THREE-MANIFOLD INVARIANTS DERIVED FROM THE KAUFFMAN BRACKET

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INTRODUCTION

In several recent articles, Lickorish [11, 12, 13] has given an elementary construction of 3-manifold invariants using the one variable Kauffman bracket [6] evaluated at 4r-th roots of unity, r ≥ 3. We show that evaluations of the bracket at 2p-th roots of unity, p odd, also give 3-manifold invariants. Moreover, we show that no other evaluations at other values lead to invariants.

In [13], Lickorish has described the precise relationship of the invariants given by his formulas, to the Jones-Witten SU(2)k invariants [18] as established by Reshetikhin and Turaev [14] using quantum groups, and further studied by Kirby and Melvin [8, 9].

In the spirit of the present paper, Blanchet has given a construction of invariants for 3-dimensional Spin manifolds. He obtains invariants for evaluations at all primitive k-th roots of unity for k ≠ 8 mod 16 [2] (compare [9], [16]).

The construction of the invariants is based on Kirby’s theorem [7] describing how two surgery descriptions of the same oriented closed 3-manifold are related. If a manifold M3 is represented by a banded link L in the 3-sphere, then the invariants are given by finite linear combinations of Kauffman brackets of tablings of L, to be evaluated at primitive 2p-th roots of unity. In fact, our invariant for odd p can be expressed by a generalization of the formula given by Lickorish in [13]. However, we do not need to use the Temperley-Lieb algebra, which was Lickorish’s main tool in establishing the existence of his invariants. Instead we systematically study which tablings and which evaluations of the Kauffman brackets of these tablings are invariant under Kirby’s calculus. This enables us to show first that non-trivial invariants can only exist for evaluations at primitive 2p-th roots of unity, and second that they do exist and are essentially unique for each such evaluation.

Here is an outline of this paper. We define the Jones-Kauffman module K(M) of an oriented 3-manifold to be the Z[A, A−1]-module generated by banded links in M, divided by the usual Kauffman relations (see [4] for an overview of more general skein modules). Let g denote the Jones-Kauffman module of the solid torus. (In [13], essentially the same object, after a change of coefficients from Z[A, A−1] to C, is called 2l.) Given a n-component banded link in S3, there is a n-linear form ⟨, . . . , ⟩L on g given by replacing the components of L by elements of g, and taking the bracket of the resulting banded link in S3. We call this n-linear form the meta-bracket of L. (It corresponds to the map 2l in [13].) In particular, we have a symmetric bilinear form ⟨, ⟩ on g given by the meta-bracket of the banded Hopf link where each component has writhe zero. Let r be the self-map of g induced by one positive twist. The following observation is due to Lickorish and appeared in [13] with a slightly different normalization. Suppose we can find an element Ω ∈ g with

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the property that

\[(*) \quad \langle t^e(\Omega), t^e(b) \rangle = \langle t^e(\Omega) \rangle \langle b \rangle \quad \text{and} \quad \langle t^e(\Omega) \rangle \text{ is invertible}\]

for all \( b \in \mathcal{B} \) and \( e = \pm 1 \). Then it follows from Kirby's calculus that for all banded links \( L \subset S^3 \), the expression

\[(**) \quad \theta_{\Omega}(L) = \frac{\langle \Omega_1, \ldots, \Omega_n \rangle_L}{\langle t(\Omega) \rangle^{b_+} \langle t^{-1}(\Omega) \rangle^{b_-}}\]

is an invariant of the manifold obtained by surgery on \( L \). (Here \( b_+ (L) \) and \( b_- (L) \) denote the number of positive and negative eigenvalues of the linking matrix of \( L \).)

In order to find such \( \Omega \)'s, we construct an orthogonal basis of \( \mathcal{B} \) with respect to the bilinear form \( \langle , \rangle_1 = \langle t( ), t( ) \rangle \). Having done so, we see that an \( \Omega \) satisfying hypothesis (*) above for \( e = +1 \) cannot exist unless we replace the coefficients \( Z[1, A, A^{-1}] \) by some ring \( \Lambda \). Moreover, if we assume \( \Lambda \) is an integral domain, then \( \Lambda \) must be a primitive \( p \)-th root of unity for some \( p \geq 1 \).

From now on, we work with coefficients in \( \Lambda_p = Z[A, A^{-1}] / \phi_{2p}(A) \), where \( \phi_p \) denotes the \( d \)-th cyclotomic polynomial in the indeterminate \( A \). Using the orthogonal basis referred to above, we show existence of an \( \omega \) satisfying condition (*) for \( e = 1 \). It follows that its conjugate \( \bar{\omega} \) satisfies condition (*) for \( e = -1 \). Using this, we show the following result.

**Theorem A.** Let \( p \geq 1 \) be an integer. Let \( \mathcal{B}_p \) denote the Jones–Kaufman module of the solid torus \( S^1 \times I \times I \) with coefficients in \( \Lambda_p \), i.e. \( \mathcal{B}_p = \mathcal{B} \otimes \Lambda_p = \Lambda_p[z] \). Set \( V_p = \mathcal{B}_p/N_p \) where \( N_p \) denotes the kernel of the bilinear form \( \langle , \rangle_1 \) on \( \mathcal{B}_p \) (with values in \( \Lambda_p \)). Then for any \( k \)-component banded link \( L \subset S^3 \), the induced meta-bracket factors through \( V_p^{\otimes k} \). Furthermore, the twist map \( t \) induces a self-map of \( V_p^k \).

It turns out that \( V_p \) is a finite-dimensional algebra (of rank \( [(p - 1)/2] \) if \( p \geq 3 \), and of rank \( p \) if \( p = 1 \) or \( 2 \)). Working in \( V_p \), we now show how to generalize the formula of [13] to the case \( p \) odd. The result may be stated as follows. Let \( z \in \mathcal{B} \) be represented by a standard untwisted band. Then \( \mathcal{B} \) is isomorphic to the polynomial algebra \( Z[A, A^{-1}] [z] \). There is a basis of monic polynomials \( e_i \) of degree \( i \) which satisfy \( e_0 = 1, e_1 = z \) and \( ze_i = e_{i+1} + e_{i-1} \).

(Lickorish had obtained this basis, called \( \Phi_i \) in [13], using idempotents in the Temperley–Lieb algebra first discovered by Jones [5].) The geometric significance of the \( e_i \) is that they are eigenvectors for the twist map \( t \) with distinct eigenvalues.

**Theorem B.** Define \( \Omega_p \in V_p \) (except \( \Omega_2 \in V_2 \otimes Z[1/2] \)) by \( \Omega_1 = 1, \Omega_2 = 1 + \frac{z}{2} \), and \( \Omega_p = \sum_{i=0}^{n-1} \langle e_i \rangle \ e_i \) for \( p \geq 3 \), where \( n = [(p - 1)/2] \). Then the expression

\[\theta_{\Omega_p}(L) = \frac{\langle \Omega_1, \ldots, \Omega_n \rangle_L}{\langle t(\Omega_p) \rangle^{b_+} \langle t^{-1}(\Omega_p) \rangle^{b_-}}\]

is an element of \( \Lambda_p[1/2] \), and an invariant, denoted by \( \theta_p(M_L) \), of the oriented 3-manifold \( M_L \) obtained by surgery on \( L \).

