Toughness, hamiltonicity and split graphs

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Abstract

Related to Chvátal's famous conjecture stating that every 2-tough graph is hamiltonian, we study the relation of toughness and hamiltonicity on special classes of graphs.

First, we consider properties of graph classes related to hamiltonicity, traceability and toughness concepts and display some algorithmic consequences. Furthermore, we present a polynomial time algorithm deciding whether the toughness of a given split graph is less than one and show that deciding whether the toughness of a bipartite graph is exactly one is coNP-complete.

We show that every 3-tough split graph is hamiltonian and that there is a sequence of non-hamiltonian split graphs with toughness converging to 3.

1. Introduction

We consider only finite undirected graphs $G = (V,E)$ without loops or multiple edges, unless stated otherwise. The cardinality of the vertex set $V$ is denoted by $n$ and the cardinality of the edge set $E$ is denoted by $m$. Throughout the paper, we assume $n \geq 3$, a standard assumption in sufficient conditions for hamiltonicity. A good reference for undefined standard graph theory terms is [7].

First we introduce a few definitions and some convenient notation. Let $G = (V,E)$ be a graph. For every vertex $v \in V$ we denote by $N(v)$ the set of all neighbours of $v$, $N(v) := \{ u \in V : \{u,v\} \in E \}$. Furthermore, let $N(V') := \bigcup_{v \in V'} N(v)$ for every set $V' \subseteq V$.

A graph $H = (V(H),E(H))$ is said to be a subgraph of the graph $G = (V(G),E(G))$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A subgraph $H$ is said to be a spanning subgraph of $G$ if $V(H) = V(G)$. Let $G = (V,E)$ be a graph and $V'$ be a subset of $V$. Then $G[V']$.

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denotes the graph induced by the vertex set $V'$, i.e. $G_{V'}$ has vertex set $V'$ and two vertices of $V'$ are adjacent in $G_{V'}$ iff they are adjacent in $G$. Furthermore, we write $G - V'$ instead of $G_{V-V'}$ and for $v \in V$ we denote the one-vertex-deleted subgraph $G_{V-v}$ by $G - v$.

A set $V' \subseteq V$ in a graph $G = (V,E)$ is said to be independent if $\{u,v\} \notin E$ for every pair $u,v \in V'$ and it is said to be a clique if $\{u,v\} \in E$ for every pair $u,v \in V'$. A graph $G = (V,E)$ is said to be bipartite if its vertex set $V$ can be partitioned into two independent sets $X$ and $Y$. Usually, a bipartite graph is denoted by $G = (X,Y,E)$ and the independent sets $X$ and $Y$ are called the colour classes of $G$. A graph $G = (V,E)$ is said to be a split graph if its vertex set $V$ can be partitioned into an independent set $I$ and a clique $C$. Usually, a split graph is denoted by $G = (C,I,E)$. For definitions and properties of graph classes not given here we refer to [8, 17].

We denote the number of components of a graph $G$ by $\omega(G)$. A set $S \subseteq V$ is said to be a cutset of $G$ if $\omega(G - S) > 1$. (Note that $S = \emptyset$ is a cutset iff $G$ is disconnected.)

The toughness $t(G)$ of a graph $G$ was defined by Chvátal in [10] in the following way: The toughness of a complete graph is infinity, $t(K_n) = \infty$. If $G$ is not complete, then $t(G) := \min\{|S|/\omega(G-S) : \omega(G-S) > 1\}$. A graph $G$ is said to be $t$-tough if $t(G) \geq t$ holds, i.e. $|S| \geq t \cdot \omega(G-S)$ for every cutset $S$ of $G$. A cutset $S$ of $G$ is said to be a tough cutset if $|S| = \omega(G-S) \cdot t(G)$ holds. A vertex $x$ is called simplicial in $G$ if $N(x)$ is a clique of $G$, and $y$ is called star vertex if $N(y) = V(G) \setminus \{y\}$. Obviously, every graph $G$ has a tough cutset containing all star vertices but no simplicial vertex of $G$. We will denote by $\kappa(G)$ and $\kappa'(G)$ the vertex connectivity and the edge connectivity of the graph $G$, respectively.

A 2-factor of a graph $G$ is a spanning subgraph $F$ of $G$ such that $F$ is 2-regular. Hence, a 2-factor of $G$ is a collection of vertex-disjoint cycles covering all vertices of $G$. A graph $G$ is said to be 2-factorable if it has a 2-factor. A hamiltonian circuit (resp. hamiltonian path) of a graph $G$ is a simple cycle (resp. path) containing all vertices of $G$. A graph is said to be hamiltonian if it has a hamiltonian circuit and said to be traceable if it has a hamiltonian path. Clearly, a hamiltonian circuit itself is a 2-factor, thus every hamiltonian graph is 2-factorable.

