Piecewise Linear Approximation of Smooth Compact Fibers

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Let $H: \mathbb{R}^N \rightarrow \mathbb{R}^m$ be a twice differentiable map. Let $N := n + m$. Assume that the fiber $F := H^{-1}(y_0)$ of $H$ over a point $y_0 \in \mathbb{R}^m$ is a compact $n$-dimensional differentiable manifold and that $\partial H$ is of maximal rank in a neighborhood of $F$. Works of Allgower and Gnutzmann, and of Allgower and Schmidt give an algorithm to find a global piecewise linear approximation of $F$ based on a triangulation of $\mathbb{R}^N$. Several authors have given algorithms for obtaining a piecewise linear approximation of $F$ in a way that does not depend on the triangulation in $\mathbb{R}^N$. This effects a reduction of the combinatorial complexity from $O(N!)$ to $O(n!)$, but the approximations are not global. In this paper, a probability one algorithm is given, which, given $H, F, N, m$ as above, uses homotopy continuation to construct a differentiable map $\tilde{H}: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}$ with a compact differentiable manifold $\tilde{F} := \tilde{H}^{-1}(0)$, which is diffeomorphic to $F$ by an explicit diffeomorphism, and with $\partial \tilde{H}$ of maximal rank in a neighborhood of $\tilde{F}$. This reduces the problem when $N > 2n + 1$ to the known piecewise linear algorithm in $\mathbb{R}^{2n+1}$ at the simpler expense of carrying out a well-behaved homotopy continuation starting at vertices of the simplices of the triangulation of a neighborhood of $\pi(F) \subset \mathbb{R}^{2n+1}$, where $\pi: \mathbb{R}^N \rightarrow \mathbb{R}^{2n+1}$ is a generic linear projection. Consequently, a global algorithm is obtained with a combinatorial complexity of $O((2n+1)!)$.

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INTRODUCTION

Let $H : \mathbb{R}^N \to \mathbb{R}^m$ be a twice differentiable map. Let $N := n + m$. Assume that the fiber $F := H^{-1}(y_0)$ of $H$ over a point $y_0 \in \mathbb{R}^m$ is a compact differentiable manifold and that $\partial H$ is of maximal rank in a neighborhood of $F$. Thus the dimension of $F$ is $n$. Without loss of generality we can assume $y_0$ is the origin $0$. In [4–6] a numerical method to compute a piecewise linear structure on $F$ is developed. Since it depends on a triangulation of a neighborhood of $F$ in the ambient $\mathbb{R}^N$, the combinatorics associated with the simplices of the triangulation is of order $O(N!)$. Rheinboldt and co-authors [7–9] have given moving frame algorithms for obtaining piecewise linear approximations of manifolds using triangulations of tangent spaces which are affine images of $\mathbb{R}^{N-m}$. However, it has so far not been possible to obtain global approximations using the moving frame algorithm without some patching together at seams.

Here we develop an alternate approach for constructing the piecewise linear structure on the manifold $F$ when $N > 2n + 1$, which reduces the construction to the known piecewise linear algorithm in $\mathbb{R}^{2n+1}$. In this context the complexity of the combinatorics is thus reduced to the order of $O((2n+1)!)$.

The main point of this paper is the explicit construction of the map $H$. The algorithm developed in this paper has the following input and output.

**Input:** A $C^2$ map $H : \mathbb{R}^N \to \mathbb{R}^m$, $N = n + m \geq 2n + 2$ with $\partial H$ of maximal rank on $F := H^{-1}(0)$.

**Output:** Explicit constructions of

1. a $C^2$ map $\tilde{H} : \mathbb{R}^{2n+1} \to \mathbb{R}^{n+1}$, with $\partial \tilde{H}$ of maximal rank on $\tilde{F} := \tilde{H}^{-1}(0)$; and
2. a generic linear surjective map $\pi : \mathbb{R}^N \to \mathbb{R}^{2n+1}$ with $\pi_F$ a diffeomorphism from $F$ to $\tilde{F}$.

The construction of the generic linear projection $\pi$ is straightforward. It is a classical fact (Lemma 1.1) that except for coefficients lying in a closed measure zero subset of the Euclidean space of the coefficients, linear projections of $\mathbb{R}^N \to \mathbb{R}^{2n+1}$ give an embedding when restricted to $F$. Fix
such a generic linear projection \( \pi \), and let \( \tilde{F} := \pi(F) \). We have to deal with trivial bundles many times so for a topological space \( X \) we let \( \mathcal{O}_X^k := X \times \mathbb{R}^k \).

