New Quadrature Formulas Based on the Zeros of the Chebyshev Polynomials of the Second Kind

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Abstract—The aim of this work is to construct a new quadrature formula based on the divided differences of the integrand at points -1, 1 and the zeros of the n th Chebyshev polynomial of the second kind. The interesting thing is that this quadrature rule is closely related to the well-known Gauss-Turán quadrature formula and includes a recent result obtained by A.K. Varma and E. Landau as a special case. © 2003 Elsevier Science Ltd. All rights reserved.

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1. INTRODUCTION

Given a weight function w which is positive and integrable on the interval [-1, 1], the zeros -1 ≤ x_n < x_{n-1} < ⋅⋅⋅ < x_2 < x_1 ≤ 1 of the nth-degree orthogonal polynomial corresponding to w provide a quadrature

\[ \int_{-1}^{1} f(x)w(x)\,dx = \sum_{k=1}^{n} \lambda_k f(x_k) + E_n(f), \]  

which is exact (that is, \( E_n(f) = 0 \)) whenever \( f(x) \) is a polynomial of degree ≤ 2n - 1. Closely related to the above classical “Gaussian quadrature” is the quadrature rule based on the zeros -1 = x_2n < x_{2n-1} < ⋅⋅⋅ < x_1 < x_0 = 1 of the polynomial \( (1-x^2)P_n^{(a,b)}(x)P_n^{(a,b)'}(x) \), where \( P_n^{(a,b)}(x) \) is the nth Jacobi polynomial corresponding to the Jacobi weight \( w_{a,b}(x) = (1-x)^a(1+x)^b \) (see [1]). Specifically, Varma and Landau established in [1] the quadrature rule

\[ \int_{-1}^{1} f(x)w_{a,b}(x)\,dx = \sum_{k=0}^{2n} \mu_k f(x_k) + E_n(f), \]  

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which is exact for all polynomials of degree less than or equal to \(2n\), where \(\mu_k\) can be explicitly determined. Particularly interesting is the case when \(\alpha = \beta = -1/2\) and (1.2) reduces to the following [1, Remark 2]:

\[
\int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx = \frac{\pi}{2n} \left\{ \sum_{k=1}^{n} f(x_k) + \frac{1}{2} (f(1) + f(-1)) + \sum_{k=1}^{n} f(y_k) \right\},
\]

a quadrature formula of maximum degree of precision \(4n - 1\), where \(-1 \leq x_n < x_{n-1} < \cdots < x_2 < x_1 \leq 1\) and \(-1 \leq y_{n-1} < \cdots < y_2 < y_1 \leq 1\) are zeros of the \(n\)th-degree Chebyshev polynomial of the first kind \(T_n(x)\) and \((n-1)\)th-degree Chebyshev polynomial of the second kind \(U_{n-1}(x)\), respectively.

In this paper, we consider quadrature formulae based on certain divided differences of the integrand at the zeros of \((1 - x^2)U_n(x)\) (see Theorems 1.1 and 1.3, below). Since \(U_{2n-1}(x) = (1/2)T_n(x)U_{n-1}(x)\), we get (1.3) as a special case. We also obtain some corollaries. It seems that all these quadratures given here are new to our best knowledge.

This paper is organized as follows. We state our main results in the rest of this section. In Section 2, some auxiliary lemmas are presented. Finally, proofs for our results are given in the last section.

To state our results, we denote by \(\mathbb{N}\) the set of all natural numbers and \(\mathbb{P}_k\) the set of all algebraic polynomials of degree less than or equal to \(k\), \(k \in \mathbb{N}\) throughout this paper. For any real number \(a\), \([a]\) designates the greatest integer less than or equal to \(a\).

The main purpose of this paper is to obtain the following quadrature formulas.