The decision problems HAMILTONIAN CIRCUIT $= \{G = (V,E) : G$ is hamiltonian$\}$ and HAMILTONIAN PATH $= \{G = (V,E) : G$ is traceable$\}$ are well-known NP-complete problems [16]. On the other hand, there is a polynomial time algorithm deciding whether a given graph is 2-factorable (cf. [24]). Consequently, HAMILTONIAN CIRCUIT is hard only on 2-factorable graphs. We believe that one should know more on non-hamiltonian 2-factorable graphs. (Indeed, some of the well-known non-hamiltonian graphs are even not 2-factorable, thus they have a good certificate of non-hamiltonicity.)

The following special types of non-hamiltonian and non-traceable graphs, respectively, were studied extensively in the literature: A graph $G = (V,E)$ is said to be hypohamiltonian if it is non-hamiltonian but for every vertex $v \in V$ the graph $G - v$
is hamiltonian. A graph $G = (V,E)$ is said to be hypotraceable if it is not traceable but for every vertex $v \in V$ the graph $G - v$ is traceable.

It is well-known that every hamiltonian graph is 1-tough [7, 10]. The main motivation of this paper is the following related conjecture of Chvátal.

**Conjecture 1.** Every 2-tough graph is hamiltonian.

Chvátal's conjecture would imply that the maximum toughness of a non-hamiltonian graph is less than 2. Indeed, Enomoto et al. [15] present a sequence of non-2-factorable graphs with toughness tending to 2. Thus, the constant 2 is the smallest nowadays possible value for the constant $t_0$ in the second (weaker) version of Chvátal's conjecture.

**Conjecture 2** (Chvátal [10]). There exists a $t_0$ such that every $t_0$-tough graph is hamiltonian.

This motivates the study of the following problems:

**Problem A.** Given a graph class $\mathcal{G}$, determine the maximum toughness of a non-hamiltonian graph in $\mathcal{G}$.

**Problem B.** Given a graph class $\mathcal{G}$, determine the maximum toughness of a non-2-factorable graph in $\mathcal{G}$.

The answers to both problems are known for the class of bipartite graphs [21] and the answer to Problem A is known for the class of planar graphs [10, 18]. We will settle both problems for split graphs.

The paper is organized as follows. In Section 2 we study properties of graph classes related to hamiltonicity and traceability of graphs. We will show that HAMILTONIAN CIRCUIT (resp. HAMILTONIAN PATH) is not likely to be NP-complete when restricted to a cycle-tough (resp. path-tough) class of graphs. In Section 3 we show that there is a sequence of non-hamiltonian split graphs with toughness converging to $\frac{3}{2}$ and that every $\frac{3}{2}$-tough split graph is hamiltonian. In Section 4 we consider the algorithmic complexity of computing the toughness when restricted to bipartite and split graphs, respectively.

2. Hamiltonicity, traceability and complexity

We are going to show some interesting relations between properties of graph classes related to more or less obvious necessary conditions for hamiltonicity and traceability. During this section we suppose that every graph class is hereditary, i.e. if a graph belongs to the class then any induced subgraph belongs to the class.
**Definition 2.1.** Let \( \mathcal{G} \) be a hereditary class of graphs. We define the following properties of \( \mathcal{G} \):

\( \mathcal{G} \) is said to be **path-tough** if for every graph \( G = (V,E) \) of \( \mathcal{G} \) we have that: \( G \) has a hamiltonian path iff the graph \( G - S \) has at most \( |S| + 1 \) components for every cutset \( S \subseteq V \) of \( G \).

\( \mathcal{G} \) is said to be **cycle-tough** if for every graph \( G = (V,E) \) of \( \mathcal{G} \) we have that: \( G \) has a hamiltonian circuit iff the graph \( G - S \) has at most \( |S| \) components for every cutset \( S \subseteq V \) of \( G \).

\( \mathcal{G} \) is said to have the **Steiner–Deogun property** if for every graph \( G = (V,E) \) of \( \mathcal{G} \) we have that: \( G \) has a hamiltonian circuit iff the graph \( G - v \) has a hamiltonian path for every vertex \( v \in V \).

Note that in all the three properties defined above we could have required 'if' instead of 'iff', since the 'only if' part holds always for every graph \( G \).

We have chosen the notion Steiner–Deogun property since Steiner and Deogun showed in [14] that a cocomparability graph (a complement of a comparability graph) is hamiltonian iff the graph \( G - v \) has a hamiltonian path for every vertex \( v \in V \), i.e. cocomparability graphs have the Steiner–Deogun property. In [23] it was shown that cocomparability graphs are path-tough without using the order-theoretic methods of [14]. Finally, cocomparability graphs are shown to be cycle-tough in [13].

It is worth mentioning that cocomparability graphs contain all interval graphs and all permutation graphs — two well-known subclasses of perfect graphs [17].

We are going to show that our three properties of graph classes are indeed strongly related to each other (without the need of any structural properties of a certain graph class).

**Theorem 2.2.** If a hereditary graph class \( \mathcal{G} \) is cycle-tough, then it also has the Steiner–Deogun property.