If \( p \geq 6 \) is even, then \( \theta_p(M) \) is \( \langle t(\Omega_p) \rangle^{b_+} \) times the invariant given by Lickorish in [13] for \( r = p/2 \). Here, \( b_1(M) \) is the first Betti number of \( M \).

Remarks.

1. If \( p \in \{1, 3, 4\} \), then \( \theta_p(M) = 1 \) for all closed connected oriented 3-manifolds \( M \).
2. The invariant \( \theta_p \) has the following properties.
(i) \( \theta_p(S^3) = 1 \)

(ii) \( \theta_p(-M) = \overline{\theta_p(M)} \), where \(-M\) denotes \( M \) with reversed orientation.

(iii) \( \theta_p(M \# N) = \theta_p(M) \theta_p(N) \)

(iv) \( \theta_p(S^1 \times S^2) = \langle \Omega_p \rangle = \begin{cases} p & \text{if } p \leq 2 \\ \frac{p}{(A^2 - A^{-2})^2} & \text{if } p \geq 3 \end{cases} \)

(3) These invariants are essentially unique. Indeed, let \( \Lambda \) be an integral domain containing a homomorphic image of \( \mathbb{Z}[A, A^{-1}] \), and suppose that \( \Omega \) satisfies condition (\( * \)) over \( \Lambda \). Let \( \theta_{\Omega}(M) \) denote the 3-manifold invariant defined by (\( ** \)). Then

\[ \theta_{\Omega}(M) = \lambda b_1(M) f(\theta_p(M)) \]

for some \( p, f, \) and \( \lambda \) independent of \( M \), where \( b_1(M) \) is the first Betti number of \( M \). See Proposition 6.10. for a precise statement.

1. JONES–KAUFFMAN MODULES AND THE META-BRACKET

Let \( M \) be a compact, oriented 3-manifold, possibly with boundary. A banded link in \( M \) is an oriented submanifold homeomorphic to a disjoint union of annuli \( S^1 \times I \) in \( M \). Up to isotopy, a banded link is the same as an unoriented link (the cores of the annuli) together with a choice, for each component, of a longitude on the boundary of a small tubular neighborhood of that component. Notice that this is precisely what is needed to do (integral) surgery on \( M \). Let \( \mathbb{Z}[A, A^{-1}] \) denote the ring of Laurent polynomials in the indeterminate \( A \).

Definition. The Jones–Kauffman module \( K(M) \) is the \( \mathbb{Z}[A, A^{-1}] \)-module generated by the set of isotopy classes of banded links in \( M \), quotiented by the following Kauffman relations:

\[
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[thick] (-1,0) .. controls (0,-1) and (0,1) .. (1,0);
\node at (0,0) {$= A$};
\end{tikzpicture}
\end{align*}
\] (\( \bigcirc \)) \\
\begin{align*}
\begin{tikzpicture}[baseline=(current bounding box.center)]
\draw[thick] (-1,0) .. controls (0,-1) and (0,1) .. (1,0);
\draw[thick] (0,0) circle (0.5);
\node at (0,0) {$L$};
\end{tikzpicture}
\end{align*}
\]

Here \( \delta = -A^2 - A^{-2} \). (More general skein modules have been defined by several authors, see [4] for an overview). Notice that \( K \) is a functor on the category of compact oriented 3-manifolds, the morphisms being isotopy classes of orientation-preserving embeddings.

Given a banded link \( L \subset S^3 \), its “value” in \( K(S^3) \) is called the Kauffman bracket of \( L \), and denoted \( \langle L \rangle \). It is a well known fact [6] that \( K(S^3) \cong \mathbb{Z}[A, A^{-1}] \). We fix an isomorphism by the convention that the bracket of the empty link is equal to 1.

(Actually, the classical Kauffman bracket [6] is a regular isotopy invariant defined for link diagrams in the plane. However, these notions are equivalent, because every banded link in \( S^3 \) may be represented unambiguously by an ordinary link diagram, since in the
plane we have a canonical band around each component. This induces a 1–1 correspondence between isotopy classes of banded links in $S^3$, and regular isotopy classes of link diagrams in $S^2$.)

For a 2-dimensional oriented surface $\Sigma$ (possibly with boundary), the Jones–Kauffman module $K(\Sigma \times I)$ is in the obvious way a $\mathbb{Z}[A, A^{-1}]$-algebra. Notice that for the solid torus $S^1 \times I \times I$, this algebra is commutative. Let $z \in K(S^1 \times I \times I)$ be represented by a standard band, e.g., by $S^1 \times J \times pt$, where $J \subset I$ is a proper subinterval.

**Proposition 1.1.** $K(S^1 \times I \times I)$ is the polynomial algebra $\mathbb{Z}[A, A^{-1}][z]$.

Indeed, using Kauffman's state model [6], one constructs an inverse of the obvious map $\mathbb{Z}[A, A^{-1}][z] \rightarrow K(S^1 \times I \times I)$. (See [15], [4].)

Write $\mathcal{B} = K(S^1 \times I \times I)$. Notice that the Jones–Kauffman module of a disjoint union of $n$ solid tori is simply the $n$-fold tensor product $\mathcal{B}^n$.

**Definition.** (cf. Lickorish [13]) Given an $n$-component banded link $L$ in $S^3$, the meta-bracket $\langle \ldots \rangle_L$ is the linear function on $\mathcal{B}^n$ defined as follows. Choose an orientation-preserving embedding of the disjoint union of $n$ solid tori into $S^3$ such that the standard bands are sent to the components of $L$. Then the meta-bracket of $L$ is the induced map $\mathcal{B}^n \rightarrow K(S^3) = \mathbb{Z}[A, A^{-1}]$.

Notice that it follows from Prop. 1.1. that the orientation-preserving automorphism of $S^1 \times I \times I$ which is orientation-reversing in the first two factors and the identity on the third, induces the identity on $\mathcal{B}$. It follows that the meta-bracket is independent of the choice of embedding.

**Remark 1.2.** As the notation indicates, we will in practice view the meta-bracket of a banded link $L$ as a multilinear function on $\mathcal{B}$. If $b_i = z^n$, then the meta-bracket $\langle b_1, \ldots, b_n \rangle_L$ is simply the Kauffman bracket of the banded link obtained from $L$ by replacing the $i$-th component by $v_i$ parallel copies. In particular, if $b_i = 1$, then the $i$-th component is removed, and if $b_i = z$, then the $i$-th component remains unchanged.