**Theorem 1.1.** For \(n, s \in \mathbb{N}\), let us denote by

\[-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1\]

the zeros of \((1 - x^2)U_n(x)\). Let \(f \in \mathbb{P}_{2(s+1)n+2s-1}\). Then we have

\[
\int_{-1}^{1} f(x) (1 - x^2)^{1/2} \, dx = \frac{\pi}{n+1} \left\{ \sum_{k=1}^{n} (1 - x_k^2) f(x_k) + \sum_{j=1}^{s} \frac{(-1)^{j+1}}{4j(n+1)} \binom{2j}{j} \right. \left. \sum_{k=0}^{n+1} f^{[j]}[x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{2j}, x_k] \right\},
\]

where we adopt the customary notation \(f^{[j]}[x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{2j}, x]\) for the divided differences of \(f\) at the points \(x_0, x_1, \ldots, x_n, x_{n+1}, x\) (\(x\) may be identical to any one of \(x_k\)s, \(k = 0, \ldots, n+1\) with \(\xi^j\) meaning that the point \(\xi\) is repeated exactly \(j\) times and the primes on the summation indicate the two terms corresponding to \(k = 0\) and \(k = n + 1\) are halved.

**Corollary 1.2.** Let \(f \in \mathbb{P}_{2(s+1)n+2s-1}\). Then we have (with other symbols as in Theorem 1.1)

\[
\int_{-1}^{1} f(x) (1 - x^2)^{1/2} \, dx = \frac{\pi}{n+1} \left\{ \sum_{k=1}^{n} (1 - x_k^2) f(x_k) + \sum_{j=1}^{s} \frac{(-1)^{j+1}}{2j4j(n+1)} \binom{2j}{j} \right. \left. \times \left\{ f'[x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{2j}] + f[x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{2j}] \right\} \right\}.
\]
THEOREM 1.3. For \( n, s \in \mathbb{N} \), let us denote by

\[-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1\]

the zeros of \((1 - x^2)U_n(x)\). Let \( f \in \mathbb{P}_{2(s+1)n+2s+1} \). Other symbols are as in Theorem 1.1. Then we have

\[
\int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx = \frac{\pi}{n + 1} \sum_{j=0}^{s} \frac{(-1)^j}{4^j(n+1)} \binom{2j}{j} \sum_{k=0}^{n+1} f \left[ x_0, x_1^2, \ldots, x_n^2, x_{n+1}^2, x_k \right].
\] (1.6)

COROLLARY 1.4. Let \( n, s \in \mathbb{N} \), \( f \in \mathbb{P}_{2(s+1)n+2s+1} \). Other symbols are as in Theorem 1.1. Then we have

\[
\int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx = \frac{\pi}{n + 1} \left\{ \sum_{k=0}^{n+1} f(x_k) + \sum_{j=1}^{s} \frac{(-1)^j}{2^j4^j(n+1)} \binom{2j}{j} \right\}.
\] (1.7)

If \( s = 0 \) and \( n \) is replaced by \( 2n - 1 \), Theorem 1.3 yields the following.

COROLLARY 1.5. For \( n, s \in \mathbb{N} \), let us denote by

\[-1 = x_{n+1} < x_n < y_{n-1} < x_{n-1} < \cdots < y_1 < x_1 < x_0 = 1\]

the zeros of \(1/2(1 - x^2)U_{2n-1}(x) = (1 - x^2)T_n(x)U_{n-1}(x)\). Let \( f \in \mathbb{P}_{4n-1} \). Other symbols are as in Theorem 1.1. Then we have

\[
\int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx = \frac{\pi}{2n} \left\{ \sum_{k=0}^{n+1} f(x_k) + \sum_{k=1}^{n-1} f(y_k) \right\}.
\] (1.8)

That is (1.3). It is also interesting to mention that (1.4)-(1.7) are closely related to the Gauss-Turán quadrature rule. For the Gauss-Turán quadrature rule, see [2-5] and references therein.

2. AUXILIARY LEMMAS

Here we shall state some lemmas which will be needed in the proofs of our theorems.