**Proof.** Let \( G \in \mathcal{G} \) be a graph such that \( G - v \) is traceable for every \( v \in V \). Then, every one-vertex-deleted subgraph fulfills the necessary condition on traceable graphs, hence we have

\[
\bigwedge_{v \in V} \bigwedge_{Y \subseteq V \setminus \{v\}} \omega((G - v) - Y) \leq |Y| + 1.
\]

This implies

\[
\bigwedge_{Z \subseteq V} \omega(G - Z) \leq |Z|.
\]

Therefore, \( G \) is hamiltonian, since \( \mathcal{G} \) is cycle-tough.

**Theorem 2.3.** If a hereditary graph class \( \mathcal{G} \) is path-tough and has the Steiner–Deogun property, then it is also cycle-tough.
Proof. Let \( G \in \mathcal{G} \) be a graph fulfilling
\[
\bigwedge_{Y \subseteq V} \omega(G - Y) \leq |Y|.
\]
This implies
\[
\bigwedge_{v \in V} \bigwedge_{Z \subseteq V \setminus \{v\}} \omega(G - (Z \cup \{v\})) \leq |Z| + 1.
\]
Since \( \mathcal{G} \) is hereditary and path-tough we get the traceability of all one-vertex-deleted subgraphs of \( G \). Thus, the Steiner–Deogun property implies that \( G \) itself is hamiltonian. \( \square \)

Remark 2.4. It would be possible to define path-tough and cycle-tough graphs as well as graphs with Steiner–Deogun property similar to Definition 2.1. Then one could prove statements similar to Theorems 2.2 and 2.3 by adding suitable conditions on the one-vertex-deleted subgraphs.

Next we consider hypohamiltonian and hypotraceable graphs in classes with the Steiner–Deogun property.

Proposition 2.5. If a hereditary graph class \( \mathcal{G} \) has the Steiner–Deogun property, then it contains neither hypohamiltonian nor hypotraceable graphs.

Proof. If a graph \( G \in \mathcal{G} \) is hypohamiltonian or hypotraceable, then \( G - v \) is traceable for every \( v \in V \) but \( G \) itself is non-hamiltonian. Hence, \( G \) itself violates the Steiner–Deogun property. \( \square \)

It is known that cycle-toughness of a graph class supports very well the design of a polynomial time algorithm deciding whether a given graph \( G \) of the class is hamiltonian or not, since any non-hamiltonian graph in the class must have a good certificate for non-hamiltonicity, namely a cutset \( S \) such that \( G - S \) has more than \( |S| \) components (cf. [22, 25]). Clearly, there may be other ways to design a polynomial time algorithm for deciding whether a graph is hamiltonian on a certain graph class, e.g. by exhaustive search in a polynomially bounded search space. The algorithm for partial \( k \)-trees, \( k \) fixed, successfully uses this approach although the class is in general not cycle-tough (cf. [1]).

However, we will show that HAMILTONIAN CIRCUIT when restricted to a cycle-tough graph class \( \mathcal{G} \) cannot remain NP-complete, unless \( \text{NP} = \text{coNP} \). The latter event is considered to be very unlikely by complexity theorists, analogously to \( \text{P} = \text{NP} \) (cf. [2, p. 67]).

Theorem 2.6. Let \( \mathcal{G} \) be a graph class whose recognition problem is in \( \text{NP} \cap \text{coNP} \). If HAMILTONIAN CIRCUIT remains \( \text{NP} \)-complete when restricted to \( \mathcal{G} \), then \( \mathcal{G} \) cannot be cycle-tough, unless \( \text{NP} = \text{coNP} \).
Proof. Let $\mathcal{G}$ be cycle-tough. Then, we have
\[
\{G \in \mathcal{G} : G \text{ hamiltonian}\} = \{G \in \mathcal{G} : t(G) \geq 1\}.
\]
The first set is clearly in NP. The second set is in coNP since $t(G) \geq 1$ iff for every cutset $S$ of $G$ holds $\omega(G - S) \leq |S|$. Thus, HAMILTONIAN CIRCUIT restricted to $\mathcal{G}$ belongs to NP $\cap$ coNP. If a NP-complete problem would belong to NP $\cap$ coNP, then NP and coNP would coincide (cf. [2, Corollary 3.2]).

Note that every graph class $\mathcal{G}$ which has a polynomial time recognition algorithm fulfills the assumptions of Theorem 2.6.

Remark 2.7. When studying restrictions of graph problems to certain graph classes we consider promise problems, i.e. the algorithm works correctly if the promise ‘the input belongs to the class’ is fulfilled. Under this model Theorem 2.6 holds without any requirement about the complexity of recognizing $\mathcal{G}$. Therefore, we omit such assumptions in the remainder of this section.

A similar statement concerning HAMILTONIAN PATH and path-toughness can be shown analogously.

Theorem 2.8. If HAMILTONIAN PATH remains NP-complete when restricted to a graph class $\mathcal{G}$ then $\mathcal{G}$ cannot be path-tough, unless NP $\cap$ coNP.