**Remark 1.3.** The meta-bracket $\langle b_1, \ldots, b_n \rangle_L$ is denoted pictorially by writing $b_i$ beneath the $i$-th component of $L$, where $L$ is represented in the plane using the convention that any line is to represent a band parallel to the plane, with orientation compatible with that of the plane. We say that the $i$-th component of $L$ is cabled by $b_i$. See Fig. 2.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure2.png}
\caption{Fig. 2.}
\end{figure}

2. THE CONSTRUCTION OF 3-MANIFOLD INVARIANTS FROM THE META-BRACKET

Recall that for a banded knot $K \subset S^3$, its writhe $w(K)$ is defined as the linking number of its two boundary components. (Here one of the two components can be oriented arbitrarily, but the other must be oriented in the same way.)
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Given a banded link \( L \subset S^3 \), let \( M_L \) denote the oriented 3-manifold obtained by surgery on \( L \). It is well-known \([10]\) that any oriented closed connected 3-manifold is oriented diffeomorphic to some \( M_L \). The theorem of Kirby \([7]\) describes the equivalence relation \( \sim \) on the set of banded links in \( S^3 \), where \( L_1 \sim L_2 \) iff \( M_{L_1} \) is oriented diffeomorphic to \( M_{L_2} \). Further refinements by Fenn–Rourke \([3]\) show that this equivalence relation is generated by isotopy and the following two moves \( K_+ \) and \( K_- \).

Move \( K_+ \) consists of adding an unknotted component \( K \) with \( w(K) = +1 \), and giving a full positive twist to the part of \( L \) passing through \( K \).

Move \( K_- \) consists of adding an unknotted component \( K \) with \( w(K) = -1 \), and giving a full negative twist to the part of \( L \) passing through \( K \).

It follows from this theorem that any invariant for banded links in \( S^3 \) which is preserved by moves \( K_+ \) and \( K_- \), gives a 3-manifold invariant.

Here is the basic idea how to construct 3-manifold invariants using the meta-bracket (cf. Lickorish \([13]\)). Let \( t \) be the self-map of \( B = K(S^1 \times I \times I) \) induced by one positive twist. Let \( \langle , \rangle \) denote the symmetric bilinear form on \( B \) given by the meta-bracket of the banded Hopf link where each component has writhe zero. Let \( \langle \rangle \) denote the linear form given by the meta-bracket of the banded un-knot with writhe zero. Notice that \( \langle b, 1 \rangle = \langle b \rangle \) for all \( b \in B \).

**Proposition 2.1.** Let the ring \( \Lambda \) contain a homomorphic image of \( \mathbb{Z}[A, A^{-1}] \). Suppose given \( \Omega \in B \otimes \Lambda = \Lambda [z] \) satisfying condition \( (*) \) over \( \Lambda \), i.e. such that for \( \epsilon = \pm 1 \), one has

(i) \( \langle t^\epsilon(\Omega), t^\epsilon(b) \rangle = \langle t^\epsilon(\Omega) \rangle \langle b \rangle \in \Lambda \) for all \( b \in B \),

(ii) \( \langle t^\epsilon(\Omega) \rangle \) is invertible in \( \Lambda \).

Then for all banded links \( L \subset S^3 \), the quantity

\[
\theta_\Omega(L) = \frac{\langle \Omega, \ldots, \Omega \rangle_L}{\langle t^0(\Omega) \rangle^{b_+(L)} \langle t^{-1}(\Omega) \rangle^{b_-(L)} \in \Lambda}
\]

is an invariant of the manifold \( M_L \).

Here \( b_+(L) \) and \( b_-(L) \) denote the number of positive and negative eigenvalues of the linking matrix of \( L \). (More precisely, one must orient the components of \( L \) to get a linking matrix, but the numbers \( b_+(L) \) and \( b_-(L) \) are independent of the chosen orientations.)

**Proof.** This follows from Kirby's calculus as described above. Indeed, let the banded links \( L \) and \( L' \) be related by a Kirby move \( K_+ \). Suppose that \( L \) has \( k \) components, and that the new component is \( L_{k+1} \). Then it is not hard to see that condition (i) implies

\[
\langle b_1, \ldots, b_k, \Omega \rangle_{L'} = \langle t^0(\Omega) \rangle \langle b_1, \ldots, b_k \rangle_L
\]

for all \( b_1, \ldots, b_k \in B \). The result follows.

3. A NATURAL BASIS FOR THE JONES–KAUFFMAN ALGEBRA OF A SOLID TORUS

Let \( c \) and \( \tau \) denote the self-maps of \( B = K(S^1 \times I \times I) \) given by adding a single band as indicated in Fig. 3.

(In these figures, the top and bottom edges of the squares are to be identified, and the annuli thus obtained represent \( S^1 \times I \times pt \). See Remarks 1.2 and 1.3.)

Recall that \( t \) denotes the effect of one positive twist.
LEMMA 3.1. For all \( u \in \mathcal{B} \), one has

(i) \( t z t^{-1}(u) = - A^3 z(u) \)
(ii) \( c(z u) = A^{-2} z c(u) + (1 - A^4) \tau(u) \)
(iii) \( \tau(z u) = - A^2 z \tau(u) + (1 - A^{-4}) c(u) \).

The proof is an easy exercise using the Kauffman relations.

Recall that \( \mathcal{B} \) is the polynomial algebra \( Z[A, A^{-1}][z] \). Notice that the self-maps \( c \) and \( t \) preserve the degree in \( z \), whereas \( \tau \) increases degrees by 1. Moreover, a moment’s thought shows that \( c \) and \( t \) preserve the sub-algebra of even polynomials in \( z \), while \( \tau \) sends even polynomials to odd ones.

Set \( \lambda_n = - (A^{2n+2} + A^{-2n-2}) \), \( \mu_n = (-1)^n A^{n^2 + 2n} \). Note that \( \lambda_0 = \delta = \langle z \rangle \).

LEMMA 3.2. For \( n \geq 0 \), one has

(i) \( \tau(z^n) = A^{2n-2} z^n + \ldots \)
(ii) \( c(z^n) = \lambda_n z^n + \ldots \)
(iii) \( t(z^n) = \mu_n z^n + \ldots \)

where the dots indicate elements of degree \( \leq n - 2 \).

The proof is by induction on \( n \), using Lemma 3.1.

Let \( Q(A) \) denote the quotient field of \( Z[A, A^{-1}] \). Since \( \mu_i \neq \mu_j \) for \( i \neq j \), \( t \) is diagonalizable when considered as a \( A \)-self-map of \( \mathcal{B} \otimes Q(A) = Q(A)[z] \). Let \( e_n \) denote the unique monic polynomial of degree \( n \) which is an eigen-vector for \( t \). (The following lemma will imply \( e_n \in Z[z] \subset \mathcal{B} \) for all \( n \), hence \( t \) is actually diagonalizable over \( Z[A, A^{-1}] \).) Notice \( e_0 = 1, e_1 = z \). Since \( c \) and \( t \) commute, \( e_n \) is also an eigen-vector for \( c \). Thus it follows from Lemma 3.2. that

\[ t(e_n) = \mu_n e_n, \quad c(e_n) = \lambda_n e_n. \]

LEMMA 3.3. For \( n \geq 1 \), one has

(i) \( \lambda_{n+k} + \lambda_{n-k} = - \lambda_{k-1} \lambda_n \) for all \( k \)
(ii) \( e_{n+1} + e_{n-1} = z e_n \).