LEMMA 2.1. Let \(-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1\) be the zeros of \((1 - x^2)U_n(x)\) and \( g \in C[1,1] \). Then we have

\[
\frac{1}{n + 1} \sum_{k=1}^{n} \left( 1 - x_k^2 \right) g[x_0, x_{n+1}, x_k] = \frac{1}{2} (g(x_0) + g(x_{n+1})) = -\frac{1}{n + 1} \sum_{k=0}^{n+1} g(x_k).
\] (2.1)

PROOF. By the well-known Newton's divided difference formula, we have

\[
g(x) = g(x_0) \frac{1 + x}{2} + g(x_{n+1}) \frac{1 - x}{2} + g(x_0, x_{n+1}, x_k) (x^2 - 1).
\]

Hence,

\[
(1 - x_k^2) g[x_0, x_{n+1}, x_k] = \frac{1}{2} (g(x_0) + g(x_{n+1})) = \frac{x_k}{2} (g(x_0) - g(x_{n+1})) - g(x_k).
\]

Summing over \( k \) from 1 to \( n \) and noting that \( \sum_{k=1}^{n} x_k = 0 \) since \( x_k = \cos k\pi/(n + 1) \), we arrive at

\[
\sum_{k=1}^{n} (1 - x_k^2) g[x_0, x_{n+1}, x_k] = \frac{n}{2} (g(x_0) + g(x_{n+1})) - \sum_{k=1}^{n} g(x_k).
\]

Adding \((-1/2)(g(x_0) + g(x_{n+1}))\) to each side of the above equation and simplifying, we obtain (2.1).
COROLLARY 2.2. Let \( s \in \mathbb{N} \) and \( f \) be a sufficiently differentiable function in \([-1, 1]\). Let \(-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1\) be the zeros of \((1 - x^2)U_n(x)\). Then we have

\[
\frac{1}{n+1} \sum_{k=1}^{n} (1 - x_k^2) f \left[ x_0^{j/2}, x_1^{j/2}, \ldots, x_{n+1}^{j/2} \right] k(x) \left( x^2 - 1 \right)^{j/2} \omega_n(x)^j + \frac{1}{2} \left\{ f \left[ x_0^{j-1}, x_1^{j-1}, \ldots, x_{n+1}^{j-1} \right] + f \left[ x_0^{j-1}, x_1^{j-1}, \ldots, x_{n+1}^{j-1} \right] \right\}
\]

(2.2)

PROOF. Setting \( g(z) := f [x_0^{j-1}, x_1^{j-1}, \ldots, x_{n+1}^{j-1}, x] \), we derive (2.2) from (2.1).

LEMMA 2.3. Let \(-1 = x_{n+1} < x_n < \cdots < x_1 < x_0 = 1\) be the zeros of \((1 - x^2)U_n(x)\) and \( s \in \mathbb{N} \). Let \( f \) be a sufficiently differentiable function in \([-1, 1]\). Then we have

\[
f(x) = \sum_{j=0}^{2s} \sum_{j=1}^{n} f \left[ x_0^{j/2}, x_1^{j/2}, x_2^{j/2}, \ldots, x_{n+1}^{j/2} \right] l_k(x) \left( x^2 - 1 \right)^{j/2} \omega_n(x)^j
\]

(2.3)

where

\[
R_{n,s}(f; x) = f \left[ x_0^{2s+1}, x_1^{2s+1}, \ldots, x_{n+1}^{2s+1} \right](x^2 - 1)^s \omega_n(x)^{2s+1},
\]

(2.4)

Remark 2.4. This lemma, which is an extension of a result in [3], is of its own interest and is still valid if the point set \( x_n < \cdots < x_1 \) is replaced by any other set of \( n \) points.