The construction of \( H \) is done in two steps:

1. Construction of an explicit diffeomorphism of a neighborhood of \( \tilde{F} \subset \mathbb{R}^{2n+1} \) with a neighborhood of \( \tilde{F} \subset \mathcal{N}_{\tilde{F}/R^{2n+1}} \), where \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \) is the normal bundle of \( \tilde{F} \) in \( \mathbb{R}^{2n+1} \).

2. Construction of an explicit trivialization \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \) by:
   (a) constructing an explicit isomorphism \( \phi : \mathcal{N}_{\tilde{F}/R^{2n+1}} \oplus \mathcal{O}_{\tilde{F}}^N \to \mathcal{O}^{N+n+1}_{\tilde{F}} \), and
   (b) constructing from \( \phi \) an explicit isomorphism \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \to \mathcal{O}^{n+1}_{\tilde{F}} \).

The first step is easy. Let \( \nu : \mathcal{N}_{\tilde{F}/R^{2n+1}} \to \tilde{F} \) denote the bundle projection. Note that \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \) is identified with the orthogonal complement of the tangent bundle \( T_{\tilde{F}} \) in \( \tilde{F} \times \mathbb{R}^{2n+1} \). Thus we have a well-defined map \( g : \mathcal{N}_{\tilde{F}/R^{2n+1}} \to \mathbb{R}^{2n+1} \) given by sending \( v \in \mathcal{N}_{\tilde{F}/R^{2n+1}} \) to \( \nu(v) + v \). Note that \( \partial g \) gives an isomorphism of \( T_{\mathcal{N}_{\tilde{F}/R^{2n+1}}} \tilde{F} \to T_{\mathbb{R}^{2n+1}} \tilde{F} \). Thus \( g^{-1} \) is a diffeomorphism of a neighborhood of \( \tilde{F} \subset \mathbb{R}^{2n+1} \) with a neighborhood of \( \tilde{F} \subset \mathcal{N}_{\tilde{F}/R^{2n+1}} \). Thus, the construction of the map \( H \) is reduced to the construction of a trivialization of \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \).

The map \( \phi \) is the composition of three explicit isomorphisms:

\[
\mathcal{N}_{\tilde{F}/R^{2n+1}} \oplus \mathcal{O}_{\tilde{F}}^N \cong \mathcal{N}_{\tilde{F}/R^{2n+1}} \oplus T_{\tilde{F}} \oplus \mathcal{N}_{\tilde{F}/R^n} \\
\cong \mathcal{O}^{2n+1}_{\tilde{F}} \oplus \mathcal{N}_{\tilde{F}/R^n} \\
\cong \mathcal{O}^{2n+1}_{\tilde{F}} \oplus \mathcal{O}_{\tilde{F}}^n.
\]

The isomorphism in line 1 is obtained by using the isomorphism \( \pi_F \) plus the orthogonal splitting \( T_{\tilde{F}} \) into \( T_{\tilde{F}} \) and \( \mathcal{N}_{\tilde{F}/R^n} \). The isomorphism in line 2 is obtained by using the isomorphism \( \pi_F \) plus the orthogonal splitting \( T_{\mathbb{R}^{2n+1}} \) into \( T_{\tilde{F}} \) and \( \mathcal{N}_{\tilde{F}/R^{2n+1}} \). The isomorphism in line 3 is obtained by using the isomorphism of \( \mathcal{N}_{\tilde{F}/R^n} \) and \( \mathcal{O}_{\tilde{F}}^n \) given by \( \partial H \). This reduces us to the explicit numerical construction of a vector bundle isomorphism \( \mathcal{O}^{n+1}_{\tilde{F}} \cong \mathcal{N}_{\tilde{F}/R^{2n+1}} \). We construct a homotopy to carry this out in Section 3.

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1. SOME USEFUL FACTS

In this section we state some useful classical results. For the convenience of the reader we give sketches of the proofs.

