SPHERICALLY SYMMETRIC THERMOELASTIC WAVES IN A TEMPERATURE-RATE DEPENDENT MEDIUM WITH A SPHERICAL CAVITY

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Abstract—The temperature-rate dependent thermoelasticity theory is employed to study spherically symmetric thermoelastic waves in an infinite medium with a spherical cavity whose inner boundary is subjected to (i) unit-step in temperature change and zero stress and (ii) unit-step in stress and zero temperature change. The effect of thermal relaxation times on the thermoelastic disturbances being short-lived, short-time approximations of the solutions for deformation, temperature and stresses have been considered. It is observed that the temperature and deformation are discontinuous at both the wave fronts while the stresses suffer delta-function singularities in case (i) and in case (ii), the deformation and temperature are continuous at both the wave fronts, while the stresses are discontinuous at these locations.

1. INTRODUCTION

The generalized thermoelasticity theory proposed by Green and Lindsay [1] and Suhhubi [2] has aroused much interest in recent years. Unlike the conventional coupled thermoelasticity theory [3], this new theory includes the rate of temperature in the constitutive equations and is referred to as the temperature-rate dependent thermoelasticity theory [2]. Like Lord and Shulman's theory [4] of generalized thermoelasticity, this theory also predicts second sound phenomenon. Some problems, revealing interesting characteristics of this new theory, have been considered by Boschi [5, 6], Agarwal [7, 8], Roy Choudhuri [9], Chandrasekhariah [10–12]. Because of the experimental evidence in support of finiteness of heat propagation speed [13, 14], the problems considered in the context of this temperature-rate dependent theory are more realistic than the counter parts in the conventional theory.

In the present article we have considered thermoelastic wave propagation in a spherically symmetric infinite medium with a spherical cavity using the temperature-rate dependent thermoelasticity theory [1], when (i) step-input of temperature and zero stress and (ii) step-input of stress and zero temperature change are applied at the boundary of the cavity. The case of conventional coupled theory is also studied and compared with the results reported by Nariboli [15]. It is observed on comparison that the temperature-rate dependent theory brings into light some discontinuities which were not encountered in the conventional coupled theory.

2. BASIC EQUATIONS AND BOUNDARY CONDITIONS

We consider an infinite isotropic medium with a spherical cavity of radius “a”. Introducing spherical polar coordinates \((r, \theta, \phi)\) with the centre of cavity as origin, infinite medium is then characterized by \(r \geq a\). If \((\tau_r, \tau_{\theta\theta}, \tau_{\phi\phi})\) are the radial and circumferential stresses, then assuming complete spherical symmetry, the radial displacement \(u\) is the only displacement so that \(\ddot{u} = [u(r, t), 0, 0]\). We then have

\[
\begin{align*}
\tau_{\theta\theta} &= \tau_{\phi\phi}, \\
\varepsilon_r &= \frac{\partial u}{\partial r}, \\
\varepsilon_{\theta\theta} &= \varepsilon_{\phi\phi} = \frac{u}{r},
\end{align*}
\]
where \(e_r, e_\theta, e_\phi\) are the radial and circumferential strains. Now the constitutive equations for isotropic temperature-rate dependent thermoelasticity theory of Green and Lindsay [1] are

\[
\begin{align*}
\tau_{ij} &= \lambda \Delta \delta_{ij} + 2\mu e_{ij} - \gamma (\theta + \alpha \dot{\theta}) \delta_{ij}, \quad (i, j = 1, 2, 3), \\
k V^2 \theta &= \rho c_s (\theta + \alpha^* \dot{\theta}) + \gamma T_0 \Delta,
\end{align*}
\]

where \(\tau_{ij}\) is the stress tensor, \(e_{ij}\) is the strain tensor, \(\Delta\) is the dilatation, \(\theta\) is the temperature increase over \(T_0\), \(T_0\) is the absolute reference temperature, \(\gamma\) is equal to \((3\lambda + 2\mu)\alpha\); \(\lambda, \mu\) are the Lamé constants, \(\alpha, \alpha^*\) are the relaxation times, \(K\) is the thermal conductivity, \(\rho\) is the mass density, \(c_s\) is the specific heat of the solid at constant strain.

In case of complete spherical symmetry, the stress–strain relations (1) and (2) along with the heat conduction equation in the temperature-rate dependent theory reduce to

\[
\begin{align*}
\tau_{rr} &= \lambda \Delta + 2\mu e_r - \gamma (\theta + \alpha \dot{\theta}), \\
\tau_{\theta\theta} &= \tau_{\phi\phi} = \lambda \Delta + 2\mu e_\theta - \gamma (\theta + \alpha \dot{\theta}), \\
K \left( \frac{\partial^2}{\partial r^2} + \frac{2}{r} \frac{\partial}{\partial r} \right) \theta &= \rho c_s (\theta + \alpha^* \dot{\theta}) + \gamma T_0 \Delta,
\end{align*}
\]

where

\[
\Delta = e_r + e_\theta + e_\phi = \frac{\partial u}{\partial r} + \frac{2\mu}{r}.
\]

The stress equations of motion, in absence of body forces, are

\[
\tau_{ij, j} = \rho \ddot{u}_i, \quad (i, j = 1, 2, 3),
\]

which, in case of complete spherical symmetry, reduce to the only stress equation of motion

\[
\frac{\partial \tau_{rr}}{\partial r} + \frac{2}{r} (\tau_{rr} - \tau_{\theta\theta}) = \rho \frac{\partial^2 u}{\partial t^2}.
\]