PROOF. Our proof is based on an idea in [3]. Let \( 1 = x_0 > x_1 > \cdots > x_n > x_{n+1} = -1 \) be the zeros of \((1 - x^2)\omega_n(x)\). Define

\[
D_j(f) := f \left[ x_0^{j/2}, x_1^{j/2}, \ldots, x_{n+1}^{j/2}, x \right].
\]

Let \( L_{n-1}(f; x) \) be the Lagrange interpolating polynomial for \( f \) with nodes \( x_1, \ldots, x_n \). Define a mapping \( L_{j,n+1} : C[-1, 1] \to \mathbb{P}_{n+1}, j \in \mathbb{N} \cup \{0\} \),

\[
L_{j,n+1}(f; x) = L_{n-1}(f; x) + \frac{1 + (-1)^{j+1}}{4} (D_j f(x_0)(1 + x) + D_j f(x_{n+1})(1 - x)) \omega_n(x).
\]

(2.5)

In other words, \( L_{j,n+1} \) is the Lagrange interpolation operator at nodes \( x_1 > \cdots > x_n \) if \( j \) is even and the Lagrange interpolation operator at nodes \( 1 = x_0 > x_1 > \cdots > x_n > x_{n+1} = -1 \) if \( j \) is odd. Applying Newton’s divided difference formula

\[
g(x) = L_{j,n+1}(g; x) + \tilde{D}_j g(x) \omega_n(x),
\]

where

\[
\tilde{D}_j g(x) = \begin{cases} g[x_1, \ldots, x_n, x], & \text{if } j \text{ is even}, \\ g[x_0, x_1, \ldots, x_n, x_{n+1}, x](x^2 - 1), & \text{otherwise}, \end{cases}
\]

consecutively to \( f, D_1 f, D_2 f, \ldots, D_{2s} f \), we obtain

\[
f(x) = \sum_{j=0}^{2s} L_{j,n+1}(D_j f; x) \left( x^2 - 1 \right)^{j/2} \omega_n(x)^j + D_{2s+1} f(x) \left( x^2 - 1 \right)^s \omega_n(x)^{2s+1}.
\]

(2.6)

A straightforward calculation together with (2.5) shows that (2.6) is equivalent to (2.3). This completes the proof.
LEMMA 2.5. Suppose that \( \xi_0, \xi_1, \ldots, \xi_n \in [-1, 1] \) are different. Let \( n, m_k \in \mathbb{N} \), \( k = 0, 1, \ldots, n \), and \( g \) be a sufficiently differentiable function in \([-1, 1]\). Then we have

\[
\sum_{k=0}^{n} m_k g[x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}, \xi_k] = g'[x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}] .
\]  

(2.7)

REMARK 2.6. It reduces to Lemma 1 in [3] when \( m_0 = m_1 = \cdots = m_n \). We therefore have \( \sum_{k=0}^{n} g[\xi_0, \xi_1, \ldots, \xi_n, \xi_k] = g'[\xi_0, \xi_1, \ldots, \xi_n] \). And it can be proved in a similar spirit to the proof in [3]. Here we use a different method.

PROOF. Proposition 95 in [6, Chapter 41] says

\[
g[x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}, \xi_k] = \frac{\prod_{v=0}^{n} D_{\xi_v}^{m_v} g[\xi_0, \xi_1, \ldots, \xi_n]}{m_k \prod_{v=0}^{n} (m_v - 1)!},
\]  

(2.8)

where \( D_{\xi_v} = \frac{\partial}{\partial \xi_v} \). If we choose \( m_0 = m_1 = \cdots = m_n = 1 \) in (2.8), we get

\[
D_{\xi_k} g[\xi_0, \xi_1, \ldots, \xi_n] = g[\xi_0, \xi_1, \ldots, \xi_n, \xi_k].
\]  

(2.9)

