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Derived invariance of Hochschild–Mitchell (co)homology and one-point extensions ☆

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Abstract

In this article we prove derived invariance of Hochschild–Mitchell homology and cohomology and we extend to k-linear categories a result by Barot and Lenzing concerning derived equivalences and one-point extensions. We also prove the existence of a long exact sequence à la Happel and we give a generalization of this result which provides an alternative approach.

Keywords: Hochschild–Mitchell cohomology; k-Category; One-point extension; Derived equivalence

1. Introduction

It is known that linear categories over a field k are a generalization of finite-dimensional k-algebras: given a finite-dimensional unitary k-algebra A and a complete system $E = \{e_1, \dots, e_n\}$ of orthogonal idempotents of A, the category $\mathcal{C}_A$ with objects indexed by E and morphisms given by $\mathrm{Hom}_{\mathcal{C}}(e_i, e_j) = e_j A e_i$ may be associated to A. Different complete sets of orthogonal idempotents of A give different categories, but all of them are Morita equivalent. Conversely given a k-linear category with a finite set of objects $\mathcal{C}_0 = \{x_1, \dots, x_n\}$,

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$a(\mathcal{C}) = \bigoplus_{i,j=1}^{n} \operatorname{Hom}_{\mathcal{C}}(x_i, x_j)$ is a k-algebra with unit $\sum_{i=1}^{n} \mathrm{id}_{x_i}$. The categorical point of view gives in our opinion a very clear insight.

In this article we study one-point extensions of linear k-categories, obtaining two main results. The first one concerns derived invariance of Hochschild–Mitchell cohomology, and the second one is the existence of a cohomological long exact sequence relating the cohomology of the category itself and the cohomology of its one-point extension.

More precisely, let A be a finite-dimensional k-algebra and M a right A-module. It has been proved by Barot and Lenzing [2] that if A is derived equivalent to another finite-dimensional k-algebra B and the equivalence maps M into a right B-module N, then the one-point extensions $A[M]$ and $B[N]$ are such that there exists a triangulated equivalence $\Phi : D^b(\mathrm{Mod}_{A[M]}) \to D^b(\mathrm{Mod}_{B[N]})$ and Φ restricts to a triangulated equivalence $\phi : D^b(\mathrm{Mod}_A) \to D^b(\mathrm{Mod}_B)$. The motivation of this article was to prove that this result holds for small k-linear categories instead of finite-dimensional k-algebras. This is achieved in Theorem 3.5.

In the way to prove this theorem, we give in Theorem 2.7 an alternative description of Morita equivalences between k-linear categories (cf. [6]) and a description of derived equivalences in this context (Theorem 2.14). As a consequence we prove in Theorem 2.15 that Hochschild–Mitchell homology and cohomology are derived invariant.

We also prove that the Hochschild–Mitchell cohomology of a one-point extension is related to the Hochschild–Mitchell cohomology of the category by a long exact sequence *à la Happel* [9]. Actually, we prove this fact in two different ways. Firstly, we provide a direct proof and secondly we reobtain the result as an example of a much more general situation (cf. Theorem 4.4). The analogue for finite-dimensional algebras is proved in [3] and [13]. This sequence has also been obtained in [7] by means of a different method, which makes use of the structural properties of the morphisms involved, and in [8] for a more general situation. Our proof is related to [3], but it is in fact simpler, even for the case of algebras.

2. Morita theory

In the first part of this section we shall give a description of equivalences between the module categories of two k-linear categories $\mathcal{C}$ and $\mathcal{D}$ that will lead to a characterization of Morita equivalences which is in fact very close to the algebraic case.

We begin by recalling the definition of a module over a linear category $\mathcal{C}$. For further references, see [4,5,14].

In this section k will be a commutative ring with unit. When we consider Hochschild–Mitchell (co)homology, we require the small category $\mathcal{C}$ to be k-projective, i.e. ${}_y\mathcal{C}_x$ is a k-projective module for all $x, y \in \mathcal{C}_0$.

Let $\mathcal{C}$ be a small category. It is a *k-linear category* if the set of morphisms between two arbitrary objects of $\mathcal{C}$ is a k-module and composition of morphisms is k-bilinear. From now on, $\mathcal{C}$ will be a k-linear category with set of objects $\mathcal{C}_0$ and given objects x, y we shall denote ${}_y\mathcal{C}_x$ the k-vector space of morphisms from x to y in $\mathcal{C}$. Given x, y, z in $\mathcal{C}_0$, the composition is a k-linear map

$$\circ_{z,y,x} : {}_z\mathcal{C}_y \otimes {}_y\mathcal{C}_x \to {}_z\mathcal{C}_x.$$

We shall denote ${}_zf_y {\cdot} {}_yg_x$, ${}_zf_y \cdot {}_yg_x$, ${}_zf_y \circ {}_yg_x$, or fg if subscripts are clear, the image of ${}_zf_y \otimes {}_yg_x$ under this map.

The simplest example of k-linear category is to look at a k-algebra A as a category with only one object and the set of morphisms equal to A.

Definition 2.1. Given two k-linear categories $\mathcal{C}$ and $\mathcal{D}$ *the (external) tensor product category*, which we denote $\mathcal{C} \boxtimes_k \mathcal{D}$, is the category with set of objects $\mathcal{C}_0 \times \mathcal{D}_0$ and given $c, c' \in \mathcal{C}_0$ and $d, d' \in \mathcal{D}_0$

$$_{(c',d)}(\mathcal{C} \boxtimes_k \mathcal{D})_{(c,d)} = {}_c\mathcal{C}_c \otimes {}_{d'}\mathcal{D}_d.$$

The functor $\mathcal{C} \boxtimes_k -$ is the left adjoint functor to $\mathrm{Func}(\mathcal{C}, -)$ (see [14, Section 2, p. 13]). We will omit the subindex k in the external tensor product. We will call the category $\mathcal{C} \boxtimes \mathcal{C}^{op}$ the *enveloping category of* $\mathcal{C}$ and denote it $\mathcal{C}^e$.

Definition 2.2. A *left $\mathcal{C}$-module M* is a covariant k-linear functor from the category $\mathcal{C}$ to the category of k-modules. Equivalently, a left $\mathcal{C}$-module M is a collection of k-modules $\{_xM\}_{x \in \mathcal{C}_0}$ provided with a left action

$$_y\mathcal{C}_x \otimes {}_xM \to {}_yM,$$

where the image of $_yf_x \otimes {}_xm$ is denoted by $_yf_x . {}_xm$ or fm, satisfying the usual axioms

$$_zf_y.({}_yg_x . {}_xm) = ({}_zf_y . {}_yg_x).{}_xm,$$
$$_x1_x . {}_xm = {}_xm.$$

Right $\mathcal{C}$-modules are defined in an analogous way. Also, a *$\mathcal{C}$-bimodule* is just a $\mathcal{C}^e$-module.

We shall denote $_{\mathcal{C}}\mathrm{Mod}$ and $\mathrm{Mod}_{\mathcal{C}}$ the categories of left $\mathcal{C}$-modules and right $\mathcal{C}$-modules, respectively.

The obvious example of $\mathcal{C}$-bimodule is given by the category itself, i.e. $_y\mathcal{C}_x$ for every $x, y \in \mathcal{C}_0$. We will denote this bimodule by $\mathcal{C}$.

In a similar way as for algebras, it is possible to define a tensor product between modules (cf. [14]):

Definition 2.3. Let M be a left $\mathcal{C}$-module and let N be a right $\mathcal{C}$-module. The *tensor product over $\mathcal{C}$ between M and N*, $M \otimes_{\mathcal{C}} N$, is defined as the k-module given by

$$M \otimes_{\mathcal{C}} N = \left(\bigoplus_{x \in \mathcal{C}_0} M_x \otimes {}_xN \right) \Big/ \big\langle \{ m.f \otimes n - m \otimes f.n\colon\ m \in M_x,\ n \in {}_yN,\ f \in {}_x\mathcal{C}_y \} \big\rangle.$$

If M and N are $\mathcal{C}$-bimodules, it is also possible to define the *$\mathcal{C}$-bimodule tensor product over $\mathcal{C}$*:

$$_y(M \otimes_{\mathcal{C}} N)_x = \left(\bigoplus_{z \in \mathcal{C}_0} {}_yM_z \otimes_k {}_zN_x \right) \Big/ \big\langle \{ m.f \otimes n - m \otimes f.n \} \big\rangle,$$

where $m \in {}_yM_{y'}$, $n \in {}_{x'}N_x$, $f \in {}_{y'}\mathcal{C}_{x'}$.

Next we recall the definition of Hochschild–Mitchell homology and cohomology. Standard (co)homological methods are available in ${}_{\mathcal{C}}\mathrm{Mod}$ (cf. [10]).

Definition 2.4. Let $(x_{n+1}, \dots, x_1)$ be an $(n+1)$-sequence of objects of $\mathcal{C}$. The *k-nerve associated to the $(n+1)$-sequence* is the k-module

$$ {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}. $$

The *k-nerve of $\mathcal{C}$ in degree n* ($n \in \mathbb{N}_0$) is

$$ \bar{N}_n = \bigoplus_{(n+1)\text{-}tuples} {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}. $$

There is a $\mathcal{C}^e$-bimodule associated to $\bar{N}_n$ defined by

$$ {}_y(N_n)_x = \bigoplus_{(n+1)\text{-}tuples} {}_y\mathcal{C}_{x_{n+1}} \otimes {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1} \otimes {}_{x_1}\mathcal{C}_x. $$

Then the associated *Hochschild–Mitchell complex* is

$$ \cdots \xrightarrow{d_{n+1}} N_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} N_1 \xrightarrow{d_1} N_0 \xrightarrow{d_0} \mathcal{C} \to 0, $$

where d_n is given by the usual formula, i.e.

$$ d_n(f_0 \otimes \cdots \otimes f_{n+1}) = \sum_{k=0}^{n} (-1)^k f_0 \otimes \cdots \otimes f_k.f_{k+1} \otimes \cdots \otimes f_{n+1}. $$

This complex is a projective resolution of the $\mathcal{C}$-bimodule $\mathcal{C}$. The proof that it is a resolution is similar to the standard proof for algebras.

