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MINIMAL TEST PATTERNS FOR CONNECTIVITY PRESERVATION IN PARALLEL THINNING ALGORITHMS FOR BINARY DIGITAL IMAGES

Christian RONSE
Philips Research Laboratory, B-1170 Brussels, Belgium

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In successive deletion stages of parallel thinning algorithms for binary digital images, one usually checks the preservation of connectivity by verifying that: (a) every removed pixel is individually deletable without modifying connectivity (well-known criteria, such as those of Rosenfeld and Yokoi, exist for that purpose); (b) every pair of 8-adjacent removed pixels is deletable without connectivity modification. In the case of the 8-connectivity for the figure (and the 4-connectivity for the background), two more patterns must be tested for connectivity preservation: an isolated triple or quadruple of mutually 8-adjacent pixels.

In this paper we give a formal characterization of these patterns for testing connectivity preservation by what we call minimal non-x-deletable sets (x-MND sets), where $x=4$, 8, or $\{4,8\}$ (the type of connectivity considered for the figure). A parallel thinning algorithm whose deletion stage cannot remove an x-MND set is guaranteed to preserve the connectivity properties of any figure. We show that an x-MND set consists in either (1) a single pixel; or (2) a pair of 8-adjacent pixels; or (3) an isolated triple or quadruple of mutually 8-adjacent pixels (for $x=8$ only).

Keywords. Binary digital images, thinning, connectivity preservation, parallel algorithms, deletability, strong deletability, minimal nondeletable sets.

1. Introduction

Many thinning algorithms for binary digital images have been published, and still continue to appear in the literature. (We do not intend to survey them.) Their aim is to reduce a figure in a digital binary picture to a one-pixel thick sketch, called a skeleton, by removing from it successive layers of pixels. The skeleton must satisfy some constraints concerning its similarity with the original figure. They are mainly of two types [1]:

(1) *Geometrical*: the skeleton must maintain the overall shape of the figure.

(2) *Topological*: the skeleton must preserve the connectivity properties of the figure and of its complement in the picture.

Geometrical constraints have always been stated in vague, intuitive terms. This vagueness can be considered as inherent if one takes into account the point of view of mathematical morphology [10]. Indeed, shape can be described in terms of basic templates called *structuring elements*. A skeleton of the figure can be obtained by taking reference pixels of all maximal structuring elements that fit into the figure [2, 3, 10]. Now there is no "objective" choice for the shape of structuring elements. For example, if one considers square structuring elements, a square will have a skeleton reduced to a single pixel; on the other hand, if one takes circular structuring elements, then that square will have a skeleton formed by its two diagonals.

On the other hand, *topological* constraints can be characterized in a precise mathematical way. This was done in [5], and we will return to it later.

Once topological and geometrical constraints have been defined, their combination can be achieved in two ways. First one can try to incorporate them into the criterion for removing a pixel in a thinning algorithm; this is what is done in most cases. Second, one can adopt the methodology proposed by Davies and Plummer [1]: one builds a geometrical skeleton (for example the distance skeleton) and mark its pixels as "nondeletable", then one applies a thinning algorithm satisfying the topological constraints, with the further condition that pixels marked "nondeletable" cannot be removed.

In this paper, which is in some way a continuation of [5], we do not consider geometrical constraints, but only topological ones. The problem that we consider is the following. There are two types of thinning algorithms: sequential ones and parallel ones. Both types consist in a repetition of deletion stages in which successive layers of pixels are removed, until further applications of that deletion stage do not remove new pixels. In a deletion stage of a sequential algorithm, one applies a removal criterion to every pixel in succession. When a pixel is removed from the figure, this is done before applying the criterion to the next pixel. Therefore the connectivity preservation in such an algorithm reduces to the connectivity preservation for the deletion of a single pixel, and there are well-known criteria for this [6, 11, 12].

On the other hand, in a deletion stage of a parallel thinning algorithm, one applies a removal criterion to every pixel, and then all pixels satisfying it are removed *at the same time*. Thus the property of connectivity preservation for the algorithm does not reduce to that property for the deletion of a single pixel, since usually several pixels are simultaneously removed. One needs thus criteria for connectivity preservation of the deletion of a group of pixels.

In most papers, authors test the connectivity preservation of their algorithm as follows. First every pixel satisfying the algorithm's removal criterion must be individually deletable without modifying the connectivity for the figure and background, as in a sequential algorithm (we mentioned above that there are well-known criteria for this [6, 11, 12]). Second, any pair of 8-adjacent pixels satisfying that algorithm's removal criterion must be deletable without connectivity modification. Note however that in the case where we consider the 8-connectivity for the figure, an algorithm satisfying these two conditions can nevertheless remove an isolated triple or quadruple of pairwise 8-adjacent pixels in the figure (this fact tends to be overlooked by authors of particular algorithms appearing in the literature).