From (2.8) and (2.9), we have successively

\[
\sum_{k=0}^{n} m_k g[x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}, \xi_k] = \frac{\prod_{v=0}^{n} D_{\xi_v}^{m_v-1} g[\xi_0, \xi_1, \ldots, \xi_n]}{m_k \prod_{v=0}^{n} (m_v - 1)!}
\]

\[
= \frac{\prod_{v=0}^{n} D_{\xi_v}^{m_v-1} \left( \sum_{k=0}^{n} g[\xi_0, \xi_1, \ldots, \xi_n, \xi_k] \right)}{m_k \prod_{v=0}^{n} (m_v - 1)!}
\]

\[
= \frac{\prod_{v=0}^{n} D_{\xi_v}^{m_v-1} g'[\xi_0, \xi_1, \ldots, \xi_n]}{m_k \prod_{v=0}^{n} (m_v - 1)!}
\]

\[
= g'[x_0^{m_0}, x_1^{m_1}, \ldots, x_n^{m_n}] .
\]

(2.10)

LEMMA 2.7. For \( m \in \mathbb{N} \), we have (see [2])

\[
\left((1 - x^2)^{1/2} U_n(x)\right)^{2m} = \sum_{\ell=0}^{\infty} \beta_{m, \ell} T_{2\ell(n+1)}(x),
\]  

(2.10)

where \( \beta_{m,0} = (1/4^m)(2m)! \).

LEMMA 2.8. We have

\[
\int_{-1}^{1} p_{n-1}(x) U_n(x)^{2m+1} (1 - x^2)^{m+1/2} \, dx = 0, \quad p_{n-1} \in \mathbb{P}_{n-1}, \quad m \in \mathbb{N}.
\]  

(2.11)

PROOF. We obtain from (2.10) that

\[
\int_{-1}^{1} p_{n-1}(x) U_n(x)^{2m+1} (1 - x^2)^{m+1/2} \, dx
\]

\[
= \int_{-1}^{1} p_{n-1}(x) U_n(x) \left((1 - x^2)^{1/2} U_n(x)\right)^{2m} (1 - x^2)^{1/2} \, dx
\]

\[
= \beta_{m,0} \int_{-1}^{1} p_{n-1}(x) U_n(x) (1 - x^2)^{1/2} \, dx
\]

\[
+ \sum_{\ell=1}^{\infty} \beta_{m,\ell} \int_{-1}^{1} (1 - x^2) p_{n-1}(x) U_n(x) T_{2\ell(n+1)}(x) (1 - x^2)^{-1/2} \, dx.
\]
Since \( p_{n-1} \in \mathbb{P}_{n-1} \) and \((1 - x^2)p_{n-1}(x)U_n(x) \in \mathbb{P}_{2n+1}\), orthogonality shows
\[
\int_{-1}^{1} p_{n-1}(x)U_n(x) (1 - x^2)^{1/2} \, dx = 0,
\]
and
\[
\int_{-1}^{1} (1 - x^2)p_{n-1}(x)U_n(x)T_{2l(n+1)}(x) (1 - x^2)^{-1/2} \, dx = 0, \quad l \in \mathbb{N}.
\]
Now the desired conclusion (2.11) follows easily.

**Lemma 2.9.** We have
\[
\int_{-1}^{1} l_k(x)U_n(x)^2 (1 - x^2)^{m+1/2} \, dx = \frac{\pi}{4^n(n + 1)} \binom{2m}{m} (1 - x_k^2), \quad m \in \mathbb{N}. \tag{2.12}
\]

**Proof.** From (2.10), orthogonality, and the well-known fact
\[
\int_{-1}^{1} l_k(x) (1 - x^2)^{1/2} \, dx = \frac{\pi}{n+1} (1 - x_k^2),
\]
we have
\[
\int_{-1}^{1} l_k(x)U_n(x)^2 (1 - x^2)^{m+1/2} \, dx = \int_{-1}^{1} l_k(x) \left( (1 - x^2)^{1/2} U_n(x) \right)^2 (1 - x^2)^{1/2} \, dx
\]
\[
= \beta_m,0 \int_{-1}^{1} l_k(x) (1 - x^2)^{1/2} \, dx
\]
\[
+ \sum_{l=1}^{\infty} \beta_m,l \int_{-1}^{1} (1 - x^2) l_k(x)T_{2l(n+1)}(x) (1 - x^2)^{-1/2} \, dx
\]
\[
= \frac{\pi}{4^n(n + 1)} \binom{2m}{m} (1 - x_k^2). \tag*{\blacksquare}
\]