We indicate a proof of part (ii). From Lemma 3.1, one obtains

\[ c(z^2 u) = - \lambda_0 z c(z u) + (2 + \lambda_1 - z^2) c(u). \]

Assuming (ii) true up to \( n \), one easily verifies that \( z e_n - e_{n-1} \) is an eigen-vector for \( c \) with eigen-value \( \lambda_{n+1} \). Since the \( \lambda_i \) are distinct, and \( e_{n+1} \) and \( z e_n - e_{n-1} \) are both monic, the result follows.

We would like to call the \( e_n \) a natural basis of \( \mathcal{B} \), since they diagonalize the geometric maps \( c \) and \( t \). Notice that \( e_n \) is the \( n \)-th Chebyshev polynomial in \( z \). Indeed, setting
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\[ z = -y - y^{-1}, \text{ part (ii) of the lemma implies} \]

\[ e_n = (-1)^n \frac{y^{n+1} - y^{-n-1}}{y - y^{-1}}. \]

Notice that if we set \( \langle y \rangle = A^2 \), then \( \langle z \rangle = \delta \) as required. Thus the following result is obvious.

**Lemma 3.4.** \( \langle e_n \rangle = (-1)^n \frac{A^{2n+2} - A^{-2n-2}}{A^2 - A^{-2}}. \)

*Remark.* In Lickorish's paper [13], this natural basis arises from certain idempotents, first considered by Jones [5] and Wenzl [17], in the Temperley-Lieb algebra.

**4. ORTHOGONAL POLYNOMIALS**

The aim of this section is to find all pairs \((\Lambda, \Omega)\) verifying condition (*) for \( \varepsilon = +1 \), i.e. invariance under move \( K \). We will assume \( \Lambda \) is an integral domain.

Recall that we have defined a symmetric bilinear form \( \langle \cdot, \cdot \rangle \) on \( \mathcal{B} \), given by the meta-bracket of the banded Hopf link where each component has writhe zero. It is convenient to first normalize \( \Omega \) by setting \( \omega = \langle t(\Omega) \rangle^{-1} \Omega \) so that \( \langle t(\omega) \rangle = 1 \). Thus we now wish to find an \( \omega \) and an integral domain \( \Lambda \) containing a homomorphic image of \( \mathbb{Z}[A, A^{-1}] \) such that

\[ \langle t(\omega), t(b) \rangle = \langle b \rangle \in \Lambda \]

for all \( b \in \mathcal{B} \). Define the bilinear form \( \langle \cdot, \cdot \rangle_1 \) on \( \mathcal{B} \) by \( \langle u, v \rangle_1 = \langle t(u), t(v) \rangle \). The idea is to express \( \omega \) with respect to a basis \( P_n \) of \( \mathcal{B} \) which is orthogonal for this bilinear form.

A moment's thought will convince the reader that for all \( u, v \in \mathcal{B} \), one has

\[ \langle zu, v \rangle_1 = \langle u, zv \rangle_1. \]

(Indeed, the bilinear form \( \langle \cdot, \cdot \rangle_1 \) is the meta-bracket of the banded Hopf link where both components have writhe \(+1\), and the assertion follows from the fact that the two components are actually parallel copies of each other.) We may express this by saying that multiplication by \( z \) is self-adjoint with respect to \( \langle \cdot, \cdot \rangle_1 \). Hence we expect the \( P_n \) to satisfy a three-term recursion formula. This is indeed the case, as Proposition 4.4. will show.

Set \( Q_n = (z - \lambda_0) \ldots (z - \lambda_{n-1}) \) and \( Q_0 = 1 \).

**Lemma 4.1.** The \( Q_n \) form an orthogonal basis of \( \mathcal{B} \) with respect to the bilinear form \( \langle \cdot, \cdot \rangle \).

*Proof.* For all \( u, v \in \mathcal{B} \), one has

\[ \langle z u, v \rangle = \langle u, c(v) \rangle. \]

In particular, \( \langle z^n, e_i \rangle - \langle 1, c^n(e_i) \rangle = -z^n \langle 1, e_i \rangle \). Hence \( \langle Q_n, e_i \rangle = Q_n(\lambda_i) \langle 1, e_i \rangle = 0 \) for \( i < n \). Since \( \text{Span}\{e_0, \ldots, e_{n-1}\} = \text{Span}\{Q_0, \ldots, Q_{n-1}\} \), this implies \( \langle Q_n, Q_i \rangle = 0 \) for \( i < n \). The result follows.

Define the monic polynomial \( P_n \) by \( t(P_n) = \mu_n Q_n \). Note that \( P_0 = 1 \) and \( P_1 = z + A^3 \lambda_0 = z - A - A^2 \).

**Lemma 4.2.** The \( P_n \) form an orthogonal basis of \( \mathcal{B} \) with respect to the bilinear form \( \langle \cdot, \cdot \rangle_1 \).

This follows immediately from Lemma 4.1.
We are now ready to state the main result of this section. Let the integral domain \( A \) contain a homomorphic image of \( \mathbb{Z}[A, A^{-1}] \). Let \( \phi_d(A) \) denote the \( d \)-th cyclotomic polynomial in the indeterminate \( A \). Define \( n(p) \) by \( n(1) = 1 \), \( n(2) = 2 \), and \( n(p) = [(p - 1)/2] \) for \( p \geq 3 \).

**Proposition 4.3.** (1) The following are equivalent.

(i) There is an \( \omega \in \mathcal{B} \otimes \Lambda = \Lambda[z] \), such that \( \langle \omega, b \rangle_1 = \langle b \rangle \) (as elements of \( \Lambda \)) for all \( b \in \mathcal{B} \).

(ii) There is an integer \( p \geq 1 \) such that \( \phi_{2p}(A) = 0 \), and \( \phi_d(A) \) is invertible in \( \Lambda \) for \( d \leq 2n(p) - 1 \) if \( n(p) > 1 \).

(2) Assume condition (ii) holds in \( \Lambda \), and suppose \( p \) is minimal with respect to this property. Then \( \langle \omega, b \rangle_1 = \langle b \rangle \) (as elements of \( \Lambda \)) for all \( b \in \mathcal{B} \) if and only if

\[
\omega = \sum_{i=0}^{n(p)-1} \prod_{j=i+1}^{2i+1} (A^j - 1) P_i \in \mathcal{B},
\]

where \( \mathcal{B} \in \text{Span}\{P_i : i \geq n(p)\} \).

The proof relies on the following result, whose proof will be given at the end of this section.

**Proposition 4.4.** For \( n \geq 1 \), one has

(i) \( P_n = (z - z_n^{-1})P_n - \beta_n^{-1}P_n^{-1} \),

where \( z_n = A^{4n+5} + A^{4n+3} - A^{2n+3} + A^{2n+1}, \beta_n^{-1} = (A^{4n+2} - 1)(A^{2n+2} + 1)(A^{2n} - 1) \).

(ii) \( \langle P_n, P_n \rangle_1 = (A^2 - 1)(A^{2n+2} + 1) \prod_{i=3}^{n} (A^{2i} - 1) \).

(iii) \( \langle P_n \rangle = (-1)^n A^{-2n} \prod_{i=3}^{n} (A^i + 1) \).