Such a method for testing connectivity preservation in a parallel thinning algorithm is empirical, and we need to have a precise formal criterion for this purpose, and to provide a mathematical justification for it. In [8] Rosenfeld characterized parallel thinning algorithms based on successive stages deleting N, W, S, and E pixels respectively; in other words he analyzed parallel thinning algorithm steps in which all removed pixels must have their neighbor in a given axial direction (N, W, S, or E) belonging to the background. In particular, he gave a criterion for the connectivity preservation for the deletion of a group of pixels each having their neighbor in that direction belonging to the background. However our aim is to give such a criterion for any parallel thinning algorithm, without assumption of a decomposition into N, W, S, and E deletion stages.

Our characterization in [5] of the topological requirements of thinning allows us to solve this problem. Given a figure F in a digital grid G, and the background $B=G \setminus F$, we suppose that in a deletion stage a subset D of F is removed from it. For technical reasons (related to finiteness) we assume that G contains a rectangular frame ϕG such that ϕG and the set of all pixels outside it either are included in F, or are included in B, and D must be in the interior of ϕG (see the next section for more details). We must ensure that F and $F \setminus D$ have the same "topology". This can be envisaged in two ways [5].

First we can simply require that F has the same number of connected components as $F \setminus D$, and B has the same number of connected components as $B \cup D$. We say then that D is *deletable* from F. In fact, according to the type of connectivity that we consider for the figure F (the 4- or 8-connectivity, or both), we say that D is *x-deletable*, where $x=4$, 8, or $\{4,8\}$ (see Definitions 2.1 and 2.4 below).

Second, we may require a stronger assumption, namely that the set inclusion relation $\subseteq$ induces a bijection from the set of connected components of $F \setminus D$ to the set of connected components of F, and from the set of connected components of B to the set of connected components of $B \cup D$. We say then that D is *strongly deletable* from F, or more precisely that D is *strongly x-deletable*, where $x=4$, 8, or $\{4,8\}$, according to the type of connectivity that we consider for F (see Definitions 2.2 and 2.4 below).

Of course, strong deletability is a stronger property than deletability, but when D is 4-connected, and in particular when D consists in a single pixel, they are equivalent (see Lemma 2.3 below).

In [5] we argued that strong deletability of D, rather than only its deletability, is the appropriate requirement for the preservation of connectivity properties. One important fact in favour of this view is the following result that we proved there (see also Proposition 2.5 below): D is strongly x-deletable from F iff $D=\{p_1,\dots,p_t\}$, where for $i=1,\dots,t$ pixel p_i is deletable from $F \setminus \{p_j \mid j<i\}$. In other words D is strongly x-deletable iff it can be removed from F by successively removing x-deletable individual pixels: this is what happens in most thinning algorithms. As we noted there, a consequence of this result is that when D is strongly x-deletable from F, then

F and $F \setminus D$ have the same *adjacency tree* [4,7], and as we explained in [4], we can then consider that they have the same topology.

Suppose now that D is the set of pixels removed by a deletion stage in a parallel thinning algorithm. If D is not deletable, then consider the smallest $U \subseteq D$ such that U is not deletable. Such a set U is not deletable, but every proper subset of it is deletable, and we call it a *minimal nondeletable* set. An interesting fact is that this property is independent of the choice of the deletability or strong deletability for U or for its subsets (see Corollary 3.2). However, it depends upon the choice of the connectivity for F, and so we speak of a *minimal non-x-deletable* set, or in brief an *x-MND* set, where $x=4$, 8, or $\{4,8\}$ (see Definition 3.3).

Clearly D is strongly x-deletable whenever it does not contain an x-MND set (otherwise we take the smallest $U \subseteq D$ which is not strongly x-deletable, and then U is a x-MND set). Thus x-MND sets are the patterns which must be tested for connectivity preservation of a deletion stage in a thinning algorithm. We show in Theorem 3.5 that an x-MND set consists in a single pixel, a pair of 8-adjacent pixels, or for $x=8$ an isolated triple or quadruple of mutually 8-adjacent pixels.