Obviously, this implies

Corollary 2.9. If HAMILTONIAN CIRCUIT remains NP-complete when restricted to the graph class $\mathcal{G}$, then there is a non-hamiltonian graph $G \in \mathcal{G}$ such that for every cutset $S$ of $G$
\[
\omega(G - S) \leq |S|,
\]
i.e. $\mathcal{G}$ contains a 1-tough non-hamiltonian graph $G$, unless NP $\cap$ coNP.

Corollary 2.10. If HAMILTONIAN PATH remains NP-complete when restricted to the graph class $\mathcal{G}$, then there is a non-traceable graph $G \in \mathcal{G}$ such that for every cutset $S$ of $G$
\[
\omega(G - S) \leq |S| + 1\text{ unless NP}=\text{coNP}.
\]

The algorithmic complexity of HAMILTONIAN PATH and HAMILTONIAN CIRCUIT when restricted to certain graph classes was studied very extensively, particularly in the last ten years. Tables summarizing the recent status can be found in [20, 11]. It is worth mentioning that for all the studied (natural) graph classes the complexity of the two problems coincides, although there is no obvious general explanation. (Artificial graph classes with different complexity of the restrictions are known, see e.g. [11].)
At the end of this section we consider Problems A and B for some special classes of graphs. From a famous result of Tutte one can deduce that the maximum toughness of a non-hamiltonian planar graph is exactly $\frac{3}{2}$ (see [10, 18]). In the next section we prove a similar result for split graphs. Bauer and Schmeichel [4] constructed a sequence of non-2-factorable graphs with toughness converging to 2 and it is not hard to see that all these graphs are perfect graphs.

Let us consider two well-known classes of perfect graphs, namely bipartite graphs and split graphs. HAMILTONIAN CIRCUIT remains NP-complete on both classes. Moreover, it remains NP-complete when restricted to proper subclasses, namely chordal bipartite graphs and strongly chordal split graphs [26]. By Corollary 2.9, there should be a bipartite graph and a split graph being 1-tough and non-hamiltonian.

Clearly, the toughness of a bipartite graph is at most 1, thus bipartite graphs do not create an interesting instance to Problem A. Anyhow, there are 1-tough non-hamiltonian bipartite graphs; an example is shown in Fig. 1. Thus, obviously the answer to Problem A is: Every bipartite graph $G$ with $t(G) > 1$ is hamiltonian and there is a non-hamiltonian 1-tough bipartite graph. (Naturally, the first part of the statement is dealing with an empty set of graphs.)

Regarding Problem B, Katerinis [21] has shown that every bipartite graph $G$ with $t(G) \geq 1$ has a 2-factor, except $K_2$. Clearly, the complete bipartite graphs $K_{n,n-1}$ have no 2-factor and $t(K_{n,n-1}) \to 1$ if $n \to \infty$.

Finally, we would like to mention that the classes of bipartite graphs and split graphs contain hypohamiltonian graphs, as e.g. the bipartite graph in Fig. 1 and the split graph constructed from this bipartite graph by adding all edges between the vertices of one colour class. Consequently, by Proposition 2.5, both graph classes do not have the Steiner–Deogun property.

3. The case of split graphs

By our observations, there should be a 1-tough non-hamiltonian split graph. Indeed, Chvátal already gave an example in [10], see Fig. 2. We will extend this example
presenting a sequence of non-hamiltonian split graphs with toughness converging to \( \frac{3}{2} \), indeed all these graphs do not even have a 2-factor.

**Theorem 3.1.** There is a sequence \( \{G_n\}_{n=1}^{\infty} \) of non-2-factorable split graphs with \( t(G_n) \to \frac{3}{2} \).

**Proof.** The split graphs are constructed as follows: For every integer \( n \geq 1 \) let \( G_n = (I, C_1 \cup C_2, E) \) with

\[
I = \{i_1, i_2, \ldots, i_{2n+1}\},
\]
\[
C_1 = \{c^1_1, c^1_2, \ldots, c^1_{2n+1}\},
\]
\[
C_2 = \{c^2_1, c^2_2, \ldots, c^2_n\},
\]
\[
E = \{\{i_r, c^1_s\} : 1 \leq r \leq 2n + 1\} \cup \{\{i_r, c^2_s\} : 1 \leq r \leq 2n + 1; 1 \leq s \leq n\}
\]
\[
\cup \{\{c^j_s, c^k_l\} : c^j_s \neq c^k_l\}.
\]

Thus, \( |V(G_n)| = 5n + 2 \). Furthermore, note that \( G_1 \) is exactly Chvátal's graph.

First, suppose that the graph \( G_n \) has a 2-factor \( F \) for some \( n \geq 1 \). Then there are \( 4n + 2 \) edges between \( I \) and \( C_1 \cup C_2 \) in \( F \). Since \( G_n \) has only \( 2n + 1 \) edges between \( I \) and \( C_1 \), \( F \) has to contain at least \( 2n + 1 \) edges between \( I \) and \( C_2 \). However, the \( n \) vertices of \( C_2 \) cannot contribute more than \( 2n \) edges between \( I \) and \( C_2 \) in \( F \). Thus, such a 2-factor \( F \) of \( G_n \) cannot exist.

