (2.1), (2.2) *and*

(2.3) *the series* $\sum_{j=n_0}^{\infty}|h_j|$ *is convergent.*

A necessary and sufficient condition for some solution x_n *of* (E2) *to have the asymptotic behavior*

(AB*) $\lim_{n\to\infty}\frac{\Delta^i x_n}{n^{1-i}}=C=\text{const.},\quad i=0,1$

is that $p_i>t_i(1-i)+1$ *for* $i=0,1$, *where* $g_i(n)$ *is not identically zero.*

Proof. *Sufficiency*: Since Eq. (E2) is a special case of Eq. (E1) it will be enough to verify if the assumption (1.3) of Theorem 1 is satisfied. If $g_0(n)$ and $g_1(n)$ are identically zero then $\sum_{j=n_0}^{\infty} j^{(1-i)t_i}g_i(j)=0$ for $i=0,1$. So, in this case condition (1.3) is fulfiled.

Suppose $g_0(n)$ is not identically zero. Then by virtue of (2.1) with $i=0$ we have $\lim_{n\to\infty} n^{p_0}g_0(n)=c_0>0$. Choose any $\varepsilon>0$ such that $c_0-\varepsilon>0$. Then $|n^{p_0}g_0(n)-c_0|\leqslant\varepsilon$ for each $n>n_1$.

Hence $0<c_0-\varepsilon\leqslant n^{p_0}g_0(n)\leqslant c_0+\varepsilon$ for each $n>n_1$.

It means that $g_0(n)\leqslant(c_0+\varepsilon)/n^{p_0}$ for each $n>n_1$. Then

$$\sum_{j=n_0}^{\infty} j^{t_0}g_0(j)=A_1+\sum_{j=n_1+1}^{\infty} j^{t_0}g_0(j)\leqslant A_1+(c_0+\varepsilon)\sum_{j=n_1+1}^{\infty} j^{t_0-p_0},$$

where

$$A_1=\sum_{j=n_0}^{n_1} j^{t_0}g_0(j).$$

But $t_0 - p_0 < -1$. Therefore, the series $\sum_{j=n_1+1}^{\infty} j^{t_0-p_0}$ is convergent and the series $\sum_{j=n_0}^{\infty} j^{t_0} g_0(j)$ is convergent, too. Hence, it follows that condition (1.3) is satisfied if $g_0(n)$ is not identically zero.

Assume $g_1(n)$ is not identically zero. Then in face of (2.1) with $i=1$ we have $\lim_{n\to\infty} n^{p_1} g_1(n) = c_1 > 0$. Choosing $\varepsilon > 0$ in a similar way as above one yields

$$|n^{p_1} g_1(n) - c_1| \leqslant \varepsilon \quad \text{for each } n > n_2$$

and

$$\sum_{j=n_0}^{\infty} g_1(j) \leqslant A_2 + (c_1+\varepsilon) \sum_{j=n_2+1}^{\infty} j^{-p_1}, \quad \text{where } A_2 = \sum_{j=n_0}^{n_2} g_1(j).$$

Since $p_1 > 1$, the series $\sum_{j=n_2+1}^{\infty} j^{-p_1}$ is convergent and the series $\sum_{j=n_0}^{\infty} g_1(j)$ is convergent, too. So, all assumptions of Theorem 1 are fulfiled. Therefore there exists a solution of (E2) which possesses the asymptotic behavior (AB*).

Necessacity: Suppose that a solution x_n of (E2) possesses the asymptotic behavior (AB*). Let $C \neq 0$. Assume there exists such j that

(2.4) $\quad p_j \leqslant t_i(1-i)+1, \quad i=0,1.$

If $C > 0$ then there exists such $\varepsilon > 0$ that $c - \varepsilon > 0$. Since $x_n/n \to C$ as $n \to \infty$, so $x_n/n \in (C-\varepsilon, C+\varepsilon)$ for $n \geqslant N(\varepsilon)$. Hence $x_n > 0$, for $n \geqslant N(\varepsilon)$. If $C < 0$ then in a similar way, one can choose such $\varepsilon > 0$ that $C+\varepsilon < 0$. Then $x_n/n \in (C-\varepsilon, C+\varepsilon)$ for $n \geqslant N(\varepsilon)$. Therefore, $x_n < 0$ for $n \geqslant N(\varepsilon)$. Hence, it follows that x_n and C are of the same sign for $n \geqslant N(\varepsilon)$. By hypothesis (2.2), if x_n is a solution of (E2), $-x_n$ satisfies (E2) with h_n replaced by $\bar{h}_n = -h_n$. Thus, one can consider behavior of a solution of one of these equations.