We introduce the following dimensionless quantities:

\[
\begin{align*}
U &= \frac{u}{a}, & R &= \frac{r}{a}, & Z &= \frac{\theta}{T_0}, & \eta &= \frac{C_1 t}{a}, & C^2 &= \frac{\lambda + 2\mu}{\rho}, \\
\alpha' &= \alpha \omega', & \alpha^* &= \alpha^* \omega^*, & \omega' &= \frac{\gamma T_0}{\alpha p G}, & \omega^* &= \frac{C_1}{a},
\end{align*}
\]

\[
(\sigma_r, \sigma_\theta, \sigma_\phi) = \frac{1}{2\mu} (\tau_{rr}, \tau_{\theta\theta}, \tau_{\phi\phi}).
\]

Then equations (3) assume the forms

\[
\begin{align*}
\sigma_r &= \frac{dU}{dR} + \frac{\lambda}{2\mu} \left( \frac{dU}{dR} + \frac{2U}{R} \right) - (\lambda + 2\mu) (a_1 Z + \alpha' \dot{Z}), \\
\sigma_\theta &= \frac{U}{R} + \frac{\lambda}{2\mu} \left( \frac{dU}{dR} + \frac{2U}{R} \right) - (\lambda + 2\mu) (a_1 Z + \alpha' \dot{Z}),
\end{align*}
\]

(7)

on introducing equations (6) and (7) into equations (4) and (5) we finally obtain

\[
\begin{align*}
D_D Z &= a_2 \left( \frac{\partial Z}{\partial \eta} + \alpha^* \frac{\partial^2 Z}{\partial \eta^2} \right) + a_3 \frac{\partial}{\partial \eta} (D_D U), \\
D_D D_D U &= a_1 \frac{\partial Z}{\partial R} - \alpha' \frac{\partial^2 Z}{\partial R \partial \eta} = \frac{\partial^3 U}{\partial \eta^3},
\end{align*}
\]

(9)
Spherically symmetric thermoelastic waves

where

\[ a_1 = \frac{\gamma T_0}{\lambda + 2\mu}, \quad a_2 = \frac{\rho c_p a C_1}{k}, \quad a_3 = \frac{\gamma a C_1}{K} \]

are non-dimensional constants and

\[ D = \frac{\partial}{\partial R}, \quad D_1 = \frac{\partial}{\partial R} + \frac{2}{R}. \]

Equations (8) and (9) readily agree with equation (2a) of Ref. [15]. The boundary of the cavity is given by \( R = 1 \). We take two types of boundary conditions as

(i) \( \sigma_R = 0, \quad R = 1, \quad \eta \geq 0, \)

\[ Z = \chi H(\eta), \quad R = 1, \quad \eta > 0, \]

(ii) \( \sigma_R = \chi_H(\eta), \quad R = 1, \quad \eta > 0, \)

\[ Z = 0, \quad R = 1, \quad \eta > 0, \]

where \( \chi \) and \( \chi_H \) are two dimensionless constants.

We assume that the medium is at rest and undisturbed initially. Thus the initial and regularity conditions are

\[ U = Z = 0, \quad \text{at} \quad \eta = 0, \quad R > 1, \quad \frac{\partial U}{\partial \eta} = 0, \quad \text{at} \quad \eta = 0 \]

and \( U = Z = 0 \), when \( R \to \infty \).

3. SOLUTION OF THE PROBLEM

In the Laplace transform domain the equations (8) and (9) lead to

\[ (D_1 D - a_2 s - \alpha' a_2 s^3)Z = a_3 s D_1 U, \quad \text{(10)} \]

\[ (D D_1 - s^2)U = (a_1 + \alpha' s)D Z, \quad \text{(11)} \]

where the “bars” denote Laplace transforms of the corresponding functions throughout and “\( s \)” is the Laplace transform parameter. Operating by \( D D_1 \) and \( D_1 D \) respectively on these equations and using the remaining one, we finally obtain

\[ [L^2 - (m_1^2 + m_2^2)L + m_1 m_2^2]U = 0, \quad \text{(12)} \]

\[ [M^2 - (m_1^2 + m_2^2)M + m_1 m_2^2]Z = 0, \quad \text{(13)} \]

where \( L = DD_1, \quad M = D_1 D \) are two operators and \( m_1^2, m_2^2 \) are the roots of the quadratic in \( m^2 \) given by

\[ m^4 - [s^2 + (1 + \epsilon) a_2 s + \alpha' a_2 s^2 + \alpha'' a_2^2 s^3]m^2 + a_2 s^4(1 + \alpha'^* s) = 0, \]

along with \( \epsilon = a_1 a_3 / a_2 \) denoting the thermoelastic coupling constant. On setting \( \alpha' = \alpha'^* = 0 \) this equation agrees with Nariboli [15] in the conventional coupled case.

The solution of equations (12) and (13) are modified Bessel functions of order 3/2 [16] and taken in terms of exponential so that

\[ U = \sum_{i=1,2} F_i \exp(-m_i R) \left( \frac{1}{R} + \frac{1}{m_i R^2} \right), \quad \text{(14)} \]

\[ Z = \sum_{i=1,2} E_i \exp(-m_i R) \frac{1}{R}, \quad \text{(15)} \]

where \( E_i, F_i \) are constants \( (i = 1, 2) \).
Substituting equations (14) and (15) in equation (10), we obtain the relations

\[
E_1[m_1^2 - a_2 s(1 + \alpha^* s)] = -F_1 a_2 s m_1,
E_2[m_2^2 - a_2 s(1 + \alpha^* s)] = -F_2 a_2 s m_2.
\]  
(16)

The two types of boundary conditions in the transformed domain reduce to

(i) \[\sigma_R = 0, \quad Z = \frac{\chi}{s} \quad R = 1,\]

(ii) \[\sigma_R = \frac{\chi}{s}, \quad Z = 0 \quad R = 1.\]