A deletion stage of a parallel thinning algorithm, which cannot remove x-MND sets from a figure, will in fact remove from it a strongly x-deletable sets of pixels, and so a succession of such stages will also remove a strongly x-deletable set. Thus this algorithm will preserve the connectivity of the figure. Hence we give a mathematical justification to the fact that, in a parallel thinning algorithm, the connectivity preservation can be verified by checking that every pixel or pair of 8-adjacent pixels satisfying the removal criterion of the algorithm is x-deletable, and moreover that for $x=8$ an isolated triple or quadruple of mutually 8-adjacent pixels cannot vanish. In particular this explains the usual empirical test mentioned above, which is correct, but incomplete for the 8-connected case.

Well-known criteria exist for checking the deletability of a pixel [6,11,12]. Now a pair $\{p,q\}$ of deletable pixels is deletable from F iff p is deletable from $F \setminus \{q\}$. We can thus very easily check whether a parallel thinning algorithm preserves connectivity.

2. Preliminaries

Let us give here the general background about binary digital images and recall the main definitions and results from [5].

Let G be a rectangular or quadruled *grid* consisting of *pixels*. Two adjacency relations can be defined on G, the *4-adjacency* and the *8-adjacency* [6,9]. For $k=4$ or 8, the k-adjacency leads to the well-known notions of *k-connected path*, *k-connectedness*, and to the partition of any subset of G into its *k-connected components* [6,9].

A *two-tone* or *binary image* on G is a map $G \to \{0,1\}$. Pixels mapped onto 1 are painted black and those mapped onto 0 are painted white. We call the set of black pixels of G the *figure* and write it F; on the other hand, the set of white pixels of G will be called the *background* and written B. Clearly $B = G \setminus F$.

For $k = 4$, 8, let $k' = 12 - k$. It is well-known [4,6] that if k-connectedness is used for F, then the opposite one, namely the k'-connectedness must be used for B.

The k-adjacency graph [4,7] of the image F has as vertices the k-connected components of F and the k'-connected components of B, and has as edges the pairs constituted by a k-connected component of F and a k'-connected component of B which are 4-adjacent (in fact they are 4-adjacent whenever they are 8-adjacent).

Although the grid G can be infinite, we will work only in a finite portion of it. We assume thus that all 4- and 8-connected components of F and B are enclosed in a bounded region, except one. Thus there is a rectangular frame ϕG such that ϕG and the portion of G surrounding it are either both included in F or both included in G. In [5] we called this requirement the *restricted frame assumption*, or in brief the *RFA*. Then the adjacency graph becomes a tree with the connected component containing ϕG on top, and whenever two vertices are adjacent, the one on the top of them corresponds to a connected component surrounding the other one [7]. We will not make any transformation outside ϕG, and so we can assume that ϕG is the boundary of G. Write $G°$ for the *interior* of G, in other words the set of pixels surrounded by ϕG. Note that $G°$ is finite.

We consider a subset D of F, and contemplate the deletion of D from F in thinning. The topological structure of F is characterized by the set of k-connected components of F, the set of k'-connected components of B, their adjacency and surrounding relations, and the relation of ϕG (and its surroundings) to F and B [4,7]. The preservation of the relation of ϕG (and its surroundings) to F and B means simply that when we delete D from F, pixels in ϕG (and around) may not change their tone. In other words we must have $D \subseteq G°$. Now the preservation of black and white connected components can be understood in two ways:

Definition 2.1 [5]. Let $D \subseteq F$ and $k = 4$, 8. Then we say that D is *k-deletable from F* if
(i) $D \subseteq G°$,
(ii) F has the same number of k-connected components as $F \setminus D$, and
(iii) B has the same number of k'-connected components as $B \cup D$.

Definition 2.2 [5]. Let $D \subseteq F$ and $k = 4$, 8. Then we say that D is *strongly k-deletable from F* if:
(i) $D \subseteq G°$;
(ii) the containment relation $\supseteq$ induces a bijection from the set of k-connected components of F to the set of k-connected components of $F \setminus D$ (in other words every k-connected component of F contains a unique k-connected component of $F \setminus D$ and every k-connected component of $F \setminus D$ is contained in a unique k-connected component of F); and
(iii) the inclusion relation $\subseteq$ induces a bijection from the set of k'-connected components of B to the set of k'-connected components of $B \cup D$ (in other words every

k'-connected component of B is contained in a unique k'-connected component of $B \cup D$ and every k'-connected component of $B \cup D$ contains a unique k'-connected component of B).