Choose constants A_1 and A_2 in the following way: $0 < A_1 < C$, $0 < A_2 < c_i$ for $i=0,1$ such that $g_i(n)$, $i=0,1$ is not identically zero. Select any $\varepsilon > 0$ such that $0 < A_1 < C - \varepsilon$ and $0 < A_2 < c_i - \varepsilon$ for $i=0,1$. Then there exist such constants $N_{C,\varepsilon}$, $N_{c_0,\varepsilon}$, $N_{c_1,\varepsilon}$ that:
if $n \geqslant N_{C,\varepsilon}$ then

(2.5) $$C+\varepsilon > \frac{\Delta^i x_n}{n^{1-i}} > C-\varepsilon > A_1 \quad \text{for } n \geqslant N_{C,\varepsilon},$$

if $n \geqslant N_{c_i,\varepsilon}$, $i=0,1$ then

(2.6) $$c_0+\varepsilon > n^{p_0} g_0(n) > c_0-\varepsilon > A_2 \quad \text{for } n \geqslant N_{c_0,\varepsilon},$$

(2.7) $$c_1+\varepsilon > n^{p_1} g_1(n) > c_1-\varepsilon > A_2 \quad \text{for } n \geqslant N_{c_1,\varepsilon}.$$

Let us denote $N_1 = \max[N_{C,\varepsilon}, N_{c_0,\varepsilon}, N_{c_1,\varepsilon}]$. Then inequalities (2.5)–(2.7) hold for $n > N_1$. Let K denote the set of those index k for which p_k associated with g_k by (2.1) have the property $p_k > t_k(1-k)+1$. Let J denote the set of those index j whose associated p_j fulfils the inequality $p_j \leqslant t_j(1-j)+1$. Then Eq. (E2) can be rewritten in the form

(2.8) $$\Delta^2 x_n + \sum_{k\in K} g_k(n)[\Delta^k x_n]^{t_k} - h_n = -\sum_{j\in J} g_j(n)[\Delta^j x_n]^{t_j}.$$

In face of (2.5)–(2.7) and (2.4) we have

$$(2.9)\qquad \sum_{j\in J} g_j(n)[\Delta^j x_n]^{t_j} = \sum_{j\in J} n^{p_j} g_j(n)\left[\frac{\Delta^j x_n}{n^{1-j}}\right]^{t_j} n^{t_j(1-j)-p_j}$$

$$> \sum_{j\in J} A_2 A_1^{t_j} n^{t_j(1-j)-p_j} > 0.$$

Using (2.9) to (2.8), one yields

$$(2.10)\qquad \Delta^2 x_n + \sum_{k\in K} g_k(n)[\Delta^k x_n]^{t_k} - h_n = -A_2 A_1^{t_q} n^{t_q(1-q)-p_q}, \quad q \in J.$$

Summation (2.10) over n gives

$$(2.11)\qquad \Delta x_n - \Delta x_{N_1} + \sum_{k\in K}\sum_{j=N_1}^{n-1} g_k(j)[\Delta^k x_j]^{t_k} - \sum_{j=N_1}^{n-1} h_j$$

$$\leqslant -A_2 A_1^{t_q} \sum_{j=N_1}^{n-1} j^{t_q(1-q)-p_q}, \quad q \in J.$$

But $t_k(1-k) - p_k < -1$ for any $k \in K$. Therefore,

$$\sum_{j=N_1}^{n-1} g_k(j)[\Delta^k x_j]^{t_k} = \sum_{j=N_1}^{n-1} j^{p_k} g_k(j)\left[\frac{\Delta^k x_j}{j^{1-k}}\right]^{t_k} j^{t_k(1-k)-p_k}$$

$$\leqslant (c_k+\varepsilon)(c+\varepsilon)^{t_k} \sum_{j=N_1}^{\infty} j^{t_k(1-k)-p_k}.$$

Hence in (2.11) as $n \to \infty$ we have

- $\Delta x_n \to C$ (by assumptions),
- Δx_{N_1} is finite,
- the series $\sum_{j=N_1}^{\infty} g_k(j)[\Delta^k x_j]^{t_k}$ and $\sum_{j=N_1}^{\infty} h_j$ are convergent.

Therefore, the left-hand side of (2.11) is finite. Since $r_q(1-q) - p_q \geqslant -1$, the right-hand side of (2.11) approaches $-\infty$ as $n \to \infty$. We have a contradiction. It means that if a solution of (E2) possesses the asymptotic behavior (AB*) then all $p_i > t_i(1-i) + 1$. □

References

[1] A. Drozdowicz, On the asymptotic behavior of solutions of second order nonhomogeneous difference equations, Ann. Mat. Pura. Appl. CLV (1989) pp. 75–84.

[2] G.M. Fichtengolc, Kurs differencial'nogo i integral'nogo iscislenija, t. 1, Moskva-Leningrad (1948), in Russian.

[3] T.G. Hallam, Asymptotic behavior of the solutions of an nth order nonhomogeneous ordinary differential equation, Trans. Amer. Math. Soc. 122 (1966) 177–196.

[4] M. Migda, Asymptotic properties of solutions of higher order difference equations, Radovi Matematički 5 (1989) 297–309.

[5] J. Popenda, On the discrete analogy of Gronwall lemma, Demonstratio Math. 16 (1) (1983) pp. 11–15.

[6] J. Popenda, J. Werbowski, On the asymptotic behavior of the solutions of difference equations of second order, Comm. Math. 22 (1980) 135–142.