The solutions in the transformed domain are found in cases (i) and (ii) respectively to be

Case (i)

\[
U(R, s) = \frac{\chi}{s} \frac{X_2 - (\lambda + 2\mu)(a_1 + \alpha's)Y_2}{X_1 Y_2 - Y_1 X_2} \exp(-m_1 R) \left(1 + \frac{1}{m_1 R^2}\right)
\]

\[
- \frac{\chi}{s} \frac{X_1 - (\lambda + 2\mu)(a_1 + \alpha's)Y_1}{X_1 Y_2 - Y_1 X_2} \exp(-m_2 R) \left(1 + \frac{1}{m_2 R^2}\right),
\]

\[
Z(R, s) = - \frac{\chi}{s} \frac{X_2 - (\lambda + 2\mu)(a_1 + \alpha's)Y_2}{X_1 Y_2 - Y_1 X_2} \frac{a_2 s m_1}{m_1^2 - a_2 s(1 + \alpha^* s)} \frac{\exp(-m_1 R)}{R}
\]

\[
+ \frac{\chi}{s} \frac{X_1 - (\lambda + 2\mu)(a_1 + \alpha's)Y_1}{X_1 Y_2 - Y_1 X_2} \frac{a_2 s m_2}{m_2^2 - a_2 s(1 + \alpha^* s)} \frac{\exp(-m_2 R)}{R},
\]

\[
\sigma_R(R, s) = \frac{\chi}{s} \exp(-m_1 R) \frac{X_2 - (\lambda + 2\mu)(a_1 + \alpha's)Y_2}{X_1 Y_2 - Y_1 X_2}
\]

\[
\times \left[ - \frac{\lambda + 2\mu m_1}{2\mu} - \frac{2}{R^2} \left(1 + \frac{1}{m_1 R}\right) - \frac{\alpha_2 s m_1 (a_1 + \alpha's)}{2\mu R} \frac{m_1^2 - a_2 s(1 + \alpha^* s)}{R} \right]
\]

\[
- \frac{\chi}{s} \exp(-m_2 R) \frac{X_1 - (\lambda + 2\mu)(a_1 + \alpha's)Y_1}{X_1 Y_2 - Y_1 X_2}
\]

\[
\times \left[ - \frac{\lambda + 2\mu m_2}{2\mu} - \frac{2}{R^2} \left(1 + \frac{1}{m_2 R}\right) - \frac{\alpha_2 s m_2 (a_1 + \alpha's)}{2\mu R} \frac{m_2^2 - a_2 s(1 + \alpha^* s)}{R} \right],
\]

\[
\sigma_0(R, s) = \frac{\chi}{s} \exp(-m_1 R) \frac{X_2 - (\lambda + 2\mu)(a_1 + \alpha's)Y_2}{X_1 Y_2 - Y_1 X_2}
\]

\[
\times \left[ - \frac{\lambda}{2\mu} - \frac{1}{R^2} \left(1 + \frac{1}{m_1 R}\right) + \frac{\alpha_2 s m_1 (a_1 + \alpha's)}{2\mu R} \frac{m_1^2 - a_2 s(1 + \alpha^* s)}{R} \right]
\]

\[
- \frac{\chi}{s} \exp(-m_2 R) \frac{X_1 - (\lambda + 2\mu)(a_1 + \alpha's)Y_1}{X_1 Y_2 - Y_1 X_2}
\]

\[
\times \left[ - \frac{\lambda}{2\mu} - \frac{1}{R^2} \left(1 + \frac{1}{m_2 R}\right) + \frac{\alpha_2 s m_2 (a_1 + \alpha's)}{2\mu R} \frac{m_2^2 - a_2 s(1 + \alpha^* s)}{R} \right].
\]
Case (ii)

\[
\begin{align*}
U(R,s) &= \frac{\chi_0 Y_2 \exp(-m_1 R)}{s(X_1 Y_2 - Y_1 X_2)} \left( \frac{1}{R} + \frac{1}{m_1 R^2} \right) - \frac{\chi_0 Y_1 \exp(-m_2 R)}{s(X_1 Y_2 - Y_1 X_2)} \left( \frac{1}{R} + \frac{1}{m_2 R^2} \right), \\
\tilde{Z}(R,s) &= -\frac{\chi_0 Y_2 \exp(-m_1 R)}{Rs(X_1 Y_2 - Y_1 X_2)} \left( \frac{a_3 s m_1}{m_1^2 - a_3 s(1 + a^* s)} + \frac{\chi_0 Y_1 \exp(-m_2 R)}{Rs(X_1 Y_2 - Y_1 X_2)} \frac{a_3 s m_2}{m_2^2 - a_3 s(1 + a^* s)} \right), \\
\tilde{\sigma}_r(R,s) &= \frac{\chi_0 Y_2 \exp(-m_1 R)}{s(X_1 Y_2 - Y_1 X_2)} \left[ -\frac{\lambda + 2\mu m_1}{2\mu R} - \frac{2}{R^2} \left( 1 + \frac{1}{m_1 R} \right) - \frac{a_3 s m_1}{m_1^2 - a_3 s(1 + a^* s)} \frac{1}{R} \right] \\
&\quad - \frac{\chi_0 Y_1 \exp(-m_2 R)}{s(X_1 Y_2 - Y_1 X_2)} \left[ -\frac{\lambda + 2\mu m_2}{2\mu R} - \frac{2}{R^2} \left( 1 + \frac{1}{m_2 R} \right) - \frac{a_3 s m_2}{m_2^2 - a_3 s(1 + a^* s)} \frac{1}{R} \right], \\
\tilde{\sigma}_\theta(R,s) &= \frac{\chi_0 Y_2 \exp(-m_1 R)}{s(X_1 Y_2 - Y_1 X_2)} \left[ -\frac{\lambda m_2}{2\mu R} + \frac{1}{R^2} \left( 1 + \frac{1}{m_1 R} \right) - \frac{a_3 s m_1}{m_1^2 - a_3 s(1 + a^* s)} \frac{1}{R} \right] \\
&\quad - \frac{\chi_0 Y_1 \exp(-m_2 R)}{s(X_1 Y_2 - Y_1 X_2)} \left[ -\frac{\lambda m_2}{2\mu R} + \frac{1}{R^2} \left( 1 + \frac{1}{m_2 R} \right) - \frac{a_3 s m_2}{m_2^2 - a_3 s(1 + a^* s)} \frac{1}{R} \right],
\end{align*}
\]