Note that the condition $D \subsetneq G^\circ$ implies in particular that D is finite. In the strong deletability, we have thus a direct one-to-one correspondence between the black and white connected components in the two binary images (before and after the deletion of D). Clearly strong deletability is a stronger property than deletability. However we have the following:

Lemma 2.3 [5]. *Let $D \subsetneq F$ and $k=4$, 8. If D is 4-connected (and in particular if D contains a single pixel), then D is k-deletable from F iff it is strongly k-deletable from F.*

We do not exclude the case where we consider both the 4- and 8-adjacencies on F. We make thus the following:

Definition 2.4 [5]. Let $D \subsetneq F$. Then we say that D is (*strongly*) $\{4,8\}$*-deletable from* F if it is both (strongly) 4-deletable from F, and (strongly) 8-deletable from F.

In [5] we argued that strong deletability is the correct requirement for the connectivity preservation of the deletion of D from F. We proved the following characterization of strongly deletable sets:

Proposition 2.5 [5]. *Let $D \subsetneq F \cap G^\circ$, $t=|D|$, and let $x=4$, 8 or $\{4,8\}$. Assume that $t>1$. Then the following two statements are equivalent:*

(i) D is strongly x-deletable from F.

(ii) The elements of D can be labelled $p_1, \dots, p_t$, in such a way that for $i=1,\dots,t$ pixel p_i is x-deletable from $F \setminus \{p_j \mid j<i\}$.

One consequence of this result is that F and $F \setminus D$ have the same adjacency tree. Indeed, when we remove a deletable pixel p from F, a connected component of $F \setminus \{p\}$ is adjacent to a connected component of $B \cup \{p\}$ iff the corresponding connected component of F is adjacent to the corresponding connected component of B; we iterate this argument with the sequence of deletable pixels $p_1, \dots, p_t$. As explained in [4], this adjacency tree characterizes the topology of F.

Another consequence of this result is that it links the requirement of strong deletability to the basic methodology underlying thinning, that is the successive deletion of individual pixels from the figure.

In the proof of Proposition 2.5 we obtained two lemmas which will be fundamental here:

Lemma 2.6 [5]. *Let $D \subseteq F \cap G^\circ$ and $x=4$, 8 or $\{4,8\}$. Suppose that $D=D' \cup D''$, where $D' \cap D'' = \emptyset$. Then any two of the following three statements imply the third:*
(i) *D' is x-deletable from F.*
(ii) *D'' is x-deletable from $F \setminus D'$.*
(iii) *D is x-deletable from F.*

Lemma 2.7 [5]. *Let $D \subseteq F \cap G^\circ$ and $x=4$, 8 or $\{4,8\}$. Suppose that $D=D' \cup D''$, where $D' \cap D'' = \emptyset$, D' is x-deletable from F, D'' is x-deletable from $F \setminus D'$, and so D is x-deletable from F (see Lemma 2.6). Then the following two statements are equivalent:*
(i) *D is strongly x-deletable from F.*
(ii) *D' is strongly x-deletable from F and D'' is strongly x-deletable from $F \setminus D'$.*

As strong deletability reduces to the existence of a sequence of deletable pixels, it is important to give deletability criteria for individual pixels:

Lemma 2.8 [5]. *Let $p \in F \cap G^\circ$ and $k=4$, 8. Then p is k-deletable from F iff there is exactly one k-connected component of $F \setminus \{p\}$ which is k-adjacent to p and there is exactly one k'-connected component of B which is k'-adjacent to p.*

An interesting fact is that the k-deletability of a pixel p depends only upon the configuration of black and white pixels in the 3×3-neighborhood centered on it. For $k=4$ or 8, let $N_k(p)$ be the set of pixels k-adjacent to p. The following result was shown by Rosenfeld under the assumption that B is k'-connected, but his proof does not require it:

Lemma 2.9 [6]. *Let $p \in F \cap G^\circ$ and $k=4$, 8. Then p is k-deletable from F iff $F \cap N_k(p) \neq \emptyset$, $B \cap N_{k'}(p) \neq \emptyset$, and $F \cap N_8(p)$ has exactly one k-connected component which is k-adjacent to p. In other words:*
(a) *p is 4-deletable from F iff $F \cap N_4(p) \neq \emptyset$, $B \cap N_8(p) \neq \emptyset$, and $F \cap N_8(p)$ has a 4-connected component containing $F \cap N_4(p)$.*
(b) *p is 8-deletable from F iff $F \cap N_8(p) \neq \emptyset$, $B \cap N_4(p) \neq \emptyset$, and $F \cap N_8(p)$ is 8-connected.*

The k-deletability of a pixel p can be tested by computing Yokoi's *k-connectivity numbers*: p is k-deletable from F iff its k-connectivity number is equal to 1 (see [11, 12] for more details).