We show that \( t(G_n) = 3n/(2n + 1) \) for every \( n \geq 1 \). W.l.o.g. we have to consider only cutsets \( S \) of \( G_n \) with \( S \cap I = \emptyset \), since otherwise \( S' = S - I \) yields \( |S'| < |S| \) and \( \omega(G_n - S') \geq \omega(G_n - S) \). Furthermore, every cutset \( S \) has to be a superset of \( C_2 \), since every vertex of \( C_2 \) is a star vertex of \( G_n \).

Hence, we may assume \( C_2 \subseteq S \subseteq C_1 \cup C_2 \). Consequently, \( \omega(G_n - S) \) depends only on \( |S \cap C_1| \). Suppose \( |S \cap C_1| = r \) with \( 1 \leq r \leq 2n + 1 \). We have, \( \omega(G_n - S) = 1 + r \) if \( r \leq 2n \) holds, and \( \omega(G_n - S) = 2n + 1 \) if \( r = 2n + 1 \) holds. Thus, we get

\[
t(G_n) = \min_{\text{cutset}} \frac{|S|}{\omega(G_n - S)} = \min_{1 \leq r \leq 2n} \frac{n + r}{1 + r} = \frac{3n}{2n + 1}.
\]
Corollary 3.2. For every \( \varepsilon > 0 \) there is a non-hamiltonian split graph \( G \) with \( t(G) > (3/2 - \varepsilon) \).

Katerinis has given a sufficient condition for the existence of a 2-factor in bipartite graphs [21, Theorem 2] implying that every split graph \( G \) with \( t(G) \geq 3 \) is 2-factorable. We are going to strengthen this and settle Problems A and B for split graphs.

Theorem 3.3. Every \( \frac{3}{2} \)-tough split graph is hamiltonian.

Proof. Let \( G = (C, I, E) \) be a split graph with \( t(G) \geq \frac{3}{2} \) on at least three vertices. We assume w.l.o.g. that \( G \) is not complete. Since \( G \) is a split graph, any tough cutset \( S \) of \( G \) does not contain a vertex of \( I \). This implies that \( t(G) \geq \frac{3}{2} \) iff the following conditions hold for every subset \( I' \subseteq I \):

(i) \( |N(I')| \geq \frac{3}{2} |I'| \),

(ii) \( |N(I')| \geq \frac{3}{2}(|I'| + 1) \), if \( N(I') \neq C \).

Let \( G^* \) be the bipartite graph obtained from \( G \) by removing all edges of the clique \( C \). Assume all the vertices in \( I \) are coloured white and all those in \( C \) are coloured black. Any path of \( G^* \) with both end points black and any trivial path consisting of one black vertex is called a \( B \)-path. Observe that \( G \) is hamiltonian if and only if the vertex set of \( G^* \) can be partitioned into \( B \)-paths. (Necessity is obvious since the hamiltonian cycle of \( G \) contains no consecutive white vertices; sufficiency follows from the fact that a \( B \)-path partition of \( G^* \) can be easily completed to a hamiltonian cycle of \( G \) by using suitable edges of \( C \) between the endvertices of distinct \( B \)-paths.)

Then by (ii), the theorem immediately follows from the next covering lemma.

Lemma 3.4 (Covering lemma). Let \( H = (W, B, E) \) be a bipartite graph. If \( |N(W')| \geq 3 |W'|/2 \) holds for every \( W' \subseteq W \), then there is a set \( P \) of vertex disjoint paths of \( H \) with end vertices belonging to \( B \) such that every vertex of \( W \) is covered by some path of \( P \).

Proof. We call vertices in \( W \) and in \( B \) white and black, respectively; and paths of \( H \) with black endpoints (also black singletons) will be called \( B \)-paths. Let \( \mathcal{F} \) be the set of all partitions of \( W \cup B \) into \( B \)-paths and edges. Observe that \( \mathcal{F} \neq \emptyset \), since by Hall’s theorem, \( H \) has a matching \( M \) which saturates all vertices of \( W \), and the non-saturated black vertices together with \( M \) form a partition belonging to \( \mathcal{F} \).

We shall show that \( \mathcal{F} \) has a partition containing only \( B \)-paths and no edges. Let \( F_0 \in \mathcal{F} \) be a partition with minimum number of edges. Denote by \( P_0 \) the set of all edges of \( F_0 \), set \( q = |P_0| \) and let \( \mathcal{F}_0 \subseteq \mathcal{F} \) be the set of all partitions containing \( P_0 \). Starting from \((P_0, \mathcal{F}_0)\) we recursively define a sequence \((P_1, \mathcal{F}_1), (P_2, \mathcal{F}_2), \ldots \) as follows.