where

\[
\begin{align*}
\chi_0 &= 2\mu \chi_0, \\
X_1 &= \exp(-m_1) \left[ m_1(\lambda + 2\mu) + 4\mu \left( 1 + \frac{1}{m_1} \right) \right], \\
X_2 &= \exp(-m_2) \left[ m_2(\lambda + 2\mu) + 4\mu \left( 1 + \frac{1}{m_2} \right) \right], \\
Y_1 &= \frac{a_3 s m_1 \exp(-m_1)}{m_1^2 - a_3 s(1 + a^* s)}, \\
Y_2 &= \frac{a_3 s m_2 \exp(-m_2)}{m_2^2 - a_3 s(1 + a^* s)}.
\end{align*}
\]

4. SHORT-TIME APPROXIMATION

As the second sound effects are short-lived, we investigate the possible discontinuities at the wave fronts of the short-time approximations of the solutions. In the limit when \( s \to \infty \), we obtain for large \( s \),

\[
m_{1,2} = \frac{s}{v_{1,2}} + k_{1,2} + O \left( \frac{1}{s} \right),
\]

where

\[
\begin{align*}
\frac{2}{v_{1,2}^2} &= 1 + \alpha a_1 + \alpha^* a_2 \pm \sqrt{\Gamma}, \\
k_{1,2} &= \frac{a_2}{4} v_{1,2} \left[ 1 + \epsilon \pm \frac{\lambda(1 + \epsilon) - 2}{\sqrt{\Gamma}} \right], \\
\Gamma &= (1 + \alpha a_1 + \alpha^* a_2)^2 - 4\alpha_2 \alpha^* , \\
\lambda &= 1 + \alpha a_1 + \alpha^* a_2.
\end{align*}
\]
Also

\[ \Gamma = (1 + \alpha' a_3 + \alpha'^* a_2)^2 - 4a_2 \alpha'^* = (1 + \alpha' a_3 - \alpha'^* a_2)^2 + 4\alpha' \alpha'^* a_2 a_3 > 0. \]

Again

\[ (1 + \alpha' a_3 + \alpha'^* a_2)^2 > \Gamma \quad \text{so that } v_1 < v_2. \]

On expanding the transformed solutions \( U, Z, \sigma, \sigma_0 \) for large \( s \) in ascending powers of \( 1/s \), retaining terms up to \( 1/s^2 \) and then taking inversion, the final solutions, valid for short-times, are for Case (i):

\[
U(R, \eta) \approx \frac{\chi \exp(-k_2 R_1)}{PR} \left[ A_2 H\left( \eta - \frac{R_1}{v_1} \right) + \left( B_2 - \frac{A_2 Q}{P} - \frac{v_1 A_2}{R} \right) \left( \eta - \frac{R_1}{v_1} \right) H\left( \eta - \frac{R_1}{v_1} \right) \right] \\
- \frac{\chi \exp(-k_2 R_1)}{PR} \left[ A_1 H\left( \eta - \frac{R_1}{v_2} \right) + \left( B_1 - \frac{A_1 Q}{P} - \frac{v_2 A_1}{R} \right) \left( \eta - \frac{R_1}{v_2} \right) H\left( \eta - \frac{R_1}{v_2} \right) \right].
\]

\[
\sigma_R(R, \eta) \approx \frac{\chi \exp(-k_2 R_1)}{PR} \left[ \frac{\lambda + 2\mu}{2\mu} \left( G_2 - \frac{2A_2 v_2}{R^2} + \frac{\lambda + 2\mu}{2\mu} \left( a_2 G_1 + \alpha' \gamma_1 - B_2 + \frac{A_2 G}{P} \right) \right) \right] \\
\times H\left( \eta - \frac{R_1}{v_1} \right) + \left\{ \frac{2A_2}{R} + \frac{\lambda + 2\mu}{2\mu} \left( a_2 G_1 + \alpha' \gamma_1 - B_2 + \frac{A_2 G}{P} \right) \right\} \\
\times \left( \eta - \frac{R_1}{v_1} \right) H\left( \eta - \frac{R_1}{v_2} \right) \left[ \frac{\lambda + 2\mu}{2\mu} \left( G_2 - \frac{A_1}{v_2} \right) \right] \delta\left( \eta - \frac{R_1}{v_2} \right) \right] \\
\times \left( \frac{2A_1}{R} - \frac{\lambda + 2\mu}{2\mu} \left( a_1 G_2 + \alpha' N_2 - B_1 + \frac{A_1 Q}{P} \right) \right) H\left( \eta - \frac{R_1}{v_2} \right) + \left\{ \frac{2A_1}{R} - \frac{A_1 Q}{P} \right\} \\
- \frac{2A_1 v_2}{R^2} + \frac{\lambda + 2\mu}{2\mu} \left( a_1 G_2 + \alpha' N_2 - B_1 + \frac{A_1 Q}{P} \right) \right\} \left( \eta - \frac{R_1}{v_2} \right) H\left( \eta - \frac{R_1}{v_2} \right).
\]