Now that we have recalled all previous results that will be needed, we can solve our problem and characterize the patterns to be tested for the connectivity preservation of a deletion stage in a parallel thinning algorithm.

3. Minimal nondeletable sets

We suppose that a parallel thinning algorithm is applied to a figure F, and that during a stage of it, pixels in a finite nonvoid subset D of $F \cap G^\circ$ are marked for deletion, and then removed together from F. In order to preserve the topology of F, the set D should be x-deletable, and even strongly x-deletable, where $x=4$, 8 or $\{4,8\}$ (the adjacency taken into account on F).

However we do not know the a priori shape of D for an arbitrary figure F. We can nevertheless describe "forbidden features" that D cannot contain. Indeed, if D is not (strongly) k-deletable from F, then it contains a smallest nonvoid subset U which is not (strongly) k-deletable. Thus U is not (strongly) k-deletable from F, but every proper subset of U is. Then U will be called a minimal non-x-deletable subset of F. We will characterize such sets, but we must beforehand show that their definition does not depend on whether we consider k-deletability or strong k-deletability:

Lemma 3.1. *Let $V \subseteq F \cap G^\circ$, with $V \neq \emptyset$, and $k=4$, 8. Then the following two statements are equivalent:*

(i) *Every nonvoid subset of V, including V itself, is k-deletable from F.*

(ii) *Every nonvoid subset of V, including V itself, is strongly k-deletable from F.*

Proof. Clearly (ii) implies (i). We show the converse by induction on the size of a subset of V. We assume that every nonvoid subset of V (including itself) is k-deletable from F. Let $W \subseteq V$, with $W \neq \emptyset$. If $|W|=1$, then by Lemma 2.3 the k-deletability of W implies its strong k-deletability.

Suppose now that $|W|=t>1$, and that every $W' \subseteq V$ such that $0<|W'|<t$, is strongly k-deletable from F. Take $p \in W$ and $W'=W \setminus \{p\}$. As $W=W' \cup \{p\}$ and W' *are* k-deletable, Lemma 2.6 implies that $\{p\}=W \setminus W'$ is k-deletable from $F \setminus W'$. Now $\{p\}$ is strongly k-deletable from $F \setminus W'$ by Lemma 2.3, and the induction hypothesis states that W' is strongly k-deletable from F. Then Lemma 2.7 implies that $W=W' \cup \{p\}$ is strongly k-deletable from F. By induction, every nonvoid subset of V, including itself, is strongly k-deletable from F. □

Corollary 3.2. *Let $U \subseteq F \cap G^\circ$, with $U \neq \emptyset$, and $x=4$, 8, or $\{4,8\}$. Then:*

(a) *The following two statements are equivalent:*

  (i) *Every nonvoid proper subset of U is x-deletable from F.*

  (ii) *Every nonvoid proper subset of U is strongly x-deletable from F.*

(b) *If U satisfies* (i) *and* (ii), *then the following two statements are equivalent:*

  (iii) *U is x-deletable from F.*

  (iv) *U is strongly x-deletable from F.*

Proof. The equivalence between (i) and (ii) follows by applying Lemma 3.1 to every nonvoid proper subset V of U for any $k=4$, 8 intervening in x. Now if we apply Lemma 3.1 to U for any $k=4$, 8 intervening in x, we obtain that (i) and (iii) together are equivalent to (ii) and (iv) together. Thus if (i) and (ii) hold, then (iii) is equivalent to (iv). □

Thanks to this last result, we can now define minimal nondeletable subsets of F without ambiguity concerning the choice of ordinary or strong deletability:

Definition 3.3. Let $U \subseteq F \cap G^\circ$, with $U \neq \emptyset$, and $x = 4$, 8 or $\{4,8\}$. Then we say that the set U is *minimal non-x-deletable from F* (or in brief, that U is *x-MND from F*) if U is not (strongly) x-deletable from F, but every nonvoid proper subset of U is (strongly) x-deletable from F.

Note that in this definition, we put the word "strongly" between parentheses in order to mean that the definition does not change whether we consider x-deletability or strong x-deletability for nonvoid proper subsets of U or for U itself.