For \( i \geq 1 \), a black vertex \( b \) not covered by paths (or edges) of \( P_{i-1} \) will be called \((i-1)\)-critical, if \( b \) is not the end vertex of any \( B \)-path in any partition of \( \mathcal{F}_{i-1} \). If there are no \((i-1)\)-critical vertices in \( B \), then \((P_{i-1}, \mathcal{F}_{i-1})\) is the last member of the sequence. Otherwise, let us call a \( B \)-path \( P \) of a fixed partition of \( \mathcal{F}_{i-1} \) \((i-1)\)-critical, whenever \( P \) has an \((i-1)\)-critical vertex.
We choose a partition $F_t \in \mathcal{F}_{t-1}$ with the following properties:

1. the number of the $(i - 1)$-critical paths of $F_t$ is maximal, moreover, if several partitions satisfy (1),
2. the total number of vertices covered by the $(i-1)$-critical paths of $F_t$ is minimal.

Let $P_t$ be the union of $P_{t-1}$ and the set of all $(i-1)$-critical paths of $F_t$, furthermore, let $\mathcal{F}_t$ be the set of all partitions of $\mathcal{F}_{t-1}$ containing $P_t$. Observe that $P_{t-1}$ is a proper subset of $P_t$, thus we obtain a finite sequence $(P_0, \mathcal{F}_0), \ldots, (P_k, \mathcal{F}_k)$. Note also that by definition, any partition of $\mathcal{F}_t$ satisfies properties (1) and (2), for every $1 \leq i \leq k$.

Let $W_k$ and $B_k$, respectively, denote the set of all white and black vertices of $H$ covered by the paths (and edges) of $P_t$.

**Claim 1.** $N(W_k) = B_k$.

**Proof.** Suppose on the contrary that for some $w \in W_k$ and $b \in B \setminus B_k$, $\{w, b\}$ is an edge of $H$. Since $(P_k, \mathcal{F}_k)$ was the last member of our sequence, $b$ is not $k$-critical. Hence, there exists a partition $F \in \mathcal{F}_k$ and a $B$-path $P \in F$ such that $b$ is an end vertex of $P$. Let $i$ be the least index $(0 \leq i \leq k)$ such that a path of $P_t$, say $Q$, contains $w$. By definition, $F \in \mathcal{F}_k \subseteq \mathcal{F}_t$, and since $P_t \subseteq P_k$, both $Q$ and $P$ belong to $F$.

If $i = 0$, then $Q$ is an edge of $F$. Using $\{w, b\}$, the concatenation of $P$ and $Q$ in $F$ would result a partition of $F$ with $q - 1$ edges, contradicting the choice of $F_0$. Hence $i > 0$, moreover, $Q \in P_t \setminus P_{t-1}$ is an $(i-1)$-critical path of $F$. Let $Q_1$ and $Q_2$ be the $B$-path components of $Q - w$. For $j = 1$ and $2$, denote by $Q_jwP$ the $B$-path obtained by concatenating $Q_j$ and $P$ at vertex $w$, see Fig. 3.

Define $F' = (F \setminus \{Q, P\}) \cup \{Q_1, Q_2wP\}$ and $F'' = (F \setminus \{Q, P\}) \cup \{Q_2, Q_1wP\}$. Obviously, $F, F', F'' \in \mathcal{F}_{t-1}$, moreover, $F$ satisfies properties (1) and (2). If both $Q_1$ and $Q_2$ are containing $(i-1)$-critical vertices of $Q$, then $F'$ has more $(i-1)$-critical paths than $F$, contradicting (1). Hence we may assume that one of $Q_1$ and $Q_2$, say $Q_1$, contains no $(i-1)$-critical vertices of $Q$. Then $Q_1wP$ is not $(i-1)$-critical, and $Q_2 \in F''$ covers fewer vertices than $Q \in F$, contradicting (2). This concludes the proof of the claim.

**Claim 2.** $|B_k| \leq 3|W_k|/2 - q/2$.

**Proof.** Observe first that every $B$-path of $P_k$ contains at least three black vertices. Indeed, let $Q$ be a $B$-path of $P_k$, and let $i$ be the least index $(0 \leq i \leq k)$ such that $Q \in P_i$. Then $i > 0$ and $Q$ has at least one $(i-1)$-critical vertex, say $b$, which is different from the endpoints of $Q$. Hence the proportion of black and white vertices covered by any $B$-path of $P_k$ is at most $\frac{2}{3}$. This observation yields $(|B_k| - q)/(|W_k| - q) \leq \frac{3}{2}$ which is equivalent with the inequality of the claim.

Using the condition of the covering lemma, and by Claim 1, we get $3|W_k|/2 \leq |N(W_k)| = |B_k|$. Thus, by Claim 2, $q = 0$ follows. Consequently, $P_0 = \emptyset$, hence the set $P$ of all non-trivial $B$-paths of $F_0$ is a covering for $W$. This completes the proof Theorem 3.3. □
4. Computing the toughness of bipartite and split graphs

We are going to study the algorithmic complexity of computing the toughness. Since we want to prove a hardness result, we use a formulation as a decision problem.