By a projection, \( p : \mathbb{R}^N \to \mathbb{R}^m \), from \( \mathbb{R}^N \) to \( \mathbb{R}^m \) we mean a linear surjection; i.e., after choosing linear coordinates on \( \mathbb{R}^N \) and \( \mathbb{R}^m \), \( p \) is a map of the form

\[
y = Ax, \quad \text{where } y = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix}, \quad x = \begin{pmatrix} x_1 \\ \vdots \\ x_N \end{pmatrix}, \quad \text{and } A = \begin{bmatrix} a_{11} & \cdots & a_{1N} \\ \vdots & \vdots & \vdots \\ a_{m1} & \cdots & a_{mN} \end{bmatrix}.
\]

Letting \( p^\perp : \mathbb{R}^N \to \mathbb{R}^{N-m} \) be the projection orthogonal to \( p \), we see that the data of \( p \) are equivalent to having an orthogonal splitting \( \mathbb{R}^N = \mathbb{R}^m \oplus \mathbb{R}^{N-m} \). If for linear coordinates on \( \mathbb{R}^N \) we use linear coordinates on \( \mathbb{R}^m \) and \( \mathbb{R}^{N-m} \), the projection takes the simple form

\[
(x_1, \ldots, x_m, \ldots, x_N) \mapsto (x_1, \ldots, x_m).
\]

Note that the data of the projection are equivalent to choosing an \((N-m)\)-dimensional linear space to be the kernel of the projection \( p \), or equivalently an \( m \)-dimensional linear space to be the kernel of the complementary projection. In either case, these spaces are parameterized by the Grassmannian of linear \((N-m)\)-dimensional subspaces of \( \mathbb{R}^N \), which is of dimension \( m(N-m) \).

The Zariski open and dense set of \( N \times m \) matrices of rank \( m \) surjects onto the Grassmannian: the map is attained by sending a matrix \( A \) to the \( m \)-dimensional linear subspace of \( \mathbb{R}^N \) spanned by the row Thus, we can work with Zariski open dense subsets of the \( Nm \)-dimensional space of \( N \times m \) matrices, rather than with Zariski open and dense subsets of Grassmannians.

**Lemma 1.1 (Whitney).** Let \( F \subset \mathbb{R}^N \) be a compact \( n \)-dimensional \( C^2 \) submanifold of \( \mathbb{R}^N \). If \( N = 2n+1 \), then for an open dense set \( U \) of the \( N(2n+1) \)-dimensional space of \( N \times (2n+1) \) matrices parameterizing projections from \( \mathbb{R}^N \) to \( \mathbb{R}^{2n+1} \), the map from \( F \) to its image \( \tilde{F} := p(F) \) is an isomorphism; i.e., \( p \) embeds \( F \) in \( \mathbb{R}^{2n+1} \). The complement of \( U \) has measure zero.

**Proof.** For simplicity we sketch the proof in the case when \( N = 2n+2 \). Let \( \mathcal{P} := \mathbb{R}^{2n+1} \) denote the Grassmannian of one dimensional linear subspaces of \( \mathbb{R}^{2n+2} \), i.e., the real projective space of dimension \( 2n+1 \).

Let \( B := \bigcup_{x \in F} B_x \), where \( B_x \) denotes the lines through the origins of the fiber \( \mathcal{F}_x \) of the tangent bundle \( \mathcal{T}_F \) of \( F \) restricted to \( x \). Note that \( B \) is naturally identified as a smooth closed submanifold of \( \mathcal{P} \). Since its dimension is \( 2n-1 \), the measure of \( B \) as a subset of \( \mathcal{P} \) is zero.

Let \( B' := F \times F \setminus \Delta \), where \( \Delta \) is the diagonal of \( F \times F \). Since \( F \) is a submanifold of \( \mathbb{R}^{2n+2} \), the quotient \( B' \) of \( B' \) under the involution \( (x, y) \mapsto (y, x) \)
is naturally identified with a smooth submanifold of $\mathcal{P}$. Since $\dim B^\pi = 2n < \dim \mathcal{P}$, the measure of $B^\pi$ as a subset of $\mathcal{P}$ is zero.

Note that the closure of $B^\pi$, as a subset of $\mathcal{P}$, equals $B^\pi \cup \partial B^\pi$, i.e., identifying points of $F \times F - \Delta$ with secants of $F \subset \mathbb{R}^N$, a sequence $z_n$ in $F \times F - \Delta$ with a cluster point $(x, x) \in \Delta$ has a subsequence of secants converging to a tangent line to $F$ at $x$. Thus $B^\pi$ is contained in the complement of an open and dense set of $\mathcal{P}$. QED

Remark 1.2. For algebraic maps, the argument shows the stronger fact that the dense open sets are Zariski open.

The use of two distinct parameter spaces in the proof is for simplicity of exposition. It is more natural to use the blowup of the quotient of $F \times F$ under the involution $(x, y) \rightarrow (y, x)$ along the image of the diagonal $\Delta$. This amounts to replacing each point $x \in \Delta$ with the projective space of lines through the origin of the fiber at $x$ of the normal bundle of $\Delta$ in $F \times F$. This normal bundle is isomorphic to the tangent bundle of $F$.