It is clear that if a nonvoid subset D of $F \cap G^\circ$ is not strongly x-deletable from F, then the smallest nonvoid subset of D (possibly D itself) which is not strongly x-deletable from F is an x-MND subset from F. Thus if we can characterize the possible shapes for an x-MND set, then we can garantee that D is strongly x-deletable whenever it does not contain some particular patterns. We will classify x-MND sets in Theorem 3.5, after the following preliminary result:

Lemma 3.4. *Let a_1, a_2, a_3, a_4 be four mutually 8-adjacent pixels of G° forming a square with diagonals $\{a_1, a_3\}$ and $\{a_2, a_4\}$. Let H be a figure such that $a_1, a_3, a_4 \in H$ and $a_2 \notin H$. Then for any $k = 4$, 8, a_3 is k-deletable from H iff it is k-deletable from $H \setminus \{a_1\}$.*

Proof. We show below $N_8(a_3)$ (up to a symmetry):

$$\begin{array}{ccc} a_1 & a_4 & \cdot \\ a_2 & a_3 & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

Then $N_8(a_3)$ forms the following two configurations of black and white pixels in H and $H \setminus \{a_1\}$ respectively, where • represents a black pixel, ○ a white one, and * one which can be either black or white:

$$\begin{array}{ccc} \bullet & \bullet & * \\ \circ & a_3 & * \\ * & * & * \end{array} \qquad \begin{array}{ccc} \circ & \bullet & * \\ \circ & a_3 & * \\ * & * & * \end{array}$$

A k-connected path in $H \cap N_8(a_3)$ which connects to a_4 a pixel $b \in H \cap N_k(a_3)$, distinct from a_1 and a_4, cannot pass through a_2 (since $a_2 \notin H$), and so it does not pass through a_1 (since a_2 is the only pixel in $N_8(a_3)$, apart from a_4, which is k-adjacent to a_1). Thus every $b \in H \cap N_k(a_3)$, either is a_1, and so is k-adjacent to a_4, or is in $H \setminus \{a_1\}$, and in this case it is k-connected to a_4 by a path in $H \cap N_8(a_3)$ iff it is k-connected to a_4 by a path in $(H \setminus \{a_1\}) \cap N_8(a_3)$. In other words, $H \cap N_k(a_3)$ is k-connected through $H \cap N_8(a_3)$ iff $(H \setminus \{a_1\}) \cap N_k(a_3)$ is k-connected through $(H \setminus \{a_1\}) \cap N_8(a_3)$. By Lemma 2.9, this means that a_3 is k-deletable from H iff it is k-deletable from $H \setminus \{a_1\}$. Note that we can also show it by checking that Yokoi's k-connectivity numbers [11, 12] are the same in both configurations shown above. □

Theorem 3.5. *Let $U \subseteq F \cap G^\circ$, with $U \neq \emptyset$, and $x = 4$, 8, or $\{4,8\}$. Then U is an x-MND set from F iff one of the following holds:*

(i) *U consists in a single pixel which is not x-deletable from F.*

(ii) *U is a pair of 8-adjacent pixels which are x-deletable from F, but U is not x-deletable from F.*

(iii) *$x = 8$, U is a triple or a quadruple of pairwise 8-adjacent pixels, and U is an 8-connected component of F (in other words, no pixel of $F \setminus U$ is 8-adjacent to U).*

Proof. It is easily seen that if (i), (ii), or (iii) holds, then U is x-MND from F. Let us now show the reverse. Let p and q be two arbitrary distinct pixels of U. Write

$$U_p = U \setminus \{p\}, \qquad U_q = U \setminus \{q\}, \qquad U_{pq} = U \setminus \{p, q\},$$

$$F_p = F \setminus U_p, \qquad F_{pq} = F \setminus U_{pq}.$$

As U_{pq} and U_q are both x-deletable from F and $U_q = U_{pq} \cup \{p\}$, Lemma 2.6 implies that p is x-deletable from $F \setminus U_{pq} = F_{pq}$. As U_p is x-deletable from F, U is not, and $U = U_p \cup \{p\}$, Lemma 2.6 again implies that p is not x-deletable from $F \setminus U_p = F_p$. It follows then by Lemma 2.9 that

$$N_8(p) \cap F_p \neq N_8(p) \cap F_{pq}.$$

But $F_{pq} = F_p \cup \{q\}$, and so we deduce that $q \in N_8(p)$.