Definition 2.5. Given a $\mathcal{C}$-bimodule M the *Hochschild–Mitchell cohomology of $\mathcal{C}$ with coefficients in M* is the cohomology of the following cochain complex

$$ 0 \to \prod_{x \in \mathcal{C}_0} {}_xM_x \xrightarrow{d^0} \mathrm{Hom}(N_1, M) \xrightarrow{d^1} \cdots \xrightarrow{d^{n-1}} \mathrm{Hom}(N_n, M) \xrightarrow{d^n} \cdots, $$

where d is given by the usual formula, and

$$ \begin{aligned} C^n(\mathcal{C}, M) &= \mathrm{Hom}(N_n, M) = \mathrm{Nat}(N_n, M) \\ &= \prod_{(n+1)\text{-}tuples} \mathrm{Hom}_k({}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}, {}_{x_{n+1}}M_{x_1}). \end{aligned} $$

We denote it $H^\bullet(\mathcal{C}, M)$ or just $HH^\bullet(\mathcal{C})$ when $M = \mathcal{C}$.

Analogously the *Hochschild–Mitchell homology of $\mathcal{C}$ with coefficients in M* is the homology of the chain complex

$$\cdots \xrightarrow{d_{n+1}} M \otimes N_n \xrightarrow{d_n} \cdots \xrightarrow{d_2} M \otimes N_1 \xrightarrow{d_1} \bigoplus_{x \in \mathcal{C}_0} {}_xM_x \to 0,$$

where d is given by the usual formula and

$$C_n(\mathcal{C}, M) = M \otimes_{\mathcal{C}^e} N_n = \bigoplus_{(n+1)\text{-}tuples} {}_{x_1}M_{x_{n+1}} \otimes {}_{x_{n+1}}\mathcal{C}_{x_n} \otimes \cdots \otimes {}_{x_2}\mathcal{C}_{x_1}.$$

We denote it $H_\bullet(\mathcal{C}, M)$ or just $HH_\bullet(\mathcal{C})$ when $M = \mathcal{C}$.

The following is a generalization of Watt's Theorem for modules over k-algebras:

Theorem 2.6. *Let $\mathcal{C}$ and $\mathcal{D}$ be k-linear categories and let $F : {}_{\mathcal{C}}\mathrm{Mod} \to {}_{\mathcal{D}}\mathrm{Mod}$ be a functor. The following statements are equivalent*

(a) *F preserves arbitrary direct sums and is right exact.*
(b) *There exists a $\mathcal{D}$–$\mathcal{C}$-bimodule T such that $F(-) = T \otimes_{\mathcal{C}} (-)$.*
(c) *F has a right adjoint.*

Proof. We trivially have that (b) implies (c), and (c) implies (a). Let us prove that (a) implies (b). For each $x \in \mathcal{C}_0$ define the left $\mathcal{D}$-module

$${}_{-}T_x = F({}_{-}\mathcal{C}_x).$$

The collection $\{{}_yT_x\}_{y \in \mathcal{D}_0, x \in \mathcal{C}_0}$ is a $\mathcal{D}$–$\mathcal{C}$-bimodule as we shall now prove: it is trivially a left $\mathcal{D}$-module by definition. Given ${}_xf_{x'} \in {}_x\mathcal{C}_{x'}$, it induces a morphism of left $\mathcal{C}$-modules

$$\cdot_x f_{x'} : {}_{-}\mathcal{C}_x \to {}_{-}\mathcal{C}_{x'},$$

given by right multiplication by ${}_xf_{x'}$, so we get a morphism of left $\mathcal{D}$-modules

$$F(\cdot_x f_{x'}) : F({}_{-}\mathcal{C}_x) \to F({}_{-}\mathcal{C}_{x'}).$$

This natural transformation gives the structure of right $\mathcal{C}$-module.

Moreover, both actions are compatible since the map $F(\cdot_x f_{x'})$ is a morphism of left $\mathcal{D}$-modules.

We have that

$${}_yT \otimes_{\mathcal{C}} \mathcal{C}_x = \left(\bigoplus_{z \in \mathcal{C}_0} {}_yT_z \otimes {}_z\mathcal{C}_x\right) \Big/ \left\langle \{t_{zz}f_{z'} \otimes {}_{z'}g - t_z \otimes {}_zf_{z'z'}g\} \right\rangle \simeq {}_yT_x,$$

using the k-linear isomorphism $\overline{t \otimes f} \mapsto tf$ (with inverse $t \mapsto \overline{t \otimes 1}$). This gives naturally a left $\mathcal{D}$-module isomorphism

$${}_{-}T \otimes_{\mathcal{C}} \mathcal{C}_x = {}_{-}T_x.$$

From now on, given $M \in {}_{\mathcal{C}}\mathrm{Mod}$, we are going to write $F(M)$ instead of $F({}_{-}M)$. We shall now prove that $F(-) = T \otimes_{\mathcal{C}} (-)$. Since F and $\otimes_{\mathcal{C}}$ commute with direct sums, there are isomorphisms of $\mathcal{D}$-modules

$$F\biggl(\bigoplus_{i\in I} \mathcal{C}_{x_i}\biggr) = \bigoplus_{i\in I} F(\mathcal{C}_{x_i}) = \bigoplus_{i\in I} T_{x_i} = \bigoplus_{i\in I} T \otimes_{\mathcal{C}} \mathcal{C}_{x_i} = T \otimes_{\mathcal{C}} \biggl(\bigoplus_{i\in I} \mathcal{C}_{x_i}\biggr).$$

Given any left $\mathcal{C}$-module M there is an exact sequence

$$\bigoplus_{j\in J} \mathcal{C}_{y_j} \to \bigoplus_{i\in I} \mathcal{C}_{x_i} \to M \to 0,$$

hence, by right exactness, we get that

$$F\biggl(\bigoplus_{j\in J} \mathcal{C}_{y_j}\biggr) \to F\biggl(\bigoplus_{i\in I} \mathcal{C}_{x_i}\biggr) \to F(M) \to 0$$

is exact. Taking into account the previous isomorphism, the following diagram has exact rows and commuting squares

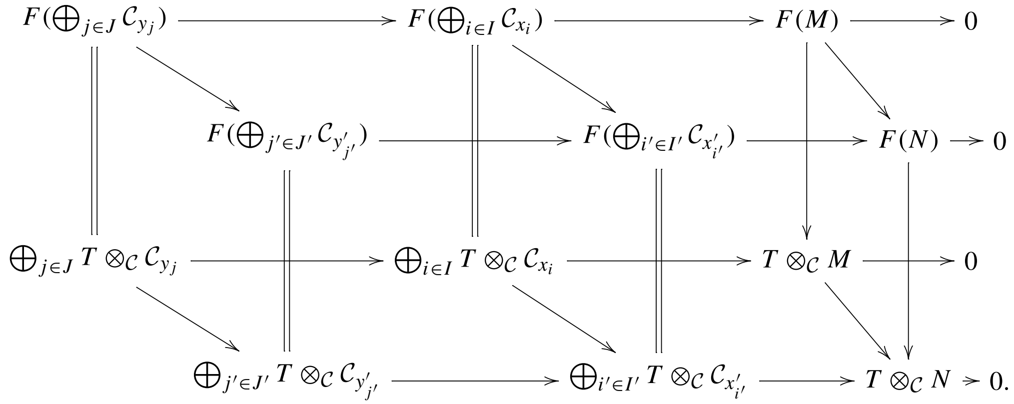
$$\begin{array}{ccccccc}
F(\bigoplus_{j\in J} \mathcal{C}_{y_j}) & \longrightarrow & F(\bigoplus_{i\in I} \mathcal{C}_{x_i}) & \longrightarrow & F(M) & \longrightarrow & 0 \\
\| & & \| & & & & \\
\bigoplus_{j\in J} T \otimes_{\mathcal{C}} \mathcal{C}_{y_j} & \longrightarrow & \bigoplus_{i\in I} T \otimes_{\mathcal{C}} \mathcal{C}_{x_i} & \longrightarrow & T \otimes_{\mathcal{C}} M & \longrightarrow & 0.
\end{array}$$

By diagrammatic considerations we get a map $F(M) \to T \otimes_{\mathcal{C}} M$ making the whole diagram commutative. Then the Five lemma assures that this map is an isomorphism.

The naturality of the map is also clear: if $f : M \to N$ is a $\mathcal{C}$-module morphism, there is a commutative diagram

$$\begin{array}{ccccccc}
\bigoplus_{j\in J} \mathcal{C}_{y_j} & \longrightarrow & \bigoplus_{i\in I} \mathcal{C}_{x_i} & \longrightarrow & M & \longrightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle f} & & \\
\bigoplus_{j'\in J'} \mathcal{C}_{y'_{j'}} & \longrightarrow & \bigoplus_{i'\in I'} \mathcal{C}_{x'_{i'}} & \longrightarrow & N & \longrightarrow & 0
\end{array}$$

and hence



Since the two left vertical faces (normal to the page) commute (by definition of T), we obtain that the right vertical face also commutes, and this fact proves the naturality. □

As an application of the characterization of such functors we obtain a description of Morita equivalences of k-linear categories. We also give an example relating this description to the one given in [6].

Theorem 2.7. *Let $\mathcal{C}$ and $\mathcal{D}$ be two k-linear categories. They are (left) Morita equivalent if and only if there are a $\mathcal{C}$–$\mathcal{D}$-bimodule P and a $\mathcal{D}$–$\mathcal{C}$-bimodule Q such that $P\otimes_{\mathcal{D}} Q\simeq\mathcal{C}$ and $Q\otimes_{\mathcal{C}} P\simeq\mathcal{D}$ as bimodules. Furthermore, these bimodules satisfy that $\{P_y\}_{y\in\mathcal{D}_0}$ and $\{Q_x\}_{x\in\mathcal{C}_0}$ are sets of projective and finitely generated generators of ${}_{\mathcal{C}}\mathrm{Mod}$ and ${}_{\mathcal{D}}\mathrm{Mod}$ respectively.*

Proof. Given two bimodules P and Q, we define the functors

$$F : {}_{\mathcal{C}}\mathrm{Mod}\to{}_{\mathcal{D}}\mathrm{Mod},$$
$$F(-) = Q\otimes_{\mathcal{C}}(-),$$

and

$$G : {}_{\mathcal{D}}\mathrm{Mod}\to{}_{\mathcal{C}}\mathrm{Mod},$$
$$G(-) = P\otimes_{\mathcal{D}}(-).$$

Since $P\otimes_{\mathcal{D}} Q$ and $Q\otimes_{\mathcal{C}} P$ are isomorphic as bimodules to $\mathcal{C}$ and $\mathcal{D}$ respectively, then $F\circ G\simeq\mathrm{id}_{\mathcal{D}}$ and $G\circ F\simeq\mathrm{id}_{\mathcal{C}}$.