(iv) \( \langle P_n, P_n \rangle_1 = \frac{(-1)^n A^{-2n}}{\prod_{i=3}^{n} (A^i - 1)} \) (in the quotient field of \( \mathbb{Z}[A, A^{-1}] \)).

**Proof of Proposition 4.3.** First recall that \( A^n - 1 \) is equal to the product of the \( \phi_d \) where \( d \) divides \( n \). Notice that it follows that \( A^n + 1 \) is equal to the product of the \( \phi_d \) where \( d \) divides \( 2n \) but not \( n \). Using this, we deduce the following Lemma.

**Lemma 4.5.** (i) Suppose \( \phi_{2d}(A) = 0 \) in \( \Lambda \) and \( \phi_{2d}(A) \neq 0 \) for \( d < p \). Then for all \( n \geq 1 \), one has \( \langle P_n \rangle = 0 \) if and only if \( n \geq n(p) \).

(ii) Suppose moreover that \( \phi_d(A) \neq 0 \) in \( \Lambda \) for \( d \leq 2n(p) - 1 \) if \( n(p) > 1 \). Then for all \( n \geq 1 \), one has \( \langle P_n, P_n \rangle_1 = 0 \) if and only if \( n \geq n(p) \).

We now show part (1) of the proposition. Write \( \omega = \sum_{i=0}^{n} u_i P_i \). Notice that condition (i) is equivalent to

\[
u_i \langle P_i, P_i \rangle_1 = \langle P_i \rangle \quad \text{for all} \quad i \geq 0
\]

(as elements of \( \Lambda \)). Assume condition (i) holds. Let \( n = \min \{ i : \langle P_i \rangle = 0 \text{ in } \Lambda \} \). Since \( \omega \) is a finite sum, condition (i) implies that \( n \) is finite. Moreover, using the integrality of \( \Lambda \), we may write \( u_i = \langle P_i, P_i \rangle_1 / \langle P_i \rangle_1 \) for \( i \leq n - 1 \). By Prop. 4.4. (iv), this implies \( \phi_d(A) \) is invertible in \( \Lambda \) for \( d \leq 2n - 1 \) if \( n > 1 \). Set \( p = \min \{ i : \phi_{2d}(A) = 0 \text{ in } \Lambda \} \). Using the integrality of \( \Lambda \), part (iii) of Prop. 4.4. shows that \( p \) is finite, and Lemma 4.5.(i) shows \( n = n(p) \). Hence we have shown the implication (i) \( \Rightarrow \) (ii). The reverse implication (ii) \( \Rightarrow \) (i), as well as part (2) of the proposition, follow immediately from Lemma 4.5.(ii).
Remark 4.6. Given a banded link $L \subset S^3$, let $\bar{L}$ denote its mirror image. Notice that $M_L$ is orientation reversing diffeomorphic to $M_{\bar{L}}$. The process of taking mirror images corresponds to a conjugation $a \to \bar{a}$ on $K(S^3) = \mathbb{Z}[A, A^{-1}]$, induced by $A \to A = A^{-1}$. This may be lifted to a conjugation on $\mathcal{B} = \mathbb{Z}[A, A^{-1}][z]$ by declaring $\bar{z} = z$. The meta-bracket commutes with conjugation in the following sense. For all $b_1, \ldots, b_k \in \mathcal{B}$ and all $k$-component banded links $L$, we have

$$\langle \overline{b_1}, \ldots, \overline{b_n} \rangle_L = \langle b_1, \ldots, b_n \rangle_{\bar{L}}.$$ 

Let the bilinear form $\langle , \rangle_{-1}$ be defined as the meta-bracket of the banded Hopf link where both components have writhe $-1$, or equivalently by the formula $\langle u, v \rangle_{-1} = \langle t^{-1}(u), t^{-1}(v) \rangle$. Notice that this banded link is the mirror image of the banded Hopf link where both components have writhe $1$. Thus, if $\omega$ is as in Proposition 4.3., then

$$\langle \bar{\omega}, b \rangle_{-1} = \langle b \rangle$$

(as elements of $\Lambda$) for all $b \in \mathcal{B}$.

The rest of this section is devoted to a proof of Proposition 4.4. We first need a few lemmas.

**Lemma 4.7.** $\langle Q_n, Q_n \rangle = (\lambda_n - \lambda_{n-1}) \sum_{i=0}^{n-1} \lambda_i \langle Q_{n-1}, Q_{n-1} \rangle$.

**Proof.** Write $Q_n = z^n - \sigma_n z^{n-1} + \ldots$, where $\sigma_n = \sum_{i=0}^{n-1} \lambda_i$. Using Lemma 3.2. and the orthogonality of the $Q_i$, we calculate

$$\langle Q_n, Q_n \rangle = \langle (z - \lambda_n)Q_{n-1}, Q_n \rangle = \langle zQ_{n-1}, Q_n \rangle$$

$$= \langle Q_{n-1}, c(Q_n) \rangle = \langle Q_{n-1}, c(z^n - \sigma_n z^{n-1}) \rangle$$

$$= \langle Q_{n-1}, \lambda_n z^n - \sigma_n \lambda_{n-1} z^{n-1} \rangle$$

$$- \langle Q_{n-1}, \lambda_n(Q_n + \sigma_n Q_{n-1}) - \sigma_n \lambda_{n-1} Q_{n-1} \rangle$$

$$= (\lambda_n - \lambda_{n-1}) \sum_{i=0}^{n-1} \lambda_i \langle Q_{n-1}, Q_{n-1} \rangle.$$

**Lemma 4.8.** $\langle P_n, P_n \rangle_1 = \beta_n \langle P_{n-1}, P_{n-1} \rangle_1$, where $\beta_n$ is given in Prop. 4.4.

**Proof.** Since $\langle P_n, P_n \rangle_1 = \mu \langle Q_n, Q_n \rangle$, this follows from the previous lemma by an easy calculation left to the reader.

**Lemma 4.9.** $\langle zQ_n, Q_n \rangle = (A^{2n} \sum_{i=0}^n \lambda_i - A^{2n-2} \sum_{i=0}^{n-1} \lambda_i) \langle Q_n, Q_n \rangle$.

**Proof.** Proceeding as in the proof of Lemma 4.7., we find

$$\langle zQ_n, Q_n \rangle = \langle (z^n - \sigma_{n-1} z^{n-1}), Q_n \rangle$$

$$= \langle A^{2n} z^n + 1 - \sigma_{n-1} A^{2n-2} z^n, Q_n \rangle$$

$$= \langle A^{2n}(Q_{n+1} + \sigma_n Q_n - \sigma_{n-1} A^{2n-2} Q_n, Q_n \rangle$$

$$= (A^{2n} \sigma_n - A^{2n-2} \sigma_{n-1}) \langle Q_n, Q_n \rangle$$

as asserted.