\[
\sigma_0(R, \eta) \approx \frac{\chi \exp(-k_2 R_1)}{PR} \left[ \left\{ \frac{\lambda + 2\mu}{2\mu} \left( G_2 - \frac{2A_2 v_2}{R^2} + \frac{\lambda + 2\mu}{2\mu} \left( a_2 G_1 + \alpha' \gamma_1 - B_2 + \frac{A_2 G}{P} \right) \right) \right\} \right] \\
\times \left( \eta - \frac{R_1}{v_1} \right) H\left( \eta - \frac{R_1}{v_1} \right) + \left\{ \frac{A_2 v_2}{R^2} + \frac{B_2}{R} - \frac{A_2 Q}{PR} \right\} \left( \eta - \frac{R_1}{v_1} \right) H\left( \eta - \frac{R_1}{v_1} \right) \\
- \frac{\lambda + 2\mu}{2\mu} \left( a_2 G_1 + \alpha' N_1 \right) \right\} H\left( \eta - \frac{R_1}{v_1} \right) + \left\{ \frac{A_2 v_2}{R^2} + \frac{B_2}{R} - \frac{A_2 Q}{PR} \right\} \left( \eta - \frac{R_1}{v_1} \right) \right] \\
\times \left( B_2 - \frac{A_2 Q}{P} \right) + \left( \frac{\lambda + 2\mu}{2\mu} a_1 N_2 \right) \right\} \left( \eta - \frac{R_1}{v_2} \right) H\left( \eta - \frac{R_1}{v_2} \right) \\
- \frac{\lambda + 2\mu}{2\mu} \left( a_2 G_2 + \alpha' N_2 \right) \right\} H\left( \eta - \frac{R_1}{v_2} \right) + \left\{ \frac{A_1 v_2}{R^2} + \frac{B_1}{R} - \frac{A_1 Q}{PR} \right\} \left( \eta - \frac{R_1}{v_2} \right) \right],
for Case (ii):

\[ U(R, \eta) \approx \frac{\chi \exp(-k \cdot R)}{PR} \frac{a_3}{v_2 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \eta - \frac{R}{v_1} \right) H \left( \eta - \frac{R}{v_1} \right) \]

\[ - \frac{\chi \exp(-k \cdot R)}{PR} \frac{a_3}{v_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \eta - \frac{R}{v_2} \right) \times H \left( \eta - \frac{R}{v_2} \right), \]

\[ Z(R, \eta) \approx - \frac{\chi \exp(k \cdot R)}{PR} \frac{a_3}{v_1 \nu_2 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \eta - \frac{R}{v_1} \right) H \left( \eta - \frac{R}{v_1} \right) \]

\[ + \frac{\chi \exp(-k \cdot R)}{PR} \frac{a_3}{v_1 \nu_2 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \eta - \frac{R}{v_2} \right) \times H \left( \eta - \frac{R}{v_2} \right), \]

\[ \sigma_n(R, \eta) \approx \frac{\chi \exp(-k \cdot R)}{PR} \left[ \left( \frac{\lambda + \mu}{1 - \nu_2^2} \right) a_3 \left( \frac{1}{\nu_2^3} - \alpha^* \right) \right] \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right) \]

\[ + \left\{ \frac{2a_3}{\nu_2^3 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} + \frac{\lambda + \mu}{1 - \nu_2^2} \frac{a_3}{v_2 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \right\} \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right) \]

\[ - \frac{Q}{P} \frac{a_3 \nu_2}{\nu_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} + \frac{a_3 \nu_2}{\nu_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \frac{1}{\nu_2^3} - \alpha^* \right) \left( \frac{1 - \nu_2^2}{\nu_2^3} \right) \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right) \]

\[ \times \frac{a_3 \nu_2}{v_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \frac{1 - \nu_2^2}{\nu_2^3} \right) \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right), \]

\[ + \left\{ \frac{2a_3}{\nu_2^3 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} + \frac{\lambda + \mu}{1 - \nu_2^2} \frac{a_3}{v_2 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \right\} \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right) \]