Conversely, let $F : {}_{\mathcal{C}}\mathrm{Mod}\to{}_{\mathcal{D}}\mathrm{Mod}$ be a functor giving the equivalence with quasi-inverse functor G. Since an equivalence preserves direct sums and is exact, Theorem 2.6 guarantees the existence of a $\mathcal{C}$–$\mathcal{D}$-bimodule P and a $\mathcal{D}$–$\mathcal{C}$-bimodule Q satisfying

$$F(-) = Q\otimes_{\mathcal{C}}(-),$$
$$G(-) = P\otimes_{\mathcal{D}}(-).$$

The isomorphism $F \circ G \simeq \mathrm{id}_{\mathcal{D}}$ implies that $\mathcal{D} \simeq F \circ G(\mathcal{D}) = Q \otimes_{\mathcal{C}} (P \otimes_{\mathcal{D}} \mathcal{D}) \simeq Q \otimes_{\mathcal{C}} P$. The other isomorphism is analogous.

Since, given $x \in \mathcal{C}_0$, Q_x is isomorphic to $F(\mathcal{C}_x)$, each Q_x is finitely generated and projective, and the same applies to P_y ($y \in \mathcal{D}_0$). Also, taking into account that $\{\mathcal{C}_x\}_{x \in \mathcal{C}_0}$ is a set of generators of ${}_{\mathcal{C}}\mathrm{Mod}$ and F is an equivalence, we get that $\{Q_x\}_{x \in \mathcal{C}_0} = \{F(\mathcal{C}_x)\}_{x \in \mathcal{C}_0}$ is a set of generators of ${}_{\mathcal{D}}\mathrm{Mod}$. The same arguments apply to $\{P_y\}_{y \in \mathcal{C}_0}$. □

Remark 2.8. We infer from the theorem above that if $\mathcal{C}$ and $\mathcal{D}$ are left Morita equivalent, then they are right Morita equivalent. This is done just by taking the functors

$$\begin{aligned} F : \mathrm{Mod}_{\mathcal{C}} &\to \mathrm{Mod}_{\mathcal{D}}, \\ F(-) &= (-) \otimes_{\mathcal{C}} P, \end{aligned}$$

and

$$\begin{aligned} G : \mathrm{Mod}_{\mathcal{D}} &\to \mathrm{Mod}_{\mathcal{C}}, \\ G(-) &= (-) \otimes_{\mathcal{D}} Q. \end{aligned}$$

The following results will complete the description.

Proposition 2.9. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be k-linear categories and ${}_{\mathcal{D}}P_{\mathcal{E}}$, ${}_{\mathcal{D}}M_{\mathcal{C}}$, ${}_{\mathcal{C}}N_{\mathcal{F}}$ be a set of bimodules. Then the following is a natural morphism of $\mathcal{E}$–$\mathcal{F}$-bimodules*

$$\eta : \mathrm{Hom}_{\mathcal{D}}(P, M) \otimes_{\mathcal{C}} N \to \mathrm{Hom}_{\mathcal{D}}(P, M \otimes_{\mathcal{C}} N),$$

defined by

$$\eta(t \otimes n)(p) = t(p) \otimes n.$$

Furthermore, if P_x is finitely generated and projective as left $\mathcal{D}$-module for each $x \in \mathcal{E}_0$, then η is an isomorphism.

Proof. The morphism is clearly well-defined and natural. To prove the second statement, let us first suppose that $\mathcal{E} = \mathcal{D}$ and $P = {}_{-}\mathcal{D}_{-}$. Since for each $x \in \mathcal{D}_0$ we have an isomorphism of right $\mathcal{D}$-modules

$$\mu_M : \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_x, M) \xrightarrow{\simeq} {}_xM$$

defined via the Yoneda isomorphism

$$\mu_M(t) = {}_xt_x(\mathrm{id}_x),$$

we get

$$\mu_M \otimes \mathrm{id} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_x, M) \otimes_{\mathcal{C}} N \xrightarrow{\simeq} {}_xM \otimes_{\mathcal{C}} N,$$

and

$$\mu_{M \otimes_{\mathcal{C}} N} : \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}_x, M \otimes_{\mathcal{C}} N) \xrightarrow{\simeq} {}_x M \otimes_{\mathcal{C}} N.$$

We see that $\eta = \mu^{-1}_{M \otimes_{\mathcal{C}} N} \circ (\mu_M \otimes \mathrm{id})$.

Now, if P_x is finitely generated and projective, there exists P' such that

$$P' \oplus P_x = \bigoplus_{i=1}^{n} \mathcal{C}_{x_i}.$$

Using Lemma (20.9) from [1], we are able to prove that η is an isomorphism. □

The proof of the following proposition is analogous:

Proposition 2.10. *Let $\mathcal{C}$, $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$ be k-linear categories and ${}_{\mathcal{E}}P_{\mathcal{D}}$, ${}_{\mathcal{C}}M_{\mathcal{D}}$, ${}_{\mathcal{C}}N_{\mathcal{F}}$ be a set of bimodules. Then the following is a natural morphism of $\mathcal{E}$–$\mathcal{F}$-bimodules*

$$\nu : P \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{C}}(M, N) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}\big(\mathrm{Hom}_{\mathcal{D}}(P, M), N\big),$$
$$\nu(p \otimes t)(u) = t\big(u(p)\big).$$

Furthermore, if ${}_xP$ is finitely generated and projective as right $\mathcal{D}$-module for each $x \in \mathcal{E}_0$, then ν is an isomorphism.

From the previous propositions we obtain:

Corollary 2.11. *Two k-linear categories $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent if and only if there exists a $\mathcal{C}$–$\mathcal{D}$-bimodule P such that $\{P_y\}_{y \in \mathcal{D}_0}$ is a set of finitely generated projective generators of ${}_{\mathcal{C}}\mathrm{Mod}$, $\{{}_xP\}_{x \in \mathcal{C}_0}$ is a set of finitely generated projective generators of $\mathrm{Mod}_{\mathcal{D}}$ and $\mathrm{Hom}_{\mathcal{C}}(P, P) = \mathcal{D}$ (as $\mathcal{D}$-bimodules).*

Proof. If $\mathcal{C}$ and $\mathcal{D}$ are Morita equivalent then we use the bimodule P defined in Theorem 2.7 which satisfies all the conditions except perhaps that $\mathrm{Hom}_{\mathcal{C}}(P, P) = \mathcal{D}$. But $Q \otimes_{\mathcal{C}} - = \mathrm{Hom}_{\mathcal{C}}(P, -)$, so we get $\mathcal{D} = Q \otimes_{\mathcal{C}} P = \mathrm{Hom}_{\mathcal{C}}(P, P)$.

Conversely, suppose that there exists a $\mathcal{C}$–$\mathcal{D}$-bimodule P such that $\{P_y\}_{y \in \mathcal{D}_0}$ is a set of finitely generated projective generators of ${}_{\mathcal{D}}\mathrm{Mod}$ and $\mathrm{Hom}_{\mathcal{C}}(P, P) = \mathcal{D}$. Then we set

$$F : {}_{\mathcal{C}}\mathrm{Mod} \to {}_{\mathcal{D}}\mathrm{Mod},$$
$$F(-) = \mathrm{Hom}_{\mathcal{C}}(P, -),$$

and

$$G : {}_{\mathcal{D}}\mathrm{Mod} \to {}_{\mathcal{C}}\mathrm{Mod},$$
$$G(-) = P \otimes_{\mathcal{D}} (-).$$

From the previous proposition we have that, for any left $\mathcal{D}$-module M,

$$\mathrm{Hom}_{\mathcal{C}}(P, P \otimes_{\mathcal{D}} M) \simeq \mathrm{Hom}_{\mathcal{C}}(P, P) \otimes_{\mathcal{D}} M \simeq \mathcal{D} \otimes_{\mathcal{D}} M \simeq M,$$

and also, for any left $\mathcal{C}$-module N,

$$P \otimes_{\mathcal{D}} \mathrm{Hom}_{\mathcal{C}}(P, N) \simeq \mathrm{Hom}_{\mathcal{D}}\big(\mathrm{Hom}_{\mathcal{C}}(P, P), N\big) \simeq \mathrm{Hom}_{\mathcal{D}}(\mathcal{D}, N) \simeq N,$$

where all isomorphisms are natural. Hence F and G are quasi-inverse functors, giving the Morita equivalence. □

Example 2.12. Suppose that E is a partition of the set of objects $\mathcal{C}_0$ of a k-linear category $\mathcal{C}$, given by $\mathcal{C}_0 = \bigsqcup_{e \in E} E_e$, with $\#(E_e) < \aleph_0$, $\forall e \in E$. It is proved in [6] that $\mathcal{C}$ is Morita equivalent to the contracted category along the partition E, $\mathcal{C}/E$. In fact the functors giving the equivalence are the following

$$\begin{aligned} F : {}_{\mathcal{C}}\mathrm{Mod} &\to {}_{\mathcal{C}/E}\mathrm{Mod}, \\ {}_eF(M) &= \bigoplus_{x \in E_e} {}_xM, \end{aligned}$$

and

$$\begin{aligned} G : {}_{\mathcal{C}/E}\mathrm{Mod} &\to {}_{\mathcal{C}}\mathrm{Mod}, \\ {}_xG(N) &= f_x.({}_eN), \end{aligned}$$

where e is the unique element of E such that $x \in E_e$ and f_x is the idempotent $|E| \times |E|$-matrix.

The bimodules giving the equivalence are:

$$\begin{aligned} {}_e({}_{\mathcal{C}/E}P_{\mathcal{C}})_x &= \bigoplus_{y \in E_e} {}_y\mathcal{C}_x, \\ {}_x({}_{\mathcal{C}}Q_{\mathcal{C}/E})_e &= \bigoplus_{y \in E_e} {}_x\mathcal{C}_y. \end{aligned}$$

It is easy to check that $P \otimes_{\mathcal{C}} Q \simeq \mathcal{C}/E$, $Q \otimes_{\mathcal{C}/E} P \simeq \mathcal{C}$ as bimodules and also $F(-) = P \otimes_{\mathcal{C}} (-)$ and $G(-) = Q \otimes_{\mathcal{C}/E} (-)$.