As p and q were arbitrarily chosen, the pixels of U are all pairwise 8-adjacent, and so it is easy to see that U contains at most 4 pixels. Hence either (i) or (ii) holds, or U is a triple or a quadruple of pairwise 8-adjacent pixels. We have only to show that (iii) holds in the latter case. We have two subcases:

(a) *U is a triple.* We can write $U = \{p_1, p_2, p_3\}$, where p_2 is 4-adjacent to p_1 and p_3. Let q be the pixel of G which is 8-adjacent to all three pixels p_1, p_2, p_3 (clearly $q \in G^\circ$). Up to a symmetry, $U \cup \{q\}$ forms the following configuration:

$$\begin{matrix} p_1 & q \\ p_2 & p_3 \end{matrix}$$

Suppose that $q \in F$. As p_2 and $\{p_2, p_3\}$ are x-deletable from F, Lemma 2.6 implies that p_3 is x-deletable from $F \setminus \{p_2\}$. Applying Lemma 3.4 with $a_1 = p_1$, $a_2 = p_2$, $a_3 = p_3$, $a_4 = q$, and $H = F \setminus \{p_2\}$, we see that for any k intervening in x,

p_3 is k-deletable from $F \setminus \{p_2\}$ iff it is k-deletable from $F \setminus \{p_1, p_2\}$. Thus p_3 is x-deletable from $F \setminus \{p_1, p_2\}$. By Lemma 2.6, as $\{p_1, p_2\}$ is x-deletable from F, this means that $U = \{p_1, p_2, p_3\}$ is x-deletable from F, a contradiction.

Hence $q \notin F$. Suppose now that $x \neq 8$, and so 4 intervenes in x. Then p_2 is 4-deletable from F, and as $p_1, p_3 \in N_4(p_2) \cap F$, Lemma 2.9 implies that p_1 and p_3 are joined by a 4-connected path in $F \cap N_8(p_2)$. We also proved above that $q \notin F$. Now we show below $N_8(p_2)$:

$$\begin{matrix} r_1 & p_1 & q \\ r_2 & p_2 & p_3 \\ r_3 & r_4 & r_5 \end{matrix}$$

The only 4-connected path in $N_8(p_2)$ which joins p_1 to p_3 without containing q is $p_1, r_1, r_2, r_3, r_4, r_5, p_3$. Therefore $r_1, \ldots, r_5 \in F$. Then

$$N_8(p_2) \cap (F \setminus \{p_1, p_3\}) = \{r_1, \ldots, r_5\},$$

which is 4-connected and intersects $N_4(p_2)$. It follows by Lemma 2.9 that p_2 is $\{4, 8\}$-deletable from $F \setminus \{p_1, p_3\}$. As $\{p_1, p_3\}$ is x-deletable from F and p_2 is x-deletable from $F \setminus \{p_1, p_3\}$, Lemma 2.6 implies that $U = \{p_1, p_2, p_3\}$ is x-deletable from F, a contradiction.

Hence $x = 8$. Suppose that U is not isolated, in other words there is some pixel $r \in F \setminus U$ which is 8-adjacent to some $p_i \in U$ ($i = 1$, 2, or 3). Take $j \in \{1, 2, 3\} \setminus \{i\}$ and $k \in \{1, 2, 3\} \setminus \{i, j\}$ (in other words $\{i, j, k\} = \{1, 2, 3\}$). As $\{p_k\}$ and $\{p_i, p_k\}$ are 8-deletable from F, Lemma 2.6 implies that p_i is 8-deletable from $F \setminus \{p_k\}$. Lemma 2.9 implies then that $N_8(p_i) \cap (F \setminus \{p_k\})$ is 8-connected. As $r, p_j \in N_8(p_i) \cap (F \setminus \{p_k\})$, they are 8-connected in $N_8(p_i) \cap (F \setminus \{p_k\})$, and in particular this means that there is some pixel $s \in N_8(p_i) \cap (F \setminus \{p_k\})$ which is 8-adjacent to p_j, in other words $s \in F \setminus U$ and s is 8-adjacent to both p_i and p_j. We repeat this argument with s instead of r, and with (j, k, i) instead of (i, j, k), and so there is a pixel $s' \in F \setminus U$ which is 8-adjacent to both p_j and p_k. We repeat this argument again, and there is a pixel $s'' \in F \setminus U$ which is 8-adjacent to both p_k and p_i. Now $\{1, 3\}$ must be one of the pairs $\{i, j\}$, $\{j, k\}$, or $\{k, i\}$, and so there is a pixel in $F \setminus U$ which is 8-adjacent to both p_1 and p_3. But we know that q is the only pixel of $G \setminus U$ which is 8-adjacent to both p_1 and p_3, and that $q \notin F$. We have thus a contradiction, and so U must be isolated. Hence (iii) holds.