\[ - \frac{Q}{P} \frac{a_3 \nu_2}{\nu_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} + \frac{a_3 \nu_2}{\nu_1 \left( \frac{1}{\nu_2^3} - \alpha^* \right)} \left( \frac{1}{\nu_2^3} - \alpha^* \right) \left( \frac{1 - \nu_2^2}{\nu_2^3} \right) \left( \eta - \frac{R}{v_1} \right) \times H \left( \eta - \frac{R}{v_1} \right) \]
\[ \begin{align*}
\sigma_s(R, \eta) & \simeq \mathcal{C} \exp(-k_1 R_1) \left[ \frac{a_1}{v_1} \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right) \right] \left[ \frac{\left( \lambda + 2\mu \right) a_1 \alpha^{'}}{2\mu v_1 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} - \frac{\lambda}{2\mu} \right] H\left( \eta - \frac{R_1}{v_1} \right) \\
& + \left\{ \frac{a_3}{\sqrt{v_2^2 - a_2 \alpha^{'}}} \left[ \frac{1}{R_2^2} - \frac{\lambda}{2\mu} \frac{a_2 v_1 (1 - \alpha^{*} k_1 v_1) - k_1}{(1 - a_2 \alpha^{*} v_1^2)} \right] - \frac{Q}{P} \frac{\lambda}{2\mu v_2} \right\} \\
& + \left\{ \frac{a_1 a_3 (\lambda + 2\mu)}{2\mu v_2 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} + \frac{a_3 (\lambda + 2\mu) \alpha^{'}}{2\mu v_2 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} \right\} \\
& - \frac{Q}{P} \frac{(\lambda + 2\mu) a_1 a_3 \alpha^{'}}{2\mu v_1 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} \left\{ \frac{a_2 v_2 (1 - \alpha^{*} k_2 v_2) - k_2}{1 - a_2 \alpha^{*} v_2^2} \right\} \left\{ \frac{a_3}{\sqrt{v_2^2 - a_2 \alpha^{'}}} \left[ \frac{1}{R_2^2} - \frac{\lambda}{2\mu} \frac{a_2 v_1 (1 - \alpha^{*} k_1 v_1) - k_1}{(1 - a_2 \alpha^{*} v_1^2)} \right] - \frac{Q}{P} \frac{\lambda}{2\mu v_2} \right\} \\
& \times \left( \eta - \frac{R_1}{v_1} \right) H\left( \eta - \frac{R_1}{v_1} \right) - \mathcal{C} \exp(-k_2 R_2) \left[ \frac{a_1}{v_1} \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right) \right] \left[ \frac{\left( \lambda + 2\mu \right) a_1 \alpha^{'}}{2\mu v_2 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} - \frac{\lambda}{2\mu} \right] H\left( \eta - \frac{R_2}{v_2} \right) \\
& + \left\{ \frac{a_3}{\sqrt{v_2^2 - a_2 \alpha^{'}}} \left[ \frac{1}{R_2^2} - \frac{\lambda}{2\mu} \frac{a_2 v_1 (1 - \alpha^{*} k_1 v_1) - k_1}{(1 - a_2 \alpha^{*} v_1^2)} \right] - \frac{Q}{P} \frac{\lambda}{2\mu v_2} \right\} \\
& + \left\{ \frac{a_1 a_3 (\lambda + 2\mu)}{2\mu v_2 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} + \frac{a_3 (\lambda + 2\mu) \alpha^{'}}{2\mu v_2 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} \right\} \\
& - \frac{Q}{P} \frac{(\lambda + 2\mu) a_1 a_3 \alpha^{'}}{2\mu v_1 \left( \frac{1}{v_1^2} - a_2 \alpha^{*} \right)} \left\{ \frac{a_2 v_2 (1 - \alpha^{*} k_2 v_2) - k_2}{1 - a_2 \alpha^{*} v_2^2} \right\} \left\{ \frac{a_3}{\sqrt{v_2^2 - a_2 \alpha^{'}}} \left[ \frac{1}{R_2^2} - \frac{\lambda}{2\mu} \frac{a_2 v_1 (1 - \alpha^{*} k_1 v_1) - k_1}{(1 - a_2 \alpha^{*} v_1^2)} \right] - \frac{Q}{P} \frac{\lambda}{2\mu v_2} \right\} \end{align*} \]

where

\[ R_1 - 1, \]

\[ P = \frac{\lambda + 2\mu}{v_1} a_1 v_2 \frac{a_2 v_2 (1 - \alpha^{*} k_2 v_2) - k_2}{(1 - a_2 \alpha^{*} v_2^2)^2} + \left\{ k_1 (\lambda + 2\mu) + 4\mu + \frac{a_3}{1 - a_2 \alpha^{*} v_2^2} \right\} \]

\[ Q = \frac{\lambda + 2\mu}{v_1} a_1 v_2 \frac{a_2 v_2 (1 - \alpha^{*} k_2 v_2) - k_2}{(1 - a_2 \alpha^{*} v_2^2)^2} + \left\{ k_1 (\lambda + 2\mu) + 4\mu \right\} \frac{a_3}{1 - a_2 \alpha^{*} v_2^2}, \]
A_{1,2} = \frac{\lambda + 2\mu}{v_{1,2}} - \frac{\alpha'(\lambda + 2\mu)a_3 v_{1,2}}{1 - a_2 \alpha^* v_{1,2}^3},

B_{1,2} = k_{1,2}(\lambda + 2\mu) + 4\mu - \frac{a_1 (a + 2\mu) a_3 v_{1,2}}{1 - a_2 \alpha^* v_{1,2}^3} \frac{a_2 v_{1,2} (1 - \alpha^* k_{1,2} v_{1,2})}{(1 - a_2 \alpha^* v_{1,2}^3)^2} - k_{1,2},

C_{1,2} = 4\mu w_{1,2} - (\lambda + 2\mu) a_1 a_3 v_{1,2}^2 \frac{a_2 v_{1,2} (1 - \alpha^* v_{1,2}^3 k_{1,2})}{(1 - a_2 \alpha^* v_{1,2}^3)^2},

G_{1,2} = \frac{A_{1,2} a_3 v_{1,2}}{1 - a_2 \alpha^* v_{1,2}^3},

N_{1,2} = \left( B_{1,2} - \frac{A_{1,2} Q}{P} \right) \frac{a_1 v_{1,2}}{1 - a_2 \alpha^* v_{1,2}^3}.

5. CONVENTIONAL COUPLED CASE

In absence of relaxation parameters, we have

\[ m^4 - [s^2 + (1 + \epsilon)a_2 s] m^2 + a_2 s^3 = 0, \]

which agrees with Ref. [15].