**Lemma 4.10.** $\langle zP_n, P_n \rangle_1 = \alpha_n \langle P_n, P_n \rangle_1$, where $\alpha_n$ is given in Prop. 4.4.
Proof. Since \( \langle zP_n, P_n \rangle_1 = \mu_n^2 \langle tz^{-1} Q_n, Q_n \rangle = -A^3 \mu_n^2 \langle tQ_n, Q_n \rangle \) by Lemma 3.1(i), this follows from the previous lemma by an easy calculation left to the reader.

Proof of Proposition 4.4. Since \( \langle P_0, P_0 \rangle_1 = 1 \), it follows from Lemma 4.7. that \( \langle P_n, P_n \rangle_1 = [\prod_{i=0}^{\beta_i} \beta_i \). Notice that this is non-zero in \( \mathbb{Z}[A,A^{-1}] \). Hence to prove part (i) of the proposition, it suffices to show

\[
\langle zP_n, P_i \rangle_1 = \langle P_{n+1}, zP_n + \beta_{n-1}P_{n-1}, P_i \rangle_1 \quad \text{for } 0 \leq i \leq n + 1.
\]

For \( i \leq n - 2 \), we have \( \langle zP_n, P_{n-1} \rangle_1 = \langle P_n, zP_{n-1} \rangle_1 = \langle P_n, P_{n-1} \rangle_1 = \beta_{n-1} \langle P_{n-1}, P_{n-1} \rangle_1 \) which is the assertion for \( i = n - 1 \). Likewise for \( i = n \) the assertion follows from Lemma 4.10., and for \( i = n + 1 \), it is obvious since the \( P_i \) are monic.

The proof of parts (ii)-(iv) is by induction and left to the reader.

5. THE QUOTIENT ALGEBRAS \( V_p \)

Throughout this section, \( p \) will be an integer \( \geq 1 \), and we will set \( n = n(p) \), where \( n(p) \) is defined as in Section 4, that is \( n(1) = 1 \), \( n(2) = 2 \), and \( n(p) = \lfloor (p - 1)/2 \rfloor \) for \( p \geq 3 \).

Let \( \Lambda_p = \mathbb{Z}[A,A^{-1}] / \mathfrak{m}_p(A) \). Set \( \mathscr{B}_p = \mathscr{B} \otimes \Lambda_p = \Lambda_p[z] \), and let \( N_p \) denote the kernel of the bilinear form \( \langle , \rangle \) on \( \mathscr{B}_p \) (with values in \( \Lambda_p \)).

**Lemma 5.1.** \( N_p = \text{Span}\{Q_i : i \geq n(p)\} \).

This follows from Lemma 4.1. and Prop. 4.4., since \( \langle Q_i, Q_i \rangle \), up to multiplication by a power of \( A \), is equal to \( \langle P_i, P_i \rangle_1 \).

Notice that \( N_p \) is a principal ideal in \( \mathscr{B}_p \), generated by \( Q_{n(p)} \).

Let \( Q(\Lambda_p) \) denote the quotient field of \( \Lambda_p \).

**Lemma 5.2.** There is an element \( \omega \in \mathscr{B} \otimes Q(\Lambda_p) = Q(\Lambda_p)[z] \) such that \( \langle t(\omega), t(b) \rangle = \langle b \rangle = \langle t^{-1}(\omega), t^{-1}(b) \rangle \) for all \( b \in \mathscr{B}_p \).

This follows immediately from Prop. 4.3. and Remark 4.6.

Define \( V_p = \mathscr{B}_p/N_p \). Notice that this is a \( \Lambda_p \)-algebra, and a free \( \Lambda_p \)-module of rank \( n(p) \).

**Proposition 5.3.** The twist map \( t \) preserves \( N_p \), and thus induces a self-map of \( V_p \).

**Proof.** Suppose \( u \in N_p \). Then for all \( v \in \mathscr{B} \), we have \( \langle t(u), v \rangle = \langle u, t^{-1}(\omega)v \rangle = 0 \). See Fig. 4.

This shows \( t(N_p) \subset N_p \). Similarly, replacing \( t \) by \( t^{-1} \) and \( \omega \) by \( \omega \), one shows \( t^{-1}(N_p) \subset N_p \), whence the result.

Fig. 4.
Proposition 5.4. For any k-component banded link $L \subset S^3$, the induced meta-bracket $\mathcal{B}_p^k \to \Lambda_p$ factors through $V_p^k$.

Proof. Let $L_1, \ldots, L_k$ denote the components of $L$, and let $b_1, \ldots, b_k \in \mathcal{B}_p$. We must show that $\langle b_1, \ldots, b_k \rangle_L$ is zero in $\Lambda_p$ whenever one of the $b_i$ is in $N_p$. Consider first the case where $L_i$ is unknotted. Then $\langle b_1, \ldots, b_k \rangle_L = \langle t^i(b_i), v \rangle$ for some $i \in \mathbb{Z}$ and $v \in \mathcal{B}_p$, and the assertion follows from Prop. 5.3. The general case reduces to this special case as follows. Recall that any knot can be unknotted by changing some of its crossings. We can change a crossing of $L_i$ by a move $\mathcal{K}_\pm$, where the sign depends on the crossing. See Fig. 5 for a move $\mathcal{K}_+$.

Fig. 5.

Let $L'$ denote the new banded link, and suppose that the new component is $L_{i+1}$. We then have

$$\langle b_1, \ldots, b_k \rangle_L = \langle b_1, \ldots, b_i, v \rangle_{L'},$$

where $v = \omega$ or $v = \bar{\omega}$ depending on the sign of the move. Repeating this process, we can unknott the $i$-th component, thereby reducing to the special case treated before. This completes the proof.

Notice that Propositions 5.3 and 5.4 imply Theorem A. Moreover, they show that from now on, we may as well work in $V_p = \mathcal{B}_p / N_p$.

Remark 5.5. Recall that we have seen in Prop. 4.3 that $\omega$ is unique up to adding an element $L \in \text{Span} \{ P_i, i \geq n(p) \}$. Notice that its image in $V_p \otimes Q(\Lambda_p)$ does not depend on $\mathcal{B}$, since it follows from Prop. 5.3 that $N_p = \text{Span} \{ P_i; i \geq n(p) \}$.

6. INVARIANTS AT $2p$-TH ROOTS OF UNITY

Define $\Omega_p \in V_p$ (except $\Omega_2 \in V_2 \otimes \mathbb{Z} [\frac{1}{2}]$) by $\Omega_1 = 1, \Omega_2 = 1 + \frac{1}{2},$ and $\Omega_p = \sum_{i=0}^{n} \langle e_i \rangle e_i$ for $p \geq 3$, where $n = \left( \frac{(p - 1)}{2} \right)$.

We will now prove Theorem B. Notice that the cases $p = 1, 3, 4$ are trivial since $\Omega_p = 1$ in these cases.