From now on we shall consider the derived category of ${}_{\mathcal{C}}\mathrm{Mod}$. This is a special case of the theory developed by Keller for DG categories. We will recall some definitions, but we refer the reader to [11] for further references.

As usual, we consider the category of $\mathcal{C}$-modules embedded into the category of cochains of complexes of $\mathcal{C}$-modules, denoted by $\mathrm{Ch}({}_{\mathcal{C}}\mathrm{Mod})$ or $\mathrm{Ch}(\mathcal{C})$, and we denote the shift of a complex $M^\bullet$ by $M^\bullet[1]$ or $SM^\bullet$, the homotopy category by $\mathcal{H}({}_{\mathcal{C}}\mathrm{Mod})$ or just by $\mathcal{H}(\mathcal{C})$, and the derived category by $D({}_{\mathcal{C}}\mathrm{Mod})$ or $D(\mathcal{C})$.

We say that a complex of $\mathcal{C}$-modules M is *relatively projective* if it is a direct summand of a direct sum of complexes of the form $\mathcal{C}_x[n]$, for $n \in \mathbb{Z}$, $x \in \mathcal{C}_0$. We also recall that a complex of $\mathcal{C}$-modules M is *homotopically projective* if it is homotopically equivalent to a complex P provided with an increasing filtration (indexed by $\mathbb{N}_0$)

$$P_{-1} = 0 \subset P_0 \subset \cdots \subset P_n \subset \cdots \subset P,$$

satisfying the following properties:

(1) $P = \bigcup_{n \in \mathbb{N}_0} P_n$.
(2) The inclusion $P_n \subset P_{n+1}$ ($n \in \mathbb{N}_0$) splits in the category of graded modules over $\mathcal{C}$.
(3) The quotient P_n / P_{n-1} ($n \in \mathbb{N}_0$) is isomorphic in $\mathrm{Ch}(\mathcal{C})$ to a relatively projective module.

As it is proved in [11], the following is a split exact sequence in the category of graded modules over $\mathcal{C}$

$$\bigoplus_{n \in \mathbb{N}_0} P_n \to \bigoplus_{n \in \mathbb{N}_0} P_n \to P, \tag{2.1}$$

and this split exact sequence gives a triangle in $\mathcal{H}(\mathcal{C})$.

We denote $\mathcal{H}_p(\mathcal{C})$ the full triangulated subcategory of $\mathcal{H}(\mathcal{C})$ formed by homotopically projective complexes of modules. We denote $\mathcal{H}_p^b(\mathcal{C})$ the smallest strictly (i.e., closed under isomorphisms) full triangulated subcategory of $\mathcal{H}_p(\mathcal{C})$ containing the $\mathcal{C}_x$, $x \in \mathcal{C}_0$.

We recall the following theorem from [11, pp. 69–70, Theorem 3.1]:

Theorem 2.13. *For any complex of $\mathcal{C}$-modules M we have the following triangle in $\mathcal{HC}$*

$$p(M) \to M \to M \to a(M) \to Sp(M),$$

where $a(M)$ is acyclic and $p(M)$ is homotopically projective.

Furthermore, this construction gives rise to triangle functors p and a on $\mathcal{H}(\mathcal{C})$ commuting with direct sums, p is the right adjoint of the inclusion functor from the full triangulated subcategory of homotopically projective complexes, and a is the left adjoint of the inclusion of the full triangulated subcategory of acyclic complexes.

Following Keller, we call $p(M)$ *the projective resolution* of the complex M.

Taking into account that any k-linear category is a DG category concentrated in degree 0 with null differential, we may apply the following theorem (cf. [11, Corollary 9.2]), adapted to the k-linear case,

Theorem 2.14. *Let $\mathcal{C}$ and $\mathcal{D}$ be two k-linear categories such that $\mathcal{D}$ is k-flat (i.e., ${}_y\mathcal{D}_x$ is k-flat, for every $x, y \in \mathcal{D}_0$). The following are equivalent*:

(i) *There is a $\mathcal{C}$–$\mathcal{D}$-bimodule P such that $P \otimes_{\mathcal{C}}^{L} - : D(\mathcal{C}) \to D(\mathcal{D})$ is an equivalence.*
(ii) *There is an S-equivalence $D(\mathcal{C}) \to D(\mathcal{D})$.*
(iii) *$\mathcal{C}$ is equivalent to a full subcategory $\mathcal{E}$ of $D(\mathcal{D})$ whose objects form a set of small generators and satisfy the following*

$$\mathrm{Hom}_{D(\mathcal{D})}\big(M, N[n]\big) = 0,$$

for all $n \neq 0$, $M, N \in \mathcal{E}$.

If any of the three equivalent conditions of the theorem is satisfied we say that $\mathcal{C}$ and $\mathcal{D}$ are derived equivalent.

We recall that a k-linear category is *projective* if ${}_y\mathcal{C}_x$ is a projective k-module for every $x, y \in \mathcal{C}_0$. We obtain the following as a corollary of the previous theorem. In particular, the hypothesis of projectivity holds when k is a field.

Theorem 2.15. *Let $\mathcal{C}$ and $\mathcal{D}$ be two small k-linear projective categories which are derived equivalent. Then the Hochschild–Mitchell homology and cohomology groups of $\mathcal{C}$ and $\mathcal{D}$ are respectively isomorphic.*

Proof. Since $\mathcal{C}$ and $\mathcal{D}$ are derived equivalent there exists a $\mathcal{D}$–$\mathcal{C}$-bimodule P and a $\mathcal{C}$–$\mathcal{D}$-bimodule Q, such that

$$P \otimes^L_{\mathcal{C}} - \otimes^L_{\mathcal{C}} Q : D(\mathcal{C}^e) \to D(\mathcal{D}^e),$$

is an equivalence, with quasi-inverse

$$Q \otimes^L_{\mathcal{C}} - \otimes^L_{\mathcal{C}} P : D(\mathcal{D}^e) \to D(\mathcal{C}^e).$$

As a consequence, $P \otimes^L_{\mathcal{C}} Q \simeq \mathcal{D}$ in $D(\mathcal{D}^e)$, and $Q \otimes^L_{\mathcal{D}} P \simeq \mathcal{C}$ in $D(\mathcal{C}^e)$.

Hence, we have the following chain of isomorphisms in $D(k)$

$$\mathcal{C} \otimes^L_{\mathcal{C}^e} \mathcal{C} \xrightarrow{\simeq} \left(Q \otimes^L_{\mathcal{D}} P\right) \otimes^L_{\mathcal{C}^e} \left(Q \otimes^L_{\mathcal{D}} P\right) \xrightarrow{\simeq} \left(P \otimes^L_{\mathcal{C}} Q\right) \otimes^L_{\mathcal{D}^e} \left(P \otimes^L_{\mathcal{C}} Q\right) \xrightarrow{\simeq} \mathcal{D} \otimes^L_{\mathcal{D}^e} \mathcal{D},$$

where the second isomorphism is induced by the isomorphism

$$\big(p(Q) \otimes_{\mathcal{D}} p(P)\big) \otimes_{\mathcal{C}^e} \big(p(Q) \otimes_{\mathcal{D}} p(P)\big) \to \big(p(P) \otimes_{\mathcal{C}} p(Q)\big) \otimes_{\mathcal{D}^e} \big(p(P) \otimes_{\mathcal{C}} p(Q)\big),$$
$$(a \otimes b) \otimes (a' \otimes b') \mapsto (b \otimes a') \otimes (b' \otimes a),$$

and the fact that $p(P) \otimes_{\mathcal{C}} p(Q)$ is a projective resolution of $P \otimes_{\mathcal{C}} Q$ in ${}_{\mathcal{D}^e}\mathrm{Mod}$ and $p(Q) \otimes_{\mathcal{D}} p(P)$ is a projective resolution of $Q \otimes_{\mathcal{C}} P$ in ${}_{\mathcal{C}^e}\mathrm{Mod}$. To prove this last statement we proceed as follows. Since $\mathcal{C}$ and $\mathcal{D}$ are k-projective categories, given a homotopically projective $\mathcal{C}$–$\mathcal{D}$-bimodule M (which we may suppose of the form $(\mathcal{C}_x \otimes_k {}_y\mathcal{D})$ for $x \in \mathcal{C}_0$, $y \in \mathcal{D}_0$) the functor $M \otimes_{\mathcal{D}} -$ sends relatively projective $\mathcal{D}$–$\mathcal{C}$-bimodules of type $(\mathcal{D}_{y'} \otimes_k {}_{x'}\mathcal{C})$ (for $x' \in \mathcal{C}_0$, $y' \in \mathcal{D}_0$) into $\mathcal{C}_x \otimes_k {}_y\mathcal{D}_{y'} \otimes_k {}_{x'}\mathcal{C}$, which are relatively projective $\mathcal{C}$-bimodules. Hence we get that $M \otimes_{\mathcal{D}} -$ sends homotopically projective $\mathcal{D}$–$\mathcal{C}$-bimodules into homotopically projective $\mathcal{C}$-bimodules.

This implies immediately the theorem for homology, since we have

$$H_n\big(\mathcal{C} \otimes^L_{\mathcal{C}^e} \mathcal{C}\big) \simeq \mathrm{Tor}^{\mathcal{C}^e}_n(\mathcal{C}, \mathcal{C}) = HH_n(\mathcal{C}).$$

For cohomology, we make use of the following isomorphism

$$\mathrm{Hom}_{D(\mathcal{C}^e)}\big(\mathcal{C}, \mathcal{C}[n]\big) \simeq \mathrm{Ext}^n_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}) = HH^n(\mathcal{C}).$$

which is proved in the second lemma of [12, Section 1.5]. This concludes the proof of the theorem. □

3. Derived equivalences between one-point extensions

Let us first state some facts concerning convex categories. From now on we shall suppose that k is a field.