Remark 1.3. An equivalent way of thinking of this is the following. Make a generic linear change of coordinates. Then under the simple projection

$$(x_1, \ldots, x_m, \ldots, x_N) \rightarrow (x_1, \ldots, x_{2n+1}),$$

$F$ is embedded in $\mathbb{R}^{2n+1}$.

2. THE BASIC SETUP

Let $H: \mathbb{R}^N \rightarrow \mathbb{R}^m$ be a $C^2$ map with $N := n+m$. Assume that the fiber $F := H^{-1}(y_0)$ of $H$ over a point $y_0 \in \mathbb{R}^m$ is a compact differentiable manifold and that $\partial H$ is of maximal rank in a neighborhood of $F$. Thus the dimension of $F$ is $n$. Without loss of generality we can assume $y_0$ is the origin 0. Assuming we have chosen our linear coordinates generically, it follows from Lemma 1.1 that if $m \geq n+1$, the projection $\pi: \mathbb{R}^{n+m} \rightarrow \mathbb{R}^{2n+1}$ onto the first $2n+1$ coordinates gives an embedding of $F$. Let $\tilde{F} := \pi(F)$. We assume that we are in the case $N := n+m > 2n+1$, since otherwise the result is done already by Allgower and Gnutzmann [4, 5] and Allgower and Schmidt [6]. Programs in C for low values of $n$ are available via the web site www.math.colostate.edu/~georg/.

The following lemma gives a key fact about $\tilde{F}$.

**Lemma 2.1.** Let $H$, $N := n+m$, $F$, $\pi$, and $\tilde{F}$ be as at the start of this section. There exists a $C^2$ map $\tilde{H}: \mathbb{R}^{2n+1} \rightarrow \mathbb{R}^{n+1}$ with $\tilde{H}^{-1}(0) = \tilde{F}$ and $\partial \tilde{H}$ of maximal rank in a neighborhood of $\tilde{F}$. 

PIECEWISE LINEAR APPROXIMATION 551
Proof. First note that a neighborhood of \( \tilde{F} \) in \( \mathbb{R}^{2n+1} \) is diffeomorphic to a neighborhood of \( F \) in \( N_{F}^{2n+1} \), e.g., using the explicit map \( g \) defined in the Introduction. Thus the proof of this lemma comes down to showing that \( N_{F}^{2n+1} \) is the trivial bundle.

Let \( v_{F}: N_{F}^{2n+1} \rightarrow F \) be the bundle projection. The condition that \( \partial H \) has maximal rank in a neighborhood of \( F \) is equivalent to \((v_{F}, \partial H)\) giving an explicit isomorphism \( N_{F}^{2n+1} \rightarrow F \times \mathbb{R}^{m} \). Thus we have

\[
C_{F}^{N} = T_{F}N_{F}^{2n+1} = T_{F}F \oplus N_{F}^{2n} = T_{F}F \oplus C_{F}^{m}.
\]

We also have the explicit orthogonal decomposition

\[
C_{F}^{2n+1} = T_{F}^{2n+1}F = T_{F}F \oplus N_{F}^{2n+1}.
\]

Thus we have

\[
N_{F}^{2n+1} \oplus C_{F}^{N} = N_{F}^{2n+1} \oplus T_{F}F \oplus C_{F}^{m} = C_{F}^{2n+1} \oplus C_{F}^{m}.
\]

Finally, it is a standard topological fact that a real vector bundle on a space, whose rank is greater than the dimension of the space, is trivial if the direct sum of the bundle and a trivial bundle is bundle isomorphic to the trivial bundle. QED

To give an explicit numerical construction of the map \( \tilde{H} \), we need to numerically construct an explicit numerical trivialization of \( N_{F}^{2n+1} \) given the decomposition from Eq. (5). This is done in the next section.

3. A RESULT FROM NUMERICAL TOPOLOGY

Let \( V \) be a differential real vector bundle of rank \( r \) on a compact \( n \)-dimensional differentiable manifold \( F \). Assume that \( r > n \) and that we have a bundle isomorphism \( \phi: C_{F}^{r} \oplus V \rightarrow C_{F}^{r+n} \). We will construct an explicit isomorphism \( \psi: V \rightarrow C_{F}^{r} \).