**Lemma 2.10.** We have
\[
\int_{-1}^{1} (1 \pm x)U_n(x)^2 (1 - x^2)^{m-1/2} \, dx = \frac{\pi}{4^n} \binom{2m}{m}, \quad m \in \mathbb{N}. \tag{2.13}
\]

**Proof.** A straightforward calculation using (2.10) and orthogonality yields
\[
\int_{-1}^{1} (1 \pm x)U_n(x)^2 (1 - x^2)^{m-1/2} \, dx
\]
\[
= \int_{-1}^{1} (1 \pm x) \left( (1 - x^2)^{1/2} U_n(x) \right)^2 (1 - x^2)^{-1/2} \, dx
\]
\[
= \beta_m,0 \int_{-1}^{1} (1 \pm x) (1 - x^2)^{-1/2} \, dx + \sum_{l=1}^{\infty} \beta_m,l \int_{-1}^{1} (1 \pm x)T_{2l(n+1)}(x) (1 - x^2)^{-1/2} \, dx
\]
\[
= \beta_m,0 \int_{0}^{\pi} (1 \pm \cos \theta) \, d\theta = \frac{\pi}{4^n} \binom{2m}{m}. \tag*{\blacksquare}
\]

**3. PROOFS OF THEOREMS**

We are now ready to prove our main results.
3.1. Proof of Theorem 1.1

**Proof.** Integrating both sides of (2.3) against the weight function $(1 - x^2)^{1/2}$ over the interval $[-1, 1]$, recalling (2.4), and noting that $\omega_n(x) = 2^{-n}U_n(x)$, we have successively, by using (2.11)-(2.13) and (2.2),

$$
\int_{-1}^{1} f(x) (1 - x^2)^{1/2} \, dx = \sum_{j=0}^{n} \sum_{k=1}^{n} 2^{-jn} f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] 
$$

\[
\times \int_{-1}^{1} \prod_{k=1}^{n} (1 - x^2)^{1/2} \, dx 
+ \frac{1}{2} \sum_{j=1}^{s} 2^{-jn} \left\{ f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \right\} \left( 1 + x \right) \left( 1 - x^2 \right)^{1/2} \, dx 
+ \int_{-1}^{1} \prod_{k=1}^{n} (1 - x^2)^{1/2} \, dx
\]

\[= \sum_{j=0}^{n} \sum_{k=1}^{n} \left[ \frac{(-1)^j}{4^n} \right] f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \int_{-1}^{1} \prod_{k=1}^{n} (1 - x^2)^{1/2} \, dx 
- \frac{1}{2} \left\{ f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] + f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \right\} \left( 1 + x \right) \left( 1 - x^2 \right)^{1/2} \, dx 
+ \int_{-1}^{1} \prod_{k=1}^{n} (1 - x^2)^{1/2} \, dx
\]

$$
= \frac{\pi}{n+1} \sum_{k=1}^{n} (1 - x_k^2) f(x_k) 
+ \frac{\pi}{n+1} \left\{ \sum_{k=1}^{n} (1 - x_k^2) f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \right\} 
- \frac{1}{2} \left\{ f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] + f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \right\} \left( 1 + x \right) \left( 1 - x^2 \right)^{1/2} \, dx 
+ \int_{-1}^{1} \prod_{k=1}^{n} (1 - x^2)^{1/2} \, dx
\]

where

$$
E_{n,s}(f) := \int_{-1}^{1} R_n,f(x) (1 - x^2)^{1/2} \, dx 
= \frac{1}{2(2s+1)n} \int_{-1}^{1} f \left[ x_0^s, x_1^s, \ldots, x_n^s, x_{n+1}, x \right] (x^2 - 1)^s U_n(x) (1 - x^2)^{1/2} \, dx.
$$

If $f \in \mathbb{P}_{2(s+1)n+2s-1}$, then $f \left[ x_0^s, x_1^s, \ldots, x_n^s, x_{n+1}, x \right] \in \mathbb{P}_{n-1}$. It follows from (2.11) that $E_{n,s}(f) = 0$.