Definition 3.1. Let $\mathcal{C}$ be a linear k-category and $\mathcal{D}$ a subcategory. We say that $\mathcal{D}$ is a *convex subcategory of* $\mathcal{C}$ if given $x_0, x_n \in \mathcal{D}_0, x_1, \dots, x_{n-1} \in \mathcal{C}_0$ such that $\exists i, 1 \leqslant i \leqslant n-1$, with $x_i \notin \mathcal{D}_0$, and morphisms $f_i \in {}_{x_{i+1}}\mathcal{C}_{x_i}$, for $0 \leqslant i \leqslant n-1$ then $f_{n-1} \circ \dots \circ f_0 = 0$.

Remark 3.2. The following facts about convex categories are easy to prove:

- If $\mathcal{C}'$ is a convex subcategory of $\mathcal{C}$, then $\mathcal{C}'^{op}$ is a convex subcategory of $\mathcal{C}^{op}$.
- If $\mathcal{C}'$ is a convex subcategory of $\mathcal{C}$ and $\mathcal{D}'$ is a convex subcategory of $\mathcal{D}$, then $\mathcal{C}' \boxtimes \mathcal{D}'$ is a convex subcategory of $\mathcal{C} \boxtimes \mathcal{D}$.

If $\mathcal{D}$ is a convex subcategory of $\mathcal{C}$ then there is a functor

$$i : \mathrm{Mod}_{\mathcal{D}} \to \mathrm{Mod}_{\mathcal{C}}$$

given by the $i(N)_x = N_x$, for $x \in \mathcal{D}_0$ and $i(N)_y = 0$, for $y \in \mathcal{C}_0 \setminus \mathcal{D}_0$. The action of $\mathcal{C}$ is induced by the action of $\mathcal{D}$ on N. It is clear that $i(N)$ is a right $\mathcal{C}$-module and it is well-defined since $\mathcal{D} \subset \mathcal{C}$ is convex.

Also, there is a functor induced by the inclusion

$$r : \mathrm{Mod}_{\mathcal{C}} \to \mathrm{Mod}_{\mathcal{D}},$$

given by $r(M) = M \circ \mathrm{inc}_{\mathcal{D} \subset \mathcal{C}}$.

They are adjoint functors, namely, we have the isomorphism

$$\theta : \mathrm{Hom}_{\mathcal{D}}\big(r(M), N\big) \to \mathrm{Hom}_{\mathcal{C}}\big(M, i(N)\big),$$
$$\theta\big(\{t_y\}_{y \in \mathcal{D}_0}\big)_x = \begin{cases} t_x & \text{if } x \in D_0, \\ 0 & \text{otherwise.} \end{cases}$$

It is easy to check that this map is well-defined and it is natural, and it is an isomorphism with inverse is given by

$$\zeta : \mathrm{Hom}_{\mathcal{C}}\big(M, i(N)\big) \to \mathrm{Hom}_{\mathcal{D}}\big(r(M), N\big),$$
$$\zeta\big(\{t_x\}_{x \in \mathcal{C}_0}\big)_y = t_y, \quad \text{for } y \in \mathcal{D}_0.$$

The adjunction says immediately that r preserves epimorphisms and i preserves monomorphisms, but we may also easily see that r preserves monomorphisms and i preserves epimorphisms. Hence both functors are exact, r preserves projective objects and i preserves injective objects.

Lemma 3.3. *Let $\mathcal{D}$ be a full convex subcategory of $\mathcal{C}$ and let M, N be two $\mathcal{D}$-modules. Then there is an isomorphism*

$$\operatorname{Ext}^{\bullet}_{\mathcal{D}}(M,N) \simeq \operatorname{Ext}^{\bullet}_{\mathcal{C}}\big(i(M), i(N)\big).$$

Proof. Choosing a projective $\mathcal{C}$-resolution $P_\bullet$ of $i(M)$, since r is exact and preserves projectives, $r(P_\bullet)$ is a projective $\mathcal{D}$-resolution of $r(i(M)) = M$. By the previous adjunction there is a morphism of complexes

$$\operatorname{Hom}_{\mathcal{C}}\big(P_\bullet, i(N)\big) \simeq \operatorname{Hom}_{\mathcal{D}}\big(r(P_\bullet), N\big),$$

implying that

$$\operatorname{Ext}^{\bullet}_{\mathcal{D}}(M,N) \simeq \operatorname{Ext}^{\bullet}_{\mathcal{C}}\big(i(M), i(N)\big). \qquad \square$$

Next let us define, given a k-linear category $\mathcal{C}$ and a right $\mathcal{C}$-module M, *the one-point extension of $\mathcal{C}$ by M* as the following small category, which we will denote $\mathcal{C}[M]$. The set of objects is $(\mathcal{C}[M])_0 = \mathcal{C}_0 \sqcup \{M\}$. The set of morphisms is given by

$${}_y\mathcal{C}[M]_x = {}_y\mathcal{C}_x, \qquad {}_M\mathcal{C}[M]_x = M_x, \qquad {}_y\mathcal{C}[M]_M = 0, \qquad {}_M\mathcal{C}[M]_M = k, \quad \text{for } x, y \in \mathcal{C}_0.$$

The composition is given by composition in $\mathcal{C}$, the action of $\mathcal{C}$ on M and the structure of k-module on each M_x. It may be easily verified that $\mathcal{C}[M]$ satisfies the axioms of a k-linear category and that $\mathcal{C}$ is a convex subcategory of $\mathcal{C}[M]$.

Remark 3.4. There is a dual definition for a left $\mathcal{C}$-module M, the only changes are ${}_x\mathcal{C}[M]_M = {}_xM$ and ${}_M\mathcal{C}[M]_y = 0$.

If $\mathcal{C}$ is finite, then $a(\mathcal{C}[M]) \simeq a(\mathcal{C})[M]$, where the last one denotes the one-point extension of the algebra $a(\mathcal{C})$ by the induced module $\bigoplus_{x\in\mathcal{C}_0} M_x$.

In this context, we define the right $\mathcal{C}[M]$-module $\bar{M}$, given by $\bar{M}_x = M_x$ $(x \in \mathcal{C}_0)$ and $\bar{M}_M = k$. The action is the following

$$\bar{\rho}_{x,y} : \bar{M}_x \otimes {}_x\mathcal{C}[M]_y = M_x \otimes {}_x\mathcal{C}_y \xrightarrow{\rho_{x,y}} M_y = \bar{M}_y,$$
$$\bar{\rho}_{x,M} : \bar{M}_x \otimes {}_x\mathcal{C}[M]_M = M_x \otimes 0 \xrightarrow{0} k = \bar{M}_M,$$
$$\bar{\rho}_{M,x} : \bar{M}_M \otimes {}_M\mathcal{C}[M]_x = k \otimes M_x \to M_x = \bar{M}_x,$$
$$\bar{\rho}_{M,M} : \bar{M}_M \otimes {}_M\mathcal{C}[M]_M = k \otimes k \to k = \bar{M}_M,$$

where the last two maps are the action of k on M_x and the product in k, respectively. Since $\bar{M} = {}_M\mathcal{C}[M]$, we get that $\bar{M}$ is a projective $\mathcal{C}[M]$-module satisfying, by Yoneda lemma, $\operatorname{Hom}_{\mathcal{C}[M]}(\bar{M}, \bar{M}) \simeq k$. Also, it is easy to see that $\bar{M}$ is small, since $\operatorname{Hom}_{\mathcal{C}[M]}(\bar{M}, N) \simeq \operatorname{Hom}_{\mathcal{C}[M]}({}_M\mathcal{C}[M], N) \simeq N_M$, for each $\mathcal{C}[M]$-module N.

Since $\mathcal{C}$ is a convex subcategory of $\mathcal{C}[M]$ there is a functor $i : \operatorname{Mod}_{\mathcal{C}} \to \operatorname{Mod}_{\mathcal{C}[M]}$, defined at the beginning of this section.

We have that $i({}_y\mathcal{C}) = {}_y\mathcal{C}[M]$, and hence i preserves relatively projectives and homotopically projectives, by definition.

Next we consider the functors

$$\mathrm{Hom}_{\mathcal{C}[M]}\big(i(-),\bar{M}\big):\mathrm{Mod}_{\mathcal{C}}\to\mathrm{Mod}_k\quad\text{and}\quad\mathrm{Hom}_{\mathcal{C}}(-,M):\mathrm{Mod}_{\mathcal{C}}\to\mathrm{Mod}_k.$$

We remark that they are isomorphic, i.e., there exists a natural isomorphism

$$\mathrm{Hom}_{\mathcal{C}[M]}\big(i(-),\bar{M}\big)\simeq\mathrm{Hom}_{\mathcal{C}}(-,M),\tag{3.1}$$

given by

$$\alpha:\mathrm{Hom}_{\mathcal{C}[M]}\big(i(-),\bar{M}\big)\to\mathrm{Hom}_{\mathcal{C}}(-,M),$$
$$\{t_{\bar{x}}\}_{\bar{x}\in\mathcal{C}[M]_0}\mapsto\{t_x\}_{x\in\mathcal{C}_0},$$

with inverse

$$\beta:\mathrm{Hom}_{\mathcal{C}}(-,M)\to\mathrm{Hom}_{\mathcal{C}[M]}\big(i(-),\bar{M}\big),$$
$$\{t_x\}_{x\in\mathcal{C}_0}\mapsto\{t_x\}_{x\in\mathcal{C}_0}\sqcup\{0_M\}.$$

Since i is an exact functor that preserves injectives, we have that $\mathrm{Ext}^{\bullet}_{\mathcal{C}[M]}(i(-),\bar{M})$ is a universal δ-functor, and it is isomorphic in degree zero to $\mathrm{Hom}_{\mathcal{C}}(-,M)$, so there is an isomorphism of δ-functors

$$\mathrm{Ext}^{\bullet}_{\mathcal{C}[M]}\big(i(-),\bar{M}\big)\simeq\mathrm{Ext}^{\bullet}_{\mathcal{C}}(-,M).$$

Also, the following identity holds

$$\mathrm{Hom}_{\mathcal{C}[M]}\big(\bar{M},i(-)\big)=0.\tag{3.2}$$

Theorem 3.5. *Let $\mathcal{C}$ and $\mathcal{D}$ be two k-linear categories, M a right $\mathcal{C}$-module and N a right $\mathcal{D}$-module. For any triangulated equivalence ϕ from $D(\mathcal{D})$ to $D(\mathcal{C})$, which maps N to M, there exists a triangulated equivalence Φ from $D(\mathcal{D}[N])$ to $D(\mathcal{C}[M])$ which restricts to ϕ.*

Proof. According to Theorem 2.14, ϕ is determined by its restriction, which is also an equivalence,

$$\phi':\mathcal{D}\to\mathcal{E}\subset D(\mathcal{C}),$$
$$\phi'(y)={}_yT,$$

where ${}_yT=\phi({}_y\mathcal{D}[0])$ denotes a complex of right $\mathcal{C}$-modules ($y\in\mathcal{D}_0$). By definition of equivalence these complexes form a set of small generators of $D(\mathcal{C})$, such that

$$\mathrm{Hom}_{D(\mathcal{C})}\big({}_yT,{}_{y'}T[n]\big)=0$$

for $n\neq 0$, and

$$\mathrm{Hom}_{D(\mathcal{C})}({}_yT,{}_{y'}T)={}_{y'}\mathcal{D}_y.$$

We are going to define an equivalence from $\mathcal{D}[N]$ to a subcategory $\bar{\mathcal{E}}$ of $D(\mathcal{C}[M])$ satisfying the hypotheses of Theorem 2.14. The following functor Φ' is fully faithful

$$\begin{aligned} \Phi' : \mathcal{D}[N] &\to \bar{\mathcal{E}} \subset D(\mathcal{C}[M]), \\ \Phi'(y) &= i({}_yT), \quad \text{if } y \in \mathcal{D}_0, \\ \Phi'(N) &= \bar{M}[0]. \end{aligned}$$

The definition on the morphisms is the natural one but it may be useful to give the precise details.