Definition 4.1. We define the following decision problems:

- $1$-TOUGH := \{G : G = (V,E); t(G) ≥ 1\}
- $t$-TOUGH := \{G : G = (V,E); t(G) ≥ t\} for every rational $t > 0$.

In [3] it was shown that the problem $t$-TOUGH is coNP-complete (under $\leq_{m}^{P}$ reductions) for any fixed rational $t > 0$.

Since the toughness of a bipartite graph is at most one it is a natural question to ask whether the computation of the toughness remains an intractable problem when restricted to bipartite graphs. Indeed, this question was raised by 'all' participants in the problem session of the Twente Workshop on Hamiltonian graph theory [9, p. 119].

Theorem 4.2. The problem 'Given a bipartite graph $G$, is $t(G) < 1$?' is NP-complete. Consequently, 1-TOUGH is coNP-complete on bipartite graphs (under $\leq_{m}^{P}$ reductions), i.e. deciding whether a bipartite graph has toughness exactly one is coNP-complete.
Proof. We show that 1-TOUGH on bipartite graphs is coNP-complete by reducing 1-TOUGH on general graphs to the problem. The \( \leq_p \) reduction used here is the classical Nash-Williams construction [27].

Let the graph \( G = (V, E) \) be an input to 1-TOUGH. Suppose \( V = \{v_1, v_2, \ldots, v_n\} \). We construct a bipartite graph \( B(G) = (W \cup X \cup Y \cup Z, \tilde{E}) \) as follows: We set \( W = \{w_1, w_2, \ldots, w_n\} \), \( X = \{x_1, x_2, \ldots, x_n\} \), \( Y = \{y_1, y_2, \ldots, y_n\} \) and \( Z = \{z_1, z_2, \ldots, z_n\} \). Furthermore,

\[
\tilde{E} = \{\{w_i, y_i\}, \{y_i, x_i\}, \{x_i, z_i\}, \{z_i, w_i\} : 1 \leq i \leq n\}
\]

\[
\cup \{\{w_i, z_j\}, \{w_j, z_i\} : \{v_i, v_j\} \in E\}.
\]

Clearly, \( B(G) \) can be constructed in polynomial time for given \( G \). It remains to show that \( t(G) < 1 \) iff \( t(B(G)) < 1 \).

Let \( S \) be a cutset of \( G \) such that \( \omega(G - S) > |S| \) holds. Let \( T := \{w_i, z_i : v_i \in S\} \). \( B(G) - T \) has components \( \{x_i, y_i\} \) for every \( v_i \in S \) and components \( \{w_i, x_i, y_i, z_i : v_i \in U\} \) for every component \( U \) of \( G - S \). Hence,

\[
\omega(B(G) - T) \geq |S| + \omega(G - S) > 2 \cdot |S| = |T|.
\]

Let \( T \) be a cutset of \( B(G) \) such that \( \omega(B(G) - T) > |T| \). Note that all vertices in \( X \cup Y \) have degree 2 in \( B(G) \) and exactly one neighbour in \( W \cup Z \). Suppose \( T \) had a non-empty intersection with \( X \cup Y \), then we replace it by \( T' = \{w_i : x_i \in T\} \cup \{z_i : y_i \in T\} \cup (T \setminus (X \cup Y)) \). This clearly implies \( \omega(B(G) - T') - |T'| \geq \omega(B(G) - T) - |T| \), thus \( \omega(B(G) - T') > |T'| \) holds. Hence, w.l.o.g. we may assume \( T \subseteq W \cup Z \).

Furthermore, we want that for every \( i \in \{1, 2, \ldots, n\} \), \( w_i \in T \) iff \( z_i \in T \) holds. Suppose this is not already true, say it is violated by some \( i \). W.l.o.g. let \( w_i \in T \) but \( z_i \notin T \). Hence, \( z_i \) belongs to the component of \( B(G) - T \) containing \( x_i \) and \( y_i \), say \( C \). If \( |C| > 3 \), then we add \( z_i \) to the cutset \( T \) and \( \omega(B(G) - T) > |T| \) still holds. Thus, for every \( i \in \{1, 2, \ldots, n\} \), with \( |\{w_i, z_i\} \cap T| = 1 \) we replace \( T \) by \( T \cup \{w_i, z_i\} \), if the component \( C \) containing \( \{w_i, z_i\} \setminus T \) has size larger than 3. This will give us a cutset \( T \) of \( B(G) \) with \( \omega(B(G) - T) > |T| \) such that \( B(G) - T \) has only the following types of components:

(a) \( \{x_i, y_i\} \), if \( \{w_i, z_i\} \subseteq T \),

(b) \( \{w_i, x_i, y_i\} \), if \( \{w_i, z_i\} \cap T = \{z_i\} \) and \( \{x_i, y_i, z_i\} \), if \( \{w_i, z_i\} \cap T = \{w_i\} \),

(c) components \( C \) with \( |C| \geq 4 \) fulfilling \( w_i \in C \) iff \( z_i \in C \) for every \( i \in \{1, 2, \ldots, n\} \).