(b) *U is a quadruple.* Take any $p \in U$. For any nonvoid proper subset V of $U \setminus \{p\}$, both $V \cup \{p\}$ and $\{p\}$ are proper subsets of U, and so are x-deletable from F; thus V is x-deletable from $F \setminus \{p\}$ by Lemma 2.6. On the other hand, $U = (U \setminus \{p\}) \cup \{p\}$ is not x-deletable from F, and as $\{p\}$ is, Lemma 2.6 implies that $U \setminus \{p\}$ is not x-deletable from $F \setminus \{p\}$. Hence $U \setminus \{p\}$ is an x-MND set from $F \setminus \{p\}$. Then (a) implies that $x = 8$ and that $U \setminus \{p\}$ is isolated in $F \setminus \{p\}$, in other words no pixel of $(F \setminus \{p\}) \setminus (U \setminus \{p\}) = F \setminus U$ is 8-adjacent to a pixel of $U \setminus \{p\}$. As p was arbitrarily chosen, this means that no pixel of $F \setminus U$ is 8-adjacent to a pixel of U. Therefore (iii) holds. □

We can say more in the case (ii) of Theorem 3.5 when $x=8$:

Proposition 3.6. *Let $U=\{p,q\}$ be a pair of diagonally adjacent pixels of $F \cap G^\circ$. If U is an 8-MND set from F, then U is an 8-connected component of F (in other words, no pixel of $F \setminus U$ is 8-adjacent to U).*

Proof. Suppose that there exist a pixel $r \in F \setminus U$ which is 8-adjacent to U, say to q. As $p, r \in N_8(q)$ and q is 8-deletable from F, Lemma 2.9 implies that there is an 8-connected path in $F \cap N_8(q)$ joining r to p. As $r \neq p$, this implies that there is some $u \in F \cap N_8(q)$ which is 8-adjacent to p, in other words u is 8-adjacent to both p and q. Let v be the other pixel 8-adjacent to both p and q (thus p, u, q, v form a square). We have two cases:

(a) $v \notin F$. We apply Lemma 3.4 with $a_1=p$, $a_2=v$, $a_3=q$, $a_4=u$, and $H=F$, and so q is 8-deletable from F iff it is 8-deletable from $F \setminus \{p\}$. As $\{p,q\}$ is an 8-MND set from F, we have a contradiction by Lemma 2.6.

(b) $v \in F$. We show below $N_8(q)$:

$$\begin{array}{ccc} p & v & \cdot \\ u & q & \cdot \\ \cdot & \cdot & \cdot \end{array}$$

As q is 8-deletable from F, Lemma 2.9 implies that every pixel $w \in N_8(q)$ such that $w \neq v$, is joined to v by an 8-connected path in $F \cap N_8(q)$. If $w \neq p$ and this path passes through p, it must also pass through u (since u is the only pixel of $N_8(q)$, apart from v, which is k-adjacent to p). As u is 8-adjacent to v, removing p from that path leaves it 8-connected. Thus every pixel $w \in N_8(q)$ distinct from p and v, is joined to v by an 8-connected path in $(F \setminus \{p\}) \cap N_8(q)$, and so Lemma 2.9 implies that q is 8-deletable from $F \setminus \{p\}$, and we have again a contradiction by Lemma 2.6. □

Note that, thanks to Lemma 2.6, in order to check that a pair $\{p,q\}$ of 8-adjacent pixels is x-MND from F, we can simply check that p and q are both x-deletable from F, but p is not x-deletable from $F \setminus \{q\}$ (or equivalently q is not x-deletable from $F \setminus \{p\}$).

Theorem 3.5 with the refinement of Proposition 3.6 gives us a simple sufficient condition for a parallel thinning algorithm to preserve the connectivity properties of a figure. A deletion stage in such an algorithm applies a local removal criterion to all pixels of the figure (inside the frame of G), and deletes in parallel from that figure all pixels satisfying that criterion. We simply check that: (1) a figure pixel satisfying that removal criterion must be x-deletable (for example satisfies the condition given in Lemma 2.9); (2) a pair of 8-adjacent figure pixels satisfying that criterion must be x-deletable; (3) if $x=8$, an isolated triple or quadruple of pairwise 8-adjacent figure pixels will not be completely removed (in other words, contains at least one pixel which does not satisfy that criterion). If these three conditions are satisfied, we know that a deletion stage in this parallel algorithm will remove from a figure F a subset D of $F \cap G^\circ$ which does not contain any x-MND set; then D will be strongly x-deletable from F. When we repeat this deletion stage, we remove from $F \cap G^\circ$ successive strongly x-deletable sets $D_1, \ldots, D_n$. Thanks to Lemmas 2.6 and 2.7, their union $D_1 \cup \cdots \cup D_n$ will be strongly x-deletable from F.

The reader can check the connectivity preservation of various algorithms presented in the literature by verifying that their removal criterion does not allow the deletion of x-MND sets.

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