Let us take $\Phi'(f) = i \circ \phi'(f)$, for $f \in {}_{y'}\mathcal{D}_y = {}_{y'}\mathcal{D}[N]_y$, and $\Phi'(f) = 0$, for $f \in {}_y\mathcal{D}[N]_N$. Given $f \in {}_N\mathcal{D}[N]_y$, we define $\Phi'(f)$ by the following chain of natural isomorphisms

$$\begin{aligned} &{}_N\mathcal{D}[N]_y = N_y \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{D}}({}_y\mathcal{D}, N) \xrightarrow{\text{inc}} \operatorname{Hom}_{D(\mathcal{D})}\big({}_y\mathcal{D}, N[0]\big) \\ &\quad \xrightarrow{\phi} \operatorname{Hom}_{D(\mathcal{C})}\big({}_yT, M[0]\big) \xrightarrow{\beta'} \operatorname{Hom}_{D(\mathcal{C}[M])}\big(i({}_yT), \bar{M}[0]\big) \xrightarrow{\simeq} \operatorname{Hom}_{\mathcal{H}(\mathcal{C}[M])}\big(i({}_yT), \bar{M}[0]\big), \end{aligned}$$

where β' is the morphism induced by β on $\mathcal{H}_p^b$. We remark that the last isomorphism holds since $\bar{M}$ is $\mathcal{C}[M]$-projective. It remains to check that β' is an isomorphism: taking into account the short exact sequence (2.1), one only needs to check that it is so on each ${}_y\mathcal{C}[n]$. This is quite simple and follows from the isomorphisms

$$\operatorname{Hom}_{D(\mathcal{C})}\big({}_y\mathcal{C}[0], M[0]\big) = \operatorname{Hom}_{\mathcal{C}}({}_y\mathcal{C}, M) \xrightarrow{\beta} \operatorname{Hom}_{\mathcal{C}[M]}\big(i({}_y\mathcal{C}), \bar{M}\big),$$

and

$$\begin{aligned} &\operatorname{Hom}_{D(\mathcal{C})}\big({}_y\mathcal{C}[n], M[0]\big) = \operatorname{Hom}_{D(\mathcal{C})}\big({}_y\mathcal{C}[0], M[-n]\big) \simeq \operatorname{Ext}_{\mathcal{C}}^{-n}({}_y\mathcal{C}, M) = 0 \\ &\quad \to \operatorname{Hom}_{D(\mathcal{C}[M])}\big(i({}_y\mathcal{C})[0], \bar{M}[-n]\big) = \operatorname{Hom}_{D(\mathcal{C}[M])}\big(i\big({}_y\mathcal{C}[n]\big), \bar{M}[0]\big), \end{aligned}$$

for $n \neq 0$. The last map is an isomorphism since $i({}_y\mathcal{C}) = {}_y\mathcal{C}[M]$ and, for $n \neq 0$, we have that

$$\operatorname{Hom}_{D(\mathcal{C}[M])}\big(i({}_y\mathcal{C})[0], \bar{M}[n]\big) = \operatorname{Ext}_{\mathcal{C}[M]}^{-n}\big(i({}_y\mathcal{C}), \bar{M}\big) = 0.$$

Finally, for $f \in {}_N\mathcal{D}[N]_N$, we define $\Phi'(f)$ by means of the isomorphisms

$${}_N\mathcal{D}[N]_N = k \simeq \operatorname{Hom}_{\mathcal{C}[M]}(\bar{M}, \bar{M}) = \operatorname{Hom}_{D(\mathcal{C}[M])}\big(\bar{M}[0], \bar{M}[0]\big).$$

The functor Φ' is fully faithful by definition. Since i is fully faithful and preserves homotopically projectives,

$$\begin{aligned} \operatorname{Hom}_{D(\mathcal{C}[M])}\big(i({}_yT), i({}_{y'}T)\big) &= \operatorname{Hom}_{\mathcal{H}(\mathcal{C}[M])}\big(i({}_yT), i({}_{y'}T)\big) = \operatorname{Hom}_{\mathcal{H}(\mathcal{C})}({}_yT, {}_{y'}T) \\ &= \operatorname{Hom}_{D(\mathcal{C})}({}_yT, {}_{y'}T) = {}_{y'}\mathcal{D}_y, \end{aligned}$$

for $y, y' \in \mathcal{D}_0$. Also, $\operatorname{Hom}_{D(\mathcal{C}[M])}(\bar{M}, i({}_{y'}T)) = 0$ as a consequence of (3.2). All other cases are straightforward.

We also need to prove that $\mathrm{Hom}_{D(\mathcal{C}[M])}(\Phi'(\bar{y}), \Phi'(\bar{y}')[n]) = 0$, for $n \neq 0$. This is achieved in exactly the same way as before, just considering a shift by n and noticing that i commutes with the shift by definition.

The image of the functor Φ' is a set of small generators: they are small since $\bar{M}[0]$ is small and $\{i({}_yT)\}_{y\in\mathcal{D}_0}$ is set of small objects. The latter is proved directly from the sequence (2.1) and the fact that $i({}_x\mathcal{C}) = {}_x\mathcal{C}[M]$ is small.

To prove that they are a set of generators we proceed as follows: $\{{}_yT\}_{y\in\mathcal{D}_0}$ is a set of generators of $D(\mathcal{C})$, then the full strictly triangulated subcategory closed under direct sums containing them also contains $\{{}_y\mathcal{C}\}_{y\in\mathcal{D}_0}$. So, the triangulated subcategory generated by $\{i({}_yT)\}_{y\in\mathcal{D}_0}$ contains $\{i({}_y\mathcal{C})\}_{y\in\mathcal{D}_0} = \{{}_y\mathcal{C}[M]\}_{y\in\mathcal{D}_0}$. As a consequence, the triangulated subcategory generated by the image of Φ' contains $\{{}_y\mathcal{C}[M]\}_{y\in\mathcal{D}_0}$ and $\bar{M} = {}_M\mathcal{C}[M]$, whence it is the whole $D(\mathcal{C}[M])$. The functor Φ in the statement of the theorem is completely determined by Φ'. □

4. Happel's cohomological long exact sequence

In this section we first generalize the long exact sequence in [9, Theorem 5.3]. to Hochschild–Mitchell cohomology. Although the proof is quite similar to the algebraic case but a little bit more technical, it is interesting to remark that in the categorical context, a more general statement (Theorem 4.4) not only holds but it is more natural. The proofs of this general statement has been inspired by an article of Cibils (cf. Theorem 4.5 in [3]) and in fact provides a simpler proof to Cibils' result.

We first state some definitions. Given a $\mathcal{C}$-bimodule N, let $j(N)$ be the $\mathcal{C}[M]$-bimodule, such that ${}_{\bar{x}}j(N)_M = {}_Mj(N)_{\bar{x}} = 0$, for $\bar{x} \in \mathcal{C}[M]_0$, and ${}_yj(N)_x = {}_yN_x$, for $x, y \in \mathcal{C}_0$. The action is induced by the action of $\mathcal{C}$ on N. Also, we will denote by S the simple right $\mathcal{C}[M]$-module satisfying $S_x = 0$, for $x \in \mathcal{C}_0$, and $S_M = k$. The action is the obvious one.

Lemma 4.1. *Let $\mathcal{C}$ be a k-linear category and M a right $\mathcal{C}$-module. The following holds*:

(1) $\mathcal{C}[M]^e \simeq \mathcal{C}[M]_M \otimes_k {}_M\mathcal{C}[M] \simeq \mathrm{Hom}_k(S, \bar{M})$, *as $\mathcal{C}[M]$-bimodules.*
(2) $\mathrm{Ext}^{n+1}_{\mathcal{C}[M]}(S, \bar{M}) \simeq \mathrm{Ext}^n_{\mathcal{C}}(M, M)$, *for $n \geqslant 1$.*
(3) $\mathrm{Ext}^1_{\mathcal{C}[M]}(S, \bar{M}) \simeq \mathrm{Hom}_{\mathcal{C}}(M, M)/k$.
(4) $\mathrm{Hom}_{\mathcal{C}[M]}(S, \bar{M}) = 0$.
(5) $\mathrm{Ext}^n_{\mathcal{C}^e}(\mathcal{C}, \mathcal{C}) \simeq \mathrm{Ext}^n_{\mathcal{C}[M]^e}(j(\mathcal{C}), j(\mathcal{C}))$, *for $n \geqslant 0$.*

Proof. (4.1). It is clear that the following morphism of $\mathcal{C}[M]$-bimodules

$$
\begin{aligned}
{}_{\bar{x}}\phi_{\bar{y}} : \mathrm{Hom}_k(S_{\bar{x}}, \bar{M}_{\bar{y}}) &\to {}_{\bar{x}}\mathcal{C}[M]_M \otimes_k {}_M\mathcal{C}[M]_{\bar{y}}, \\
{}_{\bar{x}}\phi_{\bar{y}} &= 0, \quad \text{if } \bar{x} \neq M, \\
{}_{\bar{x}}\phi_{\bar{y}}(f) &= 1 \otimes f(1), \quad \text{if } \bar{x} = M,\ f \in \mathrm{Hom}_k(k, \bar{M}_{\bar{y}}),
\end{aligned}
$$

is in fact an isomorphism.