For large \( s \),

\[ m_1 \approx s + \frac{a_2 \epsilon}{2}, \quad m_2 \approx \sqrt{a_2 s} + \frac{a_2^{3/2} \epsilon}{2 \sqrt{s}}. \]

In this case the short-time solutions for \( U, Z, \sigma_{r}, \sigma_{\theta} \) are found in a similar way to be, for Case (i):

\[ U(R, \eta) \approx \frac{\chi \exp\left(-\frac{a_2 \epsilon R_i}{2}\right)}{P'R} \left[ a H(\eta - R_i) + 2b \sqrt{\frac{\eta - R}{\pi}} H(\eta - R_i) + C(\eta - R_i) H(\eta - R_i) \right] \]

\[ - \frac{\chi}{P'R} \left[ \frac{a'}{\sqrt{\pi} \eta} \right] + b' \text{erf}\left(\frac{\sqrt{a_2 R_i}}{2\sqrt{\eta}}\right) + c' \left\{ 2 \sqrt{\eta} \exp\left[-\frac{(a_2 R_i^2)}{4\eta}\right] \right\}, \]

\[ Z(R, \eta) \approx \frac{\chi \exp\left(-\frac{a_2 \epsilon R_i}{2}\right)}{P'R} \left[ L_0 H(\eta - R_i) + 2L_0 \sqrt{\frac{\eta - R_i}{\pi}} H(\eta - R_i) - N_0(\eta - R_i) H(\eta - R_i) \right] \]

\[ + \frac{\chi}{P'R} \left[ L_0 \text{erfc}\left(\frac{\sqrt{a_2 R_i}}{2\sqrt{\eta}}\right) + M_0 \left\{ 2 \sqrt{\eta} \exp\left[-\frac{(a_2 R_i^2)}{4\eta}\right] \right\} \right] \]

\[ - R_i \sqrt{a_2} \text{erfc}\left(\frac{\sqrt{a_2 R_i}}{2\sqrt{\eta}}\right) + N_0' 4\eta^2 \text{erfc}\left(\frac{\sqrt{a_2 R_i}}{2\sqrt{\eta}}\right) \right\}. \]
\[
\sigma_\eta(R, \eta) \approx \frac{\chi \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ \frac{H(\eta - R_i) + 2m}{\sqrt{\frac{\eta - R_i}{\pi}}} \frac{H(\eta - R_i) + n(\eta - R_i)H(\eta - R_i)}{P'R} \right] \\
- \frac{\chi}{P'R} \left[ 1' \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) + m' \left\{ 2 \sqrt{\frac{\eta}{\pi}} \exp \left[ -\left(\frac{a_2 R_i^2}{4\eta}\right) \right] - R_1 \sqrt{a_2} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) \right\} \\
+ n' 4\eta^{-1} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) \right] ,
\]

\[
\sigma_\eta(R, \eta) \approx \frac{\chi \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ L_1 H(\eta - R_i) + 2M_1 \sqrt{\frac{\eta - R_i}{\pi}} H(\eta - R_i) + N_1(\eta - R_i)H(\eta - R_i) \right] \\
- \frac{\chi}{P'R} \left[ L_1' \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) + M_1' \left\{ 2 \sqrt{\frac{\eta}{\pi}} \exp \left[ -\left(\frac{a_2 R_i^2}{4\eta}\right) \right] \right\} \\
- R_1 \sqrt{a_2} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) \right] + N_1' 4\eta^{-1} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) ;
\]

for Case (ii):

\[
U(R, \eta) \approx \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} [a_1(\eta - R_i)H(\eta - R_i)] - \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ a_2 4\eta^{-1} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) \right] ,
\]

\[
Z(R, \eta) \approx \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} [1_1(\eta - R_i)H(\eta - R_i)] - \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ 1_1 4\eta^{-1} \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) \right] ,
\]

\[
\sigma_\eta(R, \eta) \approx \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ A_1 H(\eta - R_i) + 2B_1 \sqrt{\frac{\eta - R_i}{\pi}} H(\eta - R_i) + C_1(\eta - R_i)H(\eta - R_i) \right] \\
- \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ A_1' \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) + B_1' \left\{ 2 \sqrt{\frac{\eta}{\pi}} \exp \left[ -\left(\frac{a_2 R_i^2}{4\eta}\right) \right] - R_1 \sqrt{a_2} \text{erfc} \left( \frac{R_1 \sqrt{a_2}}{2\sqrt{\eta}} \right) \right\} \\
+ C_1' 4\eta^{-1} \text{erfc} \left( \frac{R_1 \sqrt{a_2}}{2\sqrt{\eta}} \right) \right] ,
\]

\[
\sigma_\eta(R, \eta) \approx \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ I_0 H(\eta - R_i) + 2m_0 \sqrt{\frac{\eta - R_i}{\pi}} H(\eta - R_i) + \eta_0(\eta - R_i)H(\eta - R_i) \right] \\
- \frac{\chi' \exp \left( -\frac{a_2 \epsilon R_i}{2} \right)}{P'R} \left[ a_0 \text{erfc} \left( \frac{\sqrt{a_2} R_i}{2\sqrt{\eta}} \right) + b_0 \left\{ 2 \sqrt{\frac{\eta}{\pi}} \exp \left[ -\left(\frac{a_2 R_i^2}{4\eta}\right) \right] - R_1 \sqrt{a_2} \text{erfc} \left( \frac{R_1 \sqrt{a_2}}{2\sqrt{\eta}} \right) \right\} \\
+ c_0 4\eta^{-1} \text{erfc} \left( \frac{R_1 \sqrt{a_2}}{2\sqrt{\eta}} \right) \right] ,
\]

where \( P', a, \ldots, c, A_1, \ldots \), etc. are constants and

\[
i^n \text{erfc}(x) = \int_{-\infty}^{x} i^{n-1} \text{erfc}(x) \, dx
\]
stands for the associated error function of \( n \)th degree and
\[
i \text{erfc}(x) = \int_x^\infty \text{erfc}(x) \, dx.
\]

6. DISCUSSION

From the short-time solutions we observe that there exist two types of waves propagating with velocities \( v_1 \) and \( v_2 \) of which \( v_1 < v_2 \). The contribution of the elastic wave in the vicinity of the wave front \( R_t = \eta v_1 \) is given by the terms containing \( H[\eta - (R_t/v_1)] \), whereas the contribution of the thermal wave in the vicinity of the wave front \( R_t = \eta v_2 \) is given by the terms containing \( H[\eta - (R_t/v_2)] \).