Suppose we had a cutset \( T \) such that \( B(G) - T \) had only components of type (a) and (c). Then we would be ready for concluding the proof by choosing \( S \).

Suppose this would not be the case. Then we choose among all cutsets \( T \) with \( \omega(B(G) - T) > |T| \) having only components of type (a), (b) and (c) one of minimum cardinality. We are going to show that such a cutset \( T \) cannot have a component of type (b) in \( B(G) - T \).

Suppose, we had a component of type (b), say w.l.o.g. \( \{z_i, x_i, y_i\} \). Then, every vertex \( w_j \) adjacent to \( z_i \) belongs to \( T \). Let \( Q \) be the set \( \{w_i\} \cup \{w_j : \{w_j, z_i\} \in \tilde{E}\} \). Now we add vertices of \( W \) to \( Q \) in the following way: While there is a \( w_j \in Q \) such that
$z_j \notin T$ then every neighbour of $z_j$ in $W$ belongs to $T$ and we add all of them to $Q$. This procedure will finally create a set $Q \subseteq T \cap W$ such that for every $w_j \in Q$ either $z_j \in T$ or all neighbours of $z_j$ in $W$ belong to $Q$. Hence, $B(G) - T$ contains exactly $|Q|$ components of type (a) and (b) on \{w_j, x_j, y_j, z_j : w_j \in Q\}. Let $R$ be the set of all vertices $w_j \in Q$ for which $z_j$ does not belong to $T$. $R = W$ is impossible, since this would imply $T \supseteq W$ and $\omega(B(G) - T) = |T|$. Thus, $T' := T - R$ is a new cutset of $B(G)$ and $\omega(B(G) - T) - \omega(B(G) - T') \leq |R| - 1$, thus $T' = T - R$ fulfills $\omega(B(G) - T') > |T'|$. Furthermore, $|T'| < |T|$ and $B(G) - T'$ has only components of type (a)–(c). This contradicts the choice of $T$.

Consequently, $B(G)$ has always a cutset $T$ such that $B(G) - T$ has only components of type (a) and (c). Now we choose $S := \{v_i : w_i \in T$ and $z_i \in T\}$. By the construction of $T$ there is no component of type (b) in $B(G) - T$. Hence, $B(G) - T$ has $|S| = |T|/2$ components of type (a) and $\omega(G - S)$ components of type (c). Hence,

$$\omega(G - S) = \omega(B(G) - T) - |S| > |T| - |S| = |T|/2 = |S|.$$ 

Thus, $t(G) < 1$ follows which concludes the proof of the theorem. $\square$

On the other hand, there is a polynomial time algorithm for deciding whether the toughness of a given split graph is less than one.

**Theorem 4.3.** There is a $O(\sqrt{n}m)$ algorithm deciding whether a given split graph has toughness less than one.

**Proof.** Let $G = (C, I, E)$ be a split graph. W.l.o.g. $|C| \geq |I|$. Then $G$ is 1-tough iff $|N(I')| \geq |I'| + 1$ holds for every non-empty proper subset $I'$ of $I$.

We construct a bipartite graph $B(G) = (C, I, E)$ from $G = (C, I, E)$ by first deleting all edges inside the clique $C$ and then adding $|C| - |I|$ dummy vertices to $I$ such that each of them is adjacent exactly to all vertices of $C$. Hence, $|C| = \hat{|I|}$. Furthermore, $t(G) \geq 1$ iff $|N(I')| \geq |I'| + 1$ holds for every non-empty proper subset $I'$ of $\hat{I}$.

Consequently, $G$ is 1-tough iff $B(G)$ is an elementary graph, i.e. the subgraph of $B(G)$ induced by all edges belonging to a perfect matching is connected (cf. [24, pp. 122–123]). By [24, Exercise 4.1.5.], a bipartite graph $B(G) = (C, I, E)$ is elementary iff $B(G)$ has a perfect matching $M$ and the digraph $B(G)$ arising from $B(G)$ by orienting all edges from $C$ to $I$ and contracting all edges of $M$ is strongly connected.

Computing a perfect matching of $B(G)$ can be done in time $O(\sqrt{|V(B(G))||E(B(G))|})$ [19]. Furthermore, there is a linear time algorithm deciding whether a given digraph is strongly connected [28].

Analysing the growth of size and order from the split graph $G$ to the bipartite graph $B(G)$ and finally to the digraph $B(G)$ shows that the described algorithm has indeed running time $O(\sqrt{nm})$, whereby the most time consuming part is the subroutine computing a maximum matching of $B(G)$. $\square$
Hence, both non-2-factorable graphs as well as graphs with toughness less than one cannot be the instances responsible for the NP-completeness of HAMILTONIAN CIRCUIT on split graphs.

The complexity of $t$-TOUGH for rationals $t \neq 1$ on split graphs is still an open problem.

References