In order to prove (2)–(4) we proceed as follows. There is a short exact sequence of right $\mathcal{C}[M]$-modules

$$0 \to i(M) \xrightarrow{f} \bar{M} \xrightarrow{g} S \to 0.$$

The morphisms are the obvious ones. Applying the functor $\mathrm{Hom}_{\mathcal{C}[M]}(-,\bar{M})$ to this short exact sequence we get the long exact sequence

$$\begin{aligned}0 &\to \mathrm{Hom}_{\mathcal{C}[M]}(S,\bar{M}) \to \mathrm{Hom}_{\mathcal{C}[M]}(\bar{M},\bar{M}) \to \mathrm{Hom}_{\mathcal{C}[M]}\big(i(M),\bar{M}\big) \to \mathrm{Ext}^1_{\mathcal{C}[M]}(S,\bar{M})\\ &\to \cdots \to \mathrm{Ext}^n_{\mathcal{C}[M]}(S,\bar{M}) \to \mathrm{Ext}^n_{\mathcal{C}[M]}(\bar{M},\bar{M}) \to \mathrm{Ext}^n_{\mathcal{C}[M]}\big(i(M),\bar{M}\big) \to \mathrm{Ext}^{n+1}_{\mathcal{C}[M]}(S,\bar{M}) \to \cdots.\end{aligned}$$

Taking into account that i preserves exactness and projectives, and the isomorphism (3.1), we have that $\mathrm{Hom}_{\mathcal{C}[M]}(i(M),\bar{M}) \simeq \mathrm{Hom}_{\mathcal{C}}(M,M)$ and $\mathrm{Ext}^n_{\mathcal{C}[M]}(i(M),\bar{M}) \simeq \mathrm{Ext}^n_{\mathcal{C}}(M,M)$, for $n \geqslant 1$. Also, we see immediately that $\mathrm{Ext}^n_{\mathcal{C}[M]}(\bar{M},\bar{M}) = 0$, for $n \geqslant 1$, since $\bar{M} = {}_M\mathcal{C}[M]$ is projective. This proves (2).

For the other statements, we recall that $\mathrm{Hom}_{\mathcal{C}[M]}(\bar{M},\bar{M}) \simeq {}_M\mathcal{C}[M]_M = k$, and notice that the map given by $f^*:\mathrm{Hom}_{\mathcal{C}[M]}(\bar{M},\bar{M}) \to \mathrm{Hom}_{\mathcal{C}[M]}(i(M),\bar{M})$ is not zero since $f^*(\mathrm{id}_{\bar{M}}) = f \neq 0$, and so f^* is injective. Hence we get (3) and (4).

In order to prove (5) we only use that $\mathcal{C}^e$ is a convex full subcategory of $\mathcal{C}[M]^e$ and apply Lemma 3.3. □

Theorem 4.2. *Let $\mathcal{C}$ be a k-linear category and M a right $\mathcal{C}$-module. There is a cohomological long exact sequence*

$$\begin{aligned}0 &\to HH^0\big(\mathcal{C}[M]\big) \to HH^0(\mathcal{C}) \to \mathrm{Hom}_{\mathcal{C}}(M,M)/k \to HH^1\big(\mathcal{C}[M]\big) \to HH^1(\mathcal{C})\\ &\to \mathrm{Ext}^1_{\mathcal{C}}(M,M) \to \cdots \to \mathrm{Ext}^{n-1}_{\mathcal{C}}(M,M) \to HH^n\big(\mathcal{C}[M]\big) \to HH^n(\mathcal{C}) \to \mathrm{Ext}^n_{\mathcal{C}}(M,M)\\ &\to HH^{n+1}\big(\mathcal{C}[M]\big) \to \cdots.\end{aligned}$$

Proof. Let us consider the following short exact sequence of $\mathcal{C}[M]$-bimodules

$$0 \to K \xrightarrow{\alpha} \mathcal{C}[M] \xrightarrow{\beta} j(\mathcal{C}) \to 0, \tag{4.1}$$

where β is given by $\beta(f) = f$, for $f \in {}_y\mathcal{C}_x \subset {}_y\mathcal{C}[M]_x$, and zero in any other case. The $\mathcal{C}[M]$-bimodule K is its kernel.

We shall see that K and $\mathcal{C}[M]_M \otimes_k {}_M\mathcal{C}[M]$ are isomorphic as $\mathcal{C}[M]$-bimodules. To prove this fact we proceed as follows: consider the map

$$\begin{aligned}\gamma:\mathcal{C}[M]_M \otimes_k {}_M\mathcal{C}[M] &\to \mathcal{C}[M],\\ \gamma(c\otimes c') &= c.c'.\end{aligned}$$

It is evident that $\beta \circ \gamma = 0$ and that γ is a monomorphism. If $c \in \mathrm{Ker}(\beta)$, then either $c = 0$ or $c \in {}_M\mathcal{C}[M]_{\bar{x}}$. In this case, $c = \gamma({}_M1_M \otimes c)$, and hence $c \in \mathrm{Im}(\gamma)$. It follows that $\mathcal{C}[M]_M \otimes_k {}_M\mathcal{C}[M]$ is also a kernel of β. As a consequence, $\mathrm{Ext}^n_{\mathcal{C}[M]^e}(K, j(\mathcal{C})) = 0$ for $n \geqslant 1$. Also $\mathrm{Hom}_{\mathcal{C}[M]^e}(K, j(\mathcal{C})) = {}_Mj(\mathcal{C})_M = 0$, so $\mathrm{Ext}^n_{\mathcal{C}[M]^e}(K, j(\mathcal{C})) = 0$, for $n \geqslant 0$.

Now, applying the functor $\mathrm{Hom}_{\mathcal{C}[M]^e}(-, j(\mathcal{C}))$ to the sequence (4.1) and using that $\mathrm{Ext}^n_{\mathcal{C}[M]^e}(K, j(\mathcal{C})) = 0$ for $n \geqslant 0$, we get $\mathrm{Ext}^n_{\mathcal{C}[M]^e}(j(\mathcal{C}), j(\mathcal{C})) = \mathrm{Ext}^n_{\mathcal{C}[M]^e}(\mathcal{C}[M], j(\mathcal{C}))$, for $n \geqslant 0$. The first one is isomorphic to $HH^n(\mathcal{C})$ using Lemma 4.1(5).

Also,

$$H^n\big(\mathcal{C}[M], K\big) = \mathrm{Ext}^n_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], K\big) \simeq \mathrm{Ext}^n_{\mathcal{C}[M]}(S,\bar{M}) = H^n\big(\mathcal{C}[M], \mathrm{Hom}_k(S,\bar{M})\big),$$

by Lemma 4.1(1). Finally, $H^n(\mathcal{C}[M], \operatorname{Hom}_k(S, \bar{M}))$ is isomorphic to $\operatorname{Ext}^n_{\mathcal{C}[M]}(S, \bar{M})$ since, by adjunction, the complex computing the Hochschild–Mitchell cohomology also gives the Ext groups. We also notice that $\operatorname{Hom}_{\mathcal{C}[M]^e}(\mathcal{C}[M], K) = 0$, $\operatorname{Ext}^1_{\mathcal{C}[M]^e}(\mathcal{C}[M], K) = \operatorname{Hom}_{\mathcal{C}}(M, M)/k$ and $\operatorname{Ext}^n_{\mathcal{C}[M]^e}(\mathcal{C}[M], K) = \operatorname{Ext}^{n-1}_{\mathcal{C}}(M, M)$, for $n \geqslant 2$, using Lemma 4.1(4), (3) and (1), respectively.

Applying now the functor $\operatorname{Hom}_{\mathcal{C}[M]^e}(\mathcal{C}[M], -)$ to (4.1), we obtain the long exact sequence

$$\begin{aligned}
0 &\to \operatorname{Hom}_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], K\big) \to \operatorname{Hom}_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], \mathcal{C}[M]\big) \to \operatorname{Hom}_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], j(\mathcal{C})\big) \\
&\to \operatorname{Ext}^1_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], K\big) \to \cdots \to \operatorname{Ext}^n_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], K\big) \to \operatorname{Ext}^n_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], \mathcal{C}[M]\big) \\
&\to \operatorname{Ext}^n_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], j(\mathcal{C})\big) \to \operatorname{Ext}^{n+1}_{\mathcal{C}[M]^e}\big(\mathcal{C}[M], K\big) \to \cdots.
\end{aligned}$$

Using the identifications above the theorem follows. □

Next we will consider a more general situation. Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two k-linear categories, and let M be a $\mathcal{C}_1$–$\mathcal{C}_2$-bimodule. We define the category $\mathcal{C} = \mathcal{C}_1 \sqcup_M \mathcal{C}_2$ with objects $\mathcal{C}_0 = (\mathcal{C}_1)_0 \sqcup (\mathcal{C}_2)_0$ and morphisms

$${}_x\mathcal{C}_y = \begin{cases} {}_x(\mathcal{C}_1)_y, & \text{for } x, y \in (\mathcal{C}_1)_0, \\ {}_x(\mathcal{C}_2)_y, & \text{for } x, y \in (\mathcal{C}_2)_0, \\ {}_xM_y, & \text{for } x \in (\mathcal{C}_1)_0,\ y \in (\mathcal{C}_2)_0, \\ 0, & \text{otherwise.} \end{cases}$$

Example 4.3. If $(\mathcal{C}_1)_0 = \{*\}$, ${}_*(\mathcal{C}_1)_* = k$ and M is a right $\mathcal{C}_2$-module, then $\mathcal{C}_1 \sqcup_M \mathcal{C}_2 = \mathcal{C}_2[M]$.

Since, for $i, j \in \{1, 2\}$, $\mathcal{C}_i \boxtimes \mathcal{C}_j^{op}$ is a convex subcategory of $\mathcal{C}^e$, there are well-defined functors of restriction. Given a $\mathcal{C}$-bimodule N, we shall denote $r_{i,j}(N)$ the corresponding restriction. We also write $r_i(N) = r_{i,i}(N)$.