In Case (i), we observe that the deformation and temperature suffer finite discontinuities at both the wave fronts whereas the stresses suffer delta-function singularities at these locations. The magnitude of the jumps in deformation and temperature at these wave fronts \( R_t = \eta v_{1,2} \) are given by
\[
[U^+ - U^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi}{PR} \frac{\exp(-R_t k_{1,2})}{A_{2,1}},
\]
\[
[Z^+ - Z^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi}{PR} \frac{\exp(-R_t k_{1,2})}{G_{1,2}}.
\]

The above jumps decay exponentially with distance. In Case (ii), we observe that deformation and temperature are continuous at both the wave fronts and stresses suffer finite discontinuities at the respective wave fronts. The magnitudes of the jumps in stresses at the wave fronts \( R_t = \eta v_{1,2} \) are given by
\[
[\sigma_n^+ - \sigma_n^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi_0}{PR} \frac{(\lambda + 2\mu)\alpha_1}{2\mu_1 v_2} 
\frac{1}{v_1,2} \left( \frac{1}{v_1,2} - a_2 \alpha_2 \right) \left( \frac{1}{v_1,2} - a_2 \alpha_2 \right)
\]
\[
[\sigma_\theta^+ - \sigma_\theta^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi_0}{PR} \frac{a_3}{v_1,2 v_2} \left( \frac{1}{v_2,1} - a_2 \alpha_2 \right)
\times \left[ \frac{(\lambda + 2\mu)\alpha_1}{2\mu_1} - \frac{\lambda}{2\mu} \right].
\]

These jumps are also seen to decay exponentially with distance.

In case of conventional coupled theory, we observe that the solutions consist of a wave part (\( E \)-wave) travelling with unit velocity and of a diffusive part. This is evident since the heat equation is the conventional coupled diffusion equation. The contribution in the vicinity of the elastic wave front (\( E \)-wave) \( R_t = \eta \) is manifested by the terms containing \( H(\eta - R_t) \). In Case (i), we observe that the deformation, temperature and stresses suffer finite jumps at the elastic wave front. The diffusive part is continuous at \( R_t = \eta \). The jumps at the elastic wave front are given by
\[
[U^+ - U^-]_{R_t = \eta} = -\frac{\chi a}{PR} \exp \left( \frac{R_t a_2 \epsilon}{2} \right),
\]
\[
[Z^+ - Z^-]_{R_t = \eta} = \frac{\chi L_0}{PR} \exp \left( \frac{R_t a_2 \epsilon}{2} \right).
\]

In Case (ii), we observe that the deformation and temperature are continuous at both the wave fronts and stresses suffer finite discontinuities at the respective wave fronts. The magnitudes of the jumps in stresses at the wave fronts \( R_t = \eta v_{1,2} \) are given by
\[
[\sigma_n^+ - \sigma_n^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi_0}{PR} \frac{(\lambda + 2\mu)\alpha_1}{2\mu_1} \left( \frac{1}{v_1,2} - a_2 \alpha_2 \right) \left( \frac{1}{v_1,2} - a_2 \alpha_2 \right)
\]
\[
[\sigma_\theta^+ - \sigma_\theta^-]_{R_t = \eta v_{1,2}} = \pm \frac{\chi_0}{PR} \frac{a_3}{v_1,2 v_2} \left( \frac{1}{v_2,1} - a_2 \alpha_2 \right)
\times \left[ \frac{(\lambda + 2\mu)\alpha_1}{2\mu_1} - \frac{\lambda}{2\mu} \right].
\]

These jumps are also seen to decay exponentially with distance.
The last results agree with equation (12) as reported in Ref. [15] apart from notation, indicating a finite jump of the hoop stress across the elastic wave front. As reported in Ref. [15], for aluminium, $a_2 \approx 10^7$, $\varepsilon = 0.0269$, so $a_2\varepsilon$ is of the order of $10^5$. In the classical coupled theory except at points very close to the cavity, the jumps are negligible. The jump is finite for small times near the cavity surface, whereas the hoop stress as well as the radial stress suffer delta-function singularity at this location for small times in the temperature-rate dependent theory.

In Case (ii), deformation and temperature are continuous at the elastic wave front, while stresses suffer finite discontinuities at this location. The jumps in the stresses at the elastic wave front are given by

$$[\sigma^+_{rr} - \sigma^-_{rr}]_{R_i = \eta} = -\frac{\chi L_{r} \exp \left( -\frac{R_i a_2 \varepsilon}{2} \right)}{P'R},$$

$$[\sigma^+_{\theta\theta} - \sigma^-_{\theta\theta}]_{R_i = \eta} = -\frac{\chi L_{\theta} \exp \left( -\frac{R_i a_2 \varepsilon}{2} \right)}{P'R},$$

showing the jumps decay exponentially with distance.

Thus delta-function singularities in stresses of the temperature-rate dependent theory disappear in Case (i) of the conventional coupled case. Also the jumps at the thermal wave front disappear in the conventional coupled case. The finite discontinuities at the thermal wave front and the delta-function singularities in stresses at both the wave fronts that appear in the present theory are due to the presence of temperature-rate among the constitutive variables and are not encountered in the conventional coupled theory. This is expected since the heat equation in the conventional coupled case does not preserve the wave like character and the constitutive equations in the conventional theory are independent of the temperature-rate. So the temperature-rate dependent theory put forth by Green and Lindsay displays some discontinuities, not occurring in the conventional coupled case.

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