3.2. Proof of Corollary 1.2

**Proof.** For $1 \leq j \leq s$, $s \in \mathbb{N}$, it is easy to check that

$$
\sum_{k=0}^{n+1} \left\{ f \left[ x_0^{j-1}, x_1^{j-1}, \ldots, x_n^{j-1}, x_{n+1}, x_k \right] + f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1}, x_k \right] \right\} 
$$

$$
+ f \left[ x_0^j, x_1^j, \ldots, x_n^j, x_{n+1} \right] + f \left[ x_0^{j-1}, x_1^{j-1}, \ldots, x_n^{j-1}, x_{n+1} \right] \right\},
$$
where
\[ m_{k,j} = \begin{cases} j - 1, & \text{if } k = 0 \text{ or } n + 1, \\ 2j, & \text{if } k = 1, \ldots, n. \end{cases} \]

It follows from Lemma 2.5 that
\[ \sum_{k=0}^{n+1} m_{k,j} f\left( x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{j-1}, x_k \right) = \sum_{k=0}^{n+1} f'\left( x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{j-1} \right). \tag{3.2} \]

Now Corollary 1.2 follows from Theorem 1.1 and (3.1),(3.2).

### 3.3. Proof of Theorem 1.3

**Proof.** We have by Newton’s divided difference formula that
\[ f(x) = \frac{1}{2} \left( f(x_0)(1 + x) + f(x_{n+1})(1 - x) \right) + f[x_0, x_{n+1}, x] (x^2 - 1). \]

Integrating both sides of the above equation with respect to weight \((1 - x^2)^{-1/2}\) and noting \((2.13)\), we obtain
\[ \int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx = \frac{1}{2} \left\{ f(x_0) \int_{-1}^{1} (1 + x) (1 - x^2)^{-1/2} \, dx + f(x_{n+1}) \right\} \times \int_{-1}^{1} (1 - x) (1 - x^2)^{-1/2} \, dx \]
\[ = \int_{-1}^{1} f[x_0, x_{n+1}, x] (1 - x^2)^{1/2} \, dx \]
\[ = \frac{\pi}{2} (f(x_0) + f(x_{n+1})) - \int_{-1}^{1} f[x_0, x_{n+1}, x] (1 - x^2)^{1/2} \, dx. \tag{3.3} \]

Now replacing \(f(x)\) in Theorem 1.1 by \(f[x_0, x_{n+1}, x]\) and a straightforward calculation, we see that Theorem 1.3 follows from Theorem 1.1 and Lemma 2.1 since \(f[x_0, x_{n+1}, x] \in \mathbb{P}_{2(s+1)n+2s-1}\) if \(f \in \mathbb{P}_{2(s+1)n+2s+1}\).

### 3.4. Proof of Corollary 1.4

**Proof.** The proof of Corollary 1.4 is similar to that of Corollary 1.2. We conclude from Theorem 1.3 and Lemma 2.5 that
\[ \int_{-1}^{1} f(x) (1 - x^2)^{-1/2} \, dx \]
\[ = \frac{\pi}{n + 1} \left\{ \sum_{k=0}^{n+1} f(x_k) + \sum_{j=1}^{s} \frac{(-1)^j}{j! (n+1)} \sum_{k=0}^{n+1} f'\left[ x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{j}, x_k \right] \right\} \]
\[ = \frac{\pi}{n + 1} \left\{ \sum_{k=0}^{n+1} f(x_k) + \sum_{j=1}^{s} \frac{(-1)^j}{2j! (n+1)} \sum_{k=0}^{n+1} f'\left[ x_0^{j-1}, x_1^{2j}, \ldots, x_n^{2j}, x_{n+1}^{j} \right] \right\}. \tag{3.3} \]

### REFERENCES