Remark 3.1. It is important to note that the condition that \( r > n \) is needed. For example, consider the unit sphere \( S^{2} \subset \mathbb{R}^{3} \). Since a global outward unit normal exists, the normal bundle \( N_{S^{2}}^{2} \subset \mathbb{R}^{3} \) is the trivial bundle \( C_{S^{2}}^{2} \). Thus we have that

\[
C_{S^{2}}^{2} = T_{S^{2}}|S^{2} = T_{S^{2}} \oplus C_{S^{2}},
\]

but since any vector field on \( S^{2} \) has a zero, the tangent bundle of \( S^{2} \) is not trivial.
The construction of $\psi$ comes down to tracking a homotopy. The homotopy, as we present it, works almost word for word for any compact CW-complex.

Let $E$ be a real vector bundle of rank $r \geq n+2$ on a compact manifold $F$. We need to know that given two distinct sections of $E$, then after replacing one of the sections by a random perturbation of it, the sections are everywhere linearly independent. Here we sketch a proof of this result adequate for our purposes. Note that for algebraic bundles, the open dense set can be taken to be Zariski open dense sets. The lemma formalizes what we mean by a random perturbation.

**Lemma 3.2.** Let $E$ be a real differentiable vector bundle of rank $r \geq n+2$ on a smooth compact connected $n$-dimensional manifold $F$. Let $\Gamma$ be a vector space of differentiable sections of $E$ with the property that the evaluation map $F \times \Gamma \to E$ is surjective. Then given two sections $s_1, s_2 \in \Gamma$, with $s_1$ nowhere zero, there is an open dense set $U \subset \Gamma$ such that $\Gamma - U$ has measure zero, and for $s \in U$, $s_1$ and $s_2 + s$ are everywhere linearly independent.

**Proof.** Let $E':= E/\mathbb{R}s_1$, let $\Gamma'$ denote the image of $\Gamma$ in the space of sections of $E'$, and let $s_2' \in \Gamma'$ denote the image of $s_2$ in $\Gamma'$. Let $r':= r-1 \geq n+1$ denote the rank of $E'$. It is clearly sufficient to show there is an open dense set $U' \subset \Gamma'$ such that $\Gamma' - U'$ has measure zero, and for $s \in U'$, $s_2' + s$ is nowhere zero. To see that this is so, let $Q \subset F \times \Gamma'$ denote the tautological space of zero sets of sections; i.e.,

$$Q := \{(x, t) \in F \times \Gamma' | t(x) = 0\}.$$ 

Note that $Q$ is nothing more than the kernel of the evaluation map

$$e: F \times \Gamma' \to E'.$$

Thus $Q$ is a differentiable manifold of dimension $n+\dim \Gamma' - r' \leq \dim \Gamma' - 1$. It follows that the image of $Q$ in $\Gamma'$ is contained in a set of measure zero, whose complement is open and dense. QED

We have the following lemma.

**Lemma 3.3.** Let $V$, $F$, $r$, $n$, $N$, and $\phi$ be as above. Let $\Gamma$ be a vector space of differentiable sections of $V$ with the property that the evaluation map $F \times \Gamma \to V$ is surjective. Assume that $\Gamma$ contains the sections $s_i, s_i'$ for $1 \leq i \leq N$, where

1. $s_i'$ denotes the section of $\mathcal{O}_F^N \oplus V$ which is the direct sum of the zero section of $V$ plus the zero sections of all components of $\mathcal{O}_F^N$ except the $i$th component, where it is the constant section $1$; and
2. \( s_i \) denotes the section of \( \mathcal{O}_F^{N+r} \) which is the direct sum of the zero sections of all components of \( \mathcal{O}_F^N \) except the \( i \)th component, where it is the constant section 1.

For \( i \) from 1 to \( N \) we can choose random perturbations \( \sigma_i \) of \( \phi(s_i') \), so that, for each \( t \in [0, 1] \), the subspace

\[
W_t := \langle t\sigma_1 + (1-t) s_1, \ldots, t\sigma_N + (1-t) s_N \rangle
\]

is \( N \)-dimensional and that \( (\mathcal{O}_F^N \oplus V)/W_t \cong V \) with \( 0 \oplus V \) going isomorphically onto \( V \).