6.1. The case $p = 2$. Then rank $V_p = 2$. Using the equality $A^2 = -1$, we find that in $V_2$, we have

$$\omega = 1 + \frac{\langle P_1 \rangle}{\langle P_1, P_1 \rangle} P_1 = 1 + \frac{1 - A}{4} (z - 2A) = \frac{1 - A}{4} (z + 2).$$

Since $\Omega_2 = 1 + \frac{1}{2} = (1 + A)\omega, \Omega_2$ satisfies condition (*) for $\varepsilon = +1$. Moreover, since $\Omega_2 = \Omega_2$, it satisfies condition (*) for $\varepsilon = -1$ as well. Thus Theorem B follows from Prop. 2.1.
For the rest of the proof of Theorem B, we suppose $p \geq 3$ and set $n = n(p) = [(p - 1)/2]$. By Proposition 2.1., it suffices to show that $\Omega_p$ satisfies condition $(\ast)$ over $\Lambda_p[\frac{1}{p}]$ for $\varepsilon = \pm 1$. This will be established in Propositions 6.5. and 6.8.

**Lemma 6.2.** If $p$ is even, then $Q_n = e_n$ in $\mathcal{B}_p$. If $p$ is odd, then $Q_n = e_n - e_{n-1}$ in $\mathcal{B}_p$.

**Proof.** Suppose first that $p = 2n + 2$ is even. For all $i \geq 0$, we have

$$
\langle e_n, z^i \rangle = \langle c^i(e_n) \rangle = \lambda_n^i \langle e_n \rangle = 0,
$$

since $\langle e_n \rangle = 0 \in \Lambda_p$ by Lemma 3.4. Hence $e_n \in N_p$. This implies that $e_n$ is a multiple of $Q_n$, since $Q_n$ generates $N_p$ as an ideal. But $e_n$ and $Q_n$ are both monic of degree $n$, whence the result.

In the case $p = 2n + 1$ odd, the result is proved similarly upon observing that $\lambda_n = \lambda_{n-1}$ and $\langle e_n \rangle = \langle e_{n-1} \rangle$ in this case.

**Remark.** A purely algebraic proof of Prop. 5.3. can be given based on Lemma 6.2. Indeed, set $v_0 = Q_n$ and for $i \geq 1$, set $v_i = e_{n+i} + e_{n-i}$ if $p$ is even, and $v_i = e_{n+i} - e_{n-i-1}$ if $p$ is odd. (Here the definition of $e_n$ is extended formally to negative $n$ by requiring that $ze_n = e_{n+1} + e_{n-1}$ holds for all $n \in \mathbb{Z}$.) Then the $v_i$ for $i \geq 0$ are eigen-vectors for the twist map $t$ on $\mathcal{B}_p$, and using Lemma 6.2., it is not hard to see that they are a basis for $N_p$. The result follows. Compare this with the next lemma.

**Lemma 6.3.** In $V_p$, the following relations hold.

(i) $e_{i+p} = (-1)^p e_i$
(ii) $e_{i+1} = -e_{i-1}$ if $p$ is even.
(iii) $e_{i+1} = e_{i-1}$ if $p$ is odd.

This follows easily from Lemma 6.2.

**Lemma 6.4.** Let $G : V_p \rightarrow V_p$ be defined by $G(x) = \sum_{i=0}^{2p-1} e_i x(t(e_i x))$. Then $G$ is $V_p$-linear, i.e. $G(x) = G(1)x$ for all $x \in V_p$.

**Proof.** It suffices to consider the case where $x$ is one of the $e_j$. The result is obvious for $e_{-1} = 0$ and $e_0 = 1$. For $j \geq 1$, it is easily proved by induction, using $ze_j = e_{j+1} + e_{j-1}$ and observing that the sum is over a period $2p$.

**Proposition 6.5.** For $\varepsilon = \pm 1$, and all $b \in V_p$, one has

$$
\langle t^\varepsilon(\Omega_p), t^\varepsilon(b) \rangle = \langle t^\varepsilon(\Omega_p), \langle b \rangle \rangle.
$$

**Proof.** Notice first that Lemma 6.3. implies that $\Omega_p = \frac{1}{4} \sum_{i=0}^{p-1} \langle e_i \rangle e_i = \frac{1}{4} \sum_{i=0}^{2p-1} \langle e_i \rangle e_i$. Hence

$$
\langle t(\Omega_p), t(b) \rangle = \langle \Omega_p, b \rangle = \langle \Omega_p, b \rangle = \langle t(\Omega_p), b \rangle
$$

- \left\langle \frac{1}{4} \sum_{i=0}^{2p-1} \langle e_i \rangle t(e_i b) \right\rangle

= \frac{1}{4} \langle G(h) \rangle = \frac{1}{4} \langle G(1) \rangle \langle b \rangle

= \langle t(\Omega_p) \rangle \langle b \rangle.
The case $\varepsilon = -1$ is proved similarly, using the map $\tilde{G}$ defined by $\tilde{G}(x) = \sum_{i=0}^{2p-1} e^{i(1-e)x}$.

Set $g(p, s) = \frac{1}{2} \sum_{k=0}^{2p-1} (-1)^k A^{sk^2} = \frac{1}{2} \sum_{k=0}^{2p-1} A^{sk^2+ksp} \in \Lambda_p$. (Sums of this type are known in Analytic Number Theory as generalized Gauss sums (see [1] for an overview)).

**Lemma 6.6.** (i) $g(p, 1)g(p, 1) = p$

(ii) $g(p, 1) = A^p(p-1)^{\frac{1}{2}} g(p, 1)$.

**Proof:** We may embed $\Lambda_p$ into $\mathbb{C}$ by setting $A = e^{\pi i/2p}$. We then have $g(p, 1) = \sqrt{p} e^{\pi i(1-p)/4}$ (see [1]). Observe that $g(p, 1)/g(p, 1) = e^{-\pi i(1-p)/2} = A^p(p-1)^{\frac{1}{2}}$ is contained in the image of $\Lambda_p$. The result follows.

**Lemma 6.7** $\langle t(\Omega_p) \rangle = \frac{A^{-3}}{A^2 - A^{-2}} g(p, 1)$.

**Proof:**

$$\langle t(\Omega_p) \rangle = \frac{1}{4} \sum_{k=0}^{2p-1} \mu_k \langle e_k \rangle^2 = \frac{1}{4} \sum_{k=0}^{2p-1} \mu_{k-1} \langle e_{k-1} \rangle^2$$

$$= \frac{1}{4} \frac{1}{(A^2 - A^{-2})^2} \sum_{k=0}^{2p-1} (-1)^{k-1} A^{k^2-1}(A^{2k} - A^{-2k})^2$$

$$= -\frac{A^{-1}}{4} \frac{1}{(A^2 - A^{-2})^2} (2A - 2 - 2) \sum_{k=0}^{2p-1} (-1)^k A^{k^2}$$

$$= \frac{A^{-3}}{A^2 - A^{-2}} \frac{1}{2} \sum_{k=0}^{2p-1} (-1)^k A^{k^2}$$

as asserted.

**Proposition 6.8.** For $\varepsilon = \pm 1$, $\langle t'(\Omega_p) \rangle$ is invertible in $\Lambda_p[\frac{1}{p}]$.

This follows from Lemmas 6.6. and 6.7. using the fact that $\langle t^{-1}(\Omega_p) \rangle = \langle t(\Omega_p) \rangle$.

The proof of Theorem B is complete.