In this situation there is a cohomological long exact sequence generalizing the previous one. The key fact of the proof is that it is possible to decompose the Hochschild–Mitchell projective resolution of $\mathcal{C}$ as $\mathcal{C}$-bimodule as follows

$$\begin{aligned}
N_n(\mathcal{C}) &= \bigoplus_{(x_0,\dots,x_n)\in\mathcal{C}_0^{n+1}} {}_-\mathcal{C}_{x_n} \otimes {}_{x_n}\mathcal{C}_{x_{n-1}} \otimes \ldots \otimes {}_{x_1}\mathcal{C}_{x_0} \otimes {}_{x_0}\mathcal{C}_- \\
&= \bigoplus_{(x_0,\dots,x_n)\in(\mathcal{C}_1)_0^{n+1}} {}_-\mathcal{C}_{x_n} \otimes {}_{x_n}(\mathcal{C}_1)_{x_{n-1}} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_1)_{x_0} \otimes {}_{x_0}\mathcal{C}_- \\
&\quad \oplus \bigoplus_{(x_0,\dots,x_n)\in(\mathcal{C}_2)_0^{n+1}} {}_-\mathcal{C}_{x_n} \otimes {}_{x_n}(\mathcal{C}_2)_{x_{n-1}} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0} \otimes {}_{x_0}\mathcal{C}_- \\
&\quad \oplus \bigoplus_{i=0}^{n-1} \bigoplus_{\substack{(x_0,\dots,x_i)\in(\mathcal{C}_2)_0^{i+1} \\ (x_{i+1},\dots,x_n)\in(\mathcal{C}_1)_0^{n-i}}} {}_-\mathcal{C}_{x_n} \otimes {}_{x_n}(\mathcal{C}_1)_{x_{n-1}} \otimes \cdots \otimes {}_{x_{i+1}}M_{x_i} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0} \otimes {}_{x_0}\mathcal{C}_-.
\end{aligned}$$

This decomposition gives

$$
\begin{aligned}
&\operatorname{Hom}_{\mathcal{C}^e}\big(N_n(\mathcal{C}), N\big)\\
&\quad= \prod_{(x_0,\ldots,x_n)\in\mathcal{C}_0^{n+1}} \operatorname{Hom}_k\big({}_{x_n}\mathcal{C}_{x_{n-1}} \otimes \cdots \otimes {}_{x_1}\mathcal{C}_{x_0}, {}_{x_n}N_{x_0}\big)\\
&\quad= \prod_{(x_0,\ldots,x_n)\in(\mathcal{C}_1)_0^{n+1}} \operatorname{Hom}_k\big({}_{x_n}(\mathcal{C}_1)_{x_{n-1}} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_1)_{x_0}, {}_{x_n}N_{x_0}\big)\\
&\qquad\oplus \prod_{(x_0,\ldots,x_n)\in(\mathcal{C}_2)_0^{n+1}} \operatorname{Hom}_k\big({}_{x_n}(\mathcal{C}_2)_{x_{n-1}} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0}, {}_{x_n}N_{x_0}\big)\\
&\qquad\oplus \bigoplus_{i=0}^{n-1} \prod_{\substack{(x_0,\ldots,x_i)\in(\mathcal{C}_2)_0^{i+1}\\ (x_{i+1},\ldots,x_n)\in(\mathcal{C}_1)_0^{n-i}}} \operatorname{Hom}_k\big({}_{x_n}(\mathcal{C}_1)_{x_{n-1}} \otimes \cdots \otimes {}_{x_{i+1}}M_{x_i} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0}, {}_{x_n}N_{x_0}\big)\\
&\quad= \operatorname{Hom}_{\mathcal{C}_1^e}\big(N_n(\mathcal{C}_1), r_1(N)\big) \oplus \operatorname{Hom}_{\mathcal{C}_2^e}\big(N_n(\mathcal{C}_2), r_2(N)\big) \oplus \operatorname{Hom}_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}\big(\tilde{M}_{n-1}, r_{1,2}(N)\big),
\end{aligned}
$$

where $(\tilde{M}_n, d_n)$ is the complex of projective $\mathcal{C}_1$–$\mathcal{C}_2$-bimodules given by

$$
\tilde{M}_n = \bigoplus_{i=0}^{n} \bigoplus_{\substack{(x_0,\ldots,x_i)\in(\mathcal{C}_1)_0^{i+1}\\ (x_{i+1},\ldots,x_{n+1})\in(\mathcal{C}_1)_0^{n+1-i}}} {}_{-}(\mathcal{C}_1)_{x_{n+1}} \otimes {}_{x_{n+1}}(\mathcal{C}_1)_{x_n} \otimes \cdots \otimes {}_{x_{i+1}}M_{x_i} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0} \otimes {}_{x_0}(\mathcal{C}_2)_{-},
$$

with differential $d_\bullet$ obtained by restricting the differential of the Hochschild–Mitchell resolution. This complex is in fact a projective resolution of M as a $\mathcal{C}_1$–$\mathcal{C}_2$-bimodule. In order to prove this statement, it is sufficient to notice that $(\tilde{M}_\bullet, d_\bullet)$ is the total complex obtained from the first quadrant double complex

$$
\tilde{M}_{i,j} = \bigoplus_{\substack{(x_0,\ldots,x_i)\in(\mathcal{C}_2)_0^{i+1}\\ (x_{i+1},\ldots,x_{i+j+1})\in(\mathcal{C}_1)_0^{j}}} {}_{-}(\mathcal{C}_1)_{x_{n+1}} \otimes {}_{x_{n+1}}(\mathcal{C}_1)_{x_n} \otimes \cdots \otimes {}_{x_{i+1}}M_{x_i} \otimes \cdots \otimes {}_{x_1}(\mathcal{C}_2)_{x_0} \otimes {}_{x_0}(\mathcal{C}_2)_{-},
$$

where the vertical and horizontal differentials are

$$
\begin{aligned}
&{}_y\big(d^h_{i,j}\big)_x\big({}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_0}(c_2)_x\big)\\
&\quad= {}_y(c_1)_{x_{n+1}} \cdot {}_{x_{n+1}}(c_1)_{x_n} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_0}(c_2)_x\\
&\qquad+ \sum_{j=i+2}^{n+1} (-1)^{j+n+1}\, {}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{j+1}}(c_1)_{x_j} \cdot {}_{x_j}(c_1)_{x_{j-1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_0}(c_2)_x\\
&\qquad+ (-1)^{i+n}\, {}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+2}}(c_1)_{x_{i+1}} \cdot {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_0}(c_2)_x
\end{aligned}
$$

and

$$
\begin{aligned}
&{}_y\big(d^v_{i,j}\big)_x\big({}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_0}(c_2)_x\big) \\
&\quad = (-1)^{i+n+1}\,{}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \cdot {}_{x_i}(c_2)_{x_{i-1}} \otimes \cdots \otimes {}_{x_0}(c_2)_x \\
&\qquad + \sum_{j=1}^{i-1}(-1)^{j+n+1}\,{}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_{j+1}}(c_2)_{x_j} \cdot {}_{x_j}(c_2)_{x_{j-1}} \otimes \cdots \otimes {}_{x_0}(c_2)_x \\
&\qquad + (-1)^{n+1}\,{}_y(c_1)_{x_{n+1}} \otimes \cdots \otimes {}_{x_{i+1}}m_{x_i} \otimes \cdots \otimes {}_{x_1}(c_2)_{x_0} \cdot {}_{x_0}(c_2)_x.
\end{aligned}
$$

This double complex has exact rows and columns using the usual homotopy arguments. Then the cohomology of the cochain complex $(\mathrm{Hom}_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}(\tilde{M}_\bullet, r_{1,2}(N)),\ d_\bullet^*)$ is exactly $\mathrm{Ext}^\bullet_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}(M, r_{1,2}(N))$.

We also notice that this cochain complex is actually a subcomplex of $\mathrm{Hom}_{\mathcal{C}^e}(N_n(\mathcal{C}), N)$ computing the Hochschild–Mitchell cohomology of $\mathcal{C}$, and its quotient is $\mathrm{Hom}_{\mathcal{C}_1^e}(N_n(\mathcal{C}_1), r_1(N)) \oplus \mathrm{Hom}_{\mathcal{C}_2^e}(N_n(\mathcal{C}_2), r_2(N))$. In other words, there is a short exact sequence of complexes of k-modules

$$
\begin{aligned}
0 &\to \mathrm{Hom}_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}\big(\tilde{M}_{\bullet-1}, r_{1,2}(N)\big) \to \mathrm{Hom}_{\mathcal{C}^e}\big(N_\bullet(\mathcal{C}), N\big) \\
&\to \mathrm{Hom}_{\mathcal{C}_1^e}\big(N_\bullet(\mathcal{C}_1), r_1(N)\big) \oplus \mathrm{Hom}_{\mathcal{C}_2^e}\big(N_\bullet(\mathcal{C}_2), r_2(N)\big) \to 0.
\end{aligned}
$$

The cohomological long exact sequence obtained from this short exact sequence yields the following theorem.

Theorem 4.4. *Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two small k-linear categories, and let M be a $\mathcal{C}_1$–$\mathcal{C}_2$-bimodule. Denoting $\mathcal{C} = \mathcal{C}_1 \sqcup_M \mathcal{C}_2$, there is cohomological long exact sequence*

$$
\begin{aligned}
&0 \to H^0(\mathcal{C}, N) \to H^0\big(\mathcal{C}_1, r_1(N)\big) \oplus H^0\big(\mathcal{C}_2, r_2(N)\big) \to \mathrm{Hom}_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}\big(M, r_{1,2}(N)\big) \to H^1(\mathcal{C}, N) \\
&\to \cdots \to H^n(\mathcal{C}, N) \to H^n\big(\mathcal{C}_1, r_1(N)\big) \oplus H^n\big(\mathcal{C}_2, r_2(N)\big) \to \mathrm{Ext}^n_{\mathcal{C}_1\boxtimes\mathcal{C}_2^{op}}\big(M, r_{1,2}(N)\big) \\
&\to H^{n+1}(\mathcal{C}, N) \to \cdots.
\end{aligned}
$$

This theorem provides a long exact sequence generalizing the one obtained by Cibils [3] and by Green and Solberg [8] for algebras and the one-point extension sequence proved before.

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