**Proof.** Using Lemma 3.2 we can assume that by choosing an element \( p_1 \in \Gamma \) of arbitrarily small norm, and setting \( \sigma_1 := \phi(s_1') + p_1 \), it follows that \( s_i \) and \( \sigma_i \) are everywhere linearly independent. Thus, the subspace \( \langle t\sigma_1 + (1-t) s_1 \rangle \) is everywhere one dimensional. Moreover, for \( p_1 \) of sufficiently small norm, the quotient of \( \mathcal{O}_F^N \oplus V \) by \( R\sigma_1 \) is isomorphic to \( \mathcal{O}_F^{N-1} \oplus V \) with \( 0 \oplus \mathcal{O}_F^{N-1} \oplus V \) mapped isomorphically onto \( \mathcal{O}_F^{N-1} \oplus V \). Repeating this construction successively we get the lemma. QED

For \( i \) from 1 to \( N \), let \( e_{it} := t\sigma_i + (1-t) s_i \). Let \( A_t : \mathcal{O}_F^{N+r} \rightarrow \mathcal{O}_F^{N+r} \) be defined by

\[
A_t(v) = v - \sum_{i=1}^{N} (v, e_{it}) e_{it}.
\]

Note that \( A_t \) restricted to \( 0 \oplus V \) is the identity and that \( A_0 \) restricted to \( 0 \oplus \mathcal{O}_F^r \) is the identity. For each \( t \), \( A_t \) projects \( \mathcal{O}_F^{N+r} \) onto the orthogonal complement of \( W_t \). Note that by compactness and the above lemma, there is an \( \varepsilon > 0 \) such that \( A_{t+\varepsilon} \) gives an isomorphism of \( A_t(\mathcal{O}_F^{N+r}) \) with \( A_{t+\varepsilon}(\mathcal{O}_F^{N+r}) \). Thus there is an integer \( K_0 > 0 \) such that for all \( K \geq K_0 \)

\[
\mathcal{A}_K := \prod_{i=1}^{K} A_i : V \rightarrow \mathcal{O}_F^r
\]

is an isomorphism. This gives the desired trivialization of \( \mathcal{N}_{F,\mathbb{R}^{2n+1}} \).

**Remark 3.4.** In practice \( \Gamma \) will be the space of all sections of \( \mathcal{O}_F^N \oplus V \) restricted to a finite set. In [5, 6], a point on \( F \) is given for the algorithm to start. Here, and in what follows, we use the notation from the Introduction. So having such a point \( x \) on our \( F := H^{-1}(0) \), we have a point on \( \tilde{F} := \pi(F) \). We choose a sufficiently fine triangulation of \( \mathbb{R}^{2n+1} \). A subroutine
in the global piecewise linear algorithm determines a sufficiently small diameter for a starting \( N\)-simplex \( \sigma \) with its barycenter at the regular point \( x \in F \) which is transversal to the fiber \( F \). From this the algorithm produces the subsequent transversal simplices. These simplices yield the global piecewise linear approximation. Analogously, the same steps may be implemented for the corresponding projected point \( \tilde{x} := \pi(x) \) in the lower dimensional environment.

The vertices of the simplices meeting \( \tilde{F} \) are images under \( g \) of a finite set of points \( \tilde{F}' \subset \tilde{F} = \pi(F) \), and \( F' \) is the space of sections of \( \mathcal{M}_{\tilde{F}'/\mathbb{R}^{2n+1}} \) restricted to this finite set \( \tilde{F}' \). Points of \( \tilde{F}' \) are computed as needed. To be more precise, starting at a vertex \( e \) near \( \pi(x) \), the equation \( g(v) = e \), i.e., \( v + \nu_p(v) = e \), is solved on \( \mathcal{M}_{\tilde{F}'/\mathbb{R}^{2n+1}} \), where \( U \) is a neighborhood of \( \pi(x) \). Note that, using \( H \) and \( \pi \), this is an explicit system of equations. Now, the homotopy on \( \mathcal{M}_{\tilde{F}'/\mathbb{R}^{2n+1}} \) is carried out to compute \( \tilde{H}(e) := A K(v) \).

Remark 3.5. The numerical tracing of the above homotopy paths can be performed via the embedding method for a finite number of sufficiently small steps in the homotopy parameter. The continuous analogue for tracing \( H(x, t) = 0 \) is given by Davidenko’s equation: \( \frac{dx}{dt} = (\partial_t H)^{-1} \partial_x H \). Expositions on numerical tracing of homotopy paths can be found in [1, 2]. A further exposition on numerical piecewise numerical methods has recently appeared in [3]. There also can be found some estimations of the approximations of the PL manifold to the actual manifold in terms of the diameter and “thickness” of the simplices of the triangulation of the ambient space (which in our case is \( \mathbb{R}^{2n+1} \)) and the norms of the derivatives of the maps. We hope to obtain such estimates and to report on an implementation of our new procedure in a sequel.

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