**Remark 6.9.** (i) It is easy to calculate

$$\theta_p(S^1 \times S^2) = \langle \Omega_p \rangle = \begin{cases} p & \text{if } p \leq 2 \\ \frac{p}{(A^2 - A^{-2})^2} & \text{if } p \geq 3. \end{cases}$$

Moreover, using Lemmas 6.6 and 6.7, one finds that for all $p \geq 1$

$$\langle \Omega_p \rangle = \langle t(\Omega_p) \rangle \langle t'(\Omega_p) \rangle.$$

(ii) For $x, y \in \Lambda_p$, write $x \sim y$ if there is a unit $u \in \Lambda_p$ such that $x = uy$. Using (i), it is not hard to see that $\langle \Omega_o \rangle \sim 2$ in $\Lambda_o$. Hence, the invariant $\theta_o(M)$ lies in $\Lambda_o[\frac{1}{3}] \subset \Lambda_o[\frac{1}{p}]$.

(iii) Similarly, it is easy to see that $\langle \Omega_6 \rangle \sim 5$ in $\Lambda_5$, a fact which will be used in the proof of Proposition 6.10.

We will now give a precise statement of Remark (3) made in the introduction. Set $\Lambda_p = \Lambda_p[\frac{1}{p}]$ for $p \notin \{1, 3, 4, 6\}$, $\Lambda_{p'} = \Lambda_p$ for $p \in \{1, 3, 4\}$, and $\Lambda_6 = \Lambda_6[\frac{1}{3}]$. Notice that by Remark 6.9., for all $p \geq 1$, one has $\theta_p(M) \in \Lambda_p$. 

THREE-MANIFOLD INVARIANTS
Proposition 6.10. Suppose \( \Lambda \) is an integral domain containing a homomorphic image of \( \mathbb{Z}[A, A^{-1}] \) such that there is an element \( \Omega \in \mathbb{Z} \otimes \Lambda = \Lambda[\frac{1}{z}] \) satisfying condition \((*)\) over \( \Lambda \). Let \( \theta_{\Omega}(M) \) denote the 3-manifold invariant defined by \( \Omega \). Then there is an integer \( p \geq 1 \) and a unit \( \lambda \in \Lambda \) such that the map \( \mathbb{Z}[A, A^{-1}] \to \Lambda \) factors through a homomorphism \( f: \Lambda_p \to \Lambda \), and such that
\[
\theta_{\Omega}(M) = \lambda^{b_1(M)} f(\theta_{\lambda}(M))
\]
for all closed connected oriented 3-manifolds \( M \), where \( b_1(M) \) is the first Betti number of \( M \).

Proof. By hypothesis \((*)\), we have condition \((i)\) of Prop. 4.3.1 (with \( \omega = \langle t\Omega \rangle^{-1} \Omega \)). Hence there is an integer \( p \geq 1 \) such that the map \( \mathbb{Z}[A, A^{-1}] \to \Lambda \) factors through a homomorphism \( f: \Lambda_p \to \Lambda \). Let \( p \) be minimal with respect to this property. Then by Prop. 4.3 and Lemma 4.5, the bilinear form \( \langle , \rangle_1 = \langle t(\cdot), t(\cdot) \rangle \) on \( V_\Lambda = V_2 \otimes \mathbb{Z}[\frac{1}{z}] \) has non-zero determinant.

Let \( f_\lambda^*(\Omega_\lambda) \) denote the image of \( \Omega_\lambda \) in \( V_\Lambda \). (If \( p = 2 \), then \( \Omega_2 = 1 + \frac{1}{2} \in V_2 \otimes \mathbb{Z}[\frac{1}{z}] \). But \( 2 \) is invertible in \( \Lambda \), because \( \phi_2(A) = 1 + A \) is invertible in \( \Lambda \) by Prop. 4.3, and \( (1 + A)^2 = 2A \). Hence \( f_\lambda^*(\Omega_2) \) exists in \( V_\Lambda \).) Notice that this is an indivisible element of \( V_\Lambda \), since \( V_\Lambda \) has basis \( \{e_0, \ldots, e_{n(p)-1}\} \), and the coefficient of \( e_0 \) in \( \Omega_2 \) is 1. Since \( \Omega \) and \( f_\lambda^*(\Omega_\lambda) \) both satisfy the equation
\[
\langle \omega, b \rangle_1 = \langle t(\omega), t(b) \rangle
\]
for all \( b \in V_\Lambda \), it follows that
\[
\Omega = \lambda f_\lambda^*(\Omega_\lambda)
\]
for some \( \lambda \in \Lambda \).

Next, we claim that \( f \) factors through \( \Lambda_p \), and hence \( f(\theta_{\lambda}(M_L)) \) makes sense in \( \Lambda \). Indeed, let \( \Lambda^* \) denote the group of units of \( \Lambda \). By hypothesis, we have \( \langle t(\Omega) \rangle \in \Lambda^* \), hence \( \lambda \in \Lambda^* \) and \( \langle t(f_\lambda^*(\Omega_\lambda)) \rangle = f(\langle t(\Omega_\lambda) \rangle) \in \Lambda^* \). By Remark 6.9 (i), it follows that \( f(\langle t(\Omega_\lambda) \rangle) \in \Lambda^* \).

If \( p \geq 7 \), this implies \( p \in \Lambda^* \) by Remark 6.9 (i) together with the fact that \( A^2 - A^{-2} \in \Lambda^* \) by Prop. 4.3. If \( p = 5 \), this implies \( 5 \in \Lambda^* \) by Remark 6.9 (iii), and if \( p = 6 \), it implies \( 2 \in \Lambda^* \) by Remark 6.9 (ii). If \( p = 2 \), then we have already shown that \( 2 \in \Lambda^* \). Hence \( f \) factors through \( \Lambda_p \) as asserted.

Finally, recall that for a banded link \( L \subset S^3 \), the first Betti number \( b_1(M_L) \) is equal to the number \( b_0(L) \) of zero eigenvalues of the linking matrix of \( L \). Since \( b_+(L) + b_-(L) + b_0(L) \) is equal to the number of components of \( I \), the equation \( \Omega = \lambda f_\lambda^*(\Omega_\lambda) \) implies \( \theta_{\Omega}(M_L) = \lambda^{b_1(M_L)} f(\theta_{\lambda}(M_L)) \) as asserted.

Remark 6.11. Let \( s > 0 \) and consider the lens space \( L(s, 1) \). It is obtained by surgery on a 1-component banded unknot with writhe \( s \). Hence
\[
\theta_{\lambda}(L(s, 1)) = \frac{\langle t^i(\Omega_\lambda) \rangle}{\langle t(\Omega_\lambda) \rangle}.
\]
In general, this can be evaluated using well-known reciprocity formulae for Gauss sums. In particular, if \( (p, s) = 1 \), then we may proceed as in the proof of Lemma 6.7. to find the following formula
\[
\theta_p(L(s, 1)) = A^{1-s} \frac{A^{-4s'} - 1}{A^4 - 1} \frac{g(p, s)}{g(p, 1)}.
\]
Here \( ss' \equiv 1 \mod p \).
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