Discrete Applied Mathematics 28 (1990) 265-269
North-Holland

THE DOUBLE COMPETITION NUMBER OF SOME TRIANGLE-FREE GRAPHS

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Received 7 September 1988
Revised 15 June 1989

The competition graph of a digraph was introduced by Joel Cohen in 1968 in the study of ecological niches. It was generalized by Debra Scott in 1985 to the competition-common enemy graph. In this paper, we study some triangle-free competition-common enemy graphs.

1. Introduction

In 1968 Cohen [1] introduced the competition graph of the digraph corresponding to a food web in his study of ecological niches. In mathematical terms a *food web* is an acyclic digraph $D=(V,A)$. For any arc $xy\in A$, x is a *predator* of y and y is a *prey* of x. The *competition graph* of D is the graph $G=(V,E)$ where $xy\in E$ if and only if $x\neq y$ and for some $z\in V$, $xz, yz\in A$ (i.e., x and y have a common prey in D). In 1978 Roberts [3] defined the *competition number* of a graph G, $k(G)$, to be the smallest integer such that G together with k isolated nodes, $G\cup I_k$, is the competition graph of some acyclic digraph, and showed that it was well defined. In 1985 Scott [4] defined the *competition-common enemy graph* of a digraph $D=(V,A)$ to be the graph $G=(V,E)$ where $xy\in E$ if and only if $x\neq y$ and for some $z, w\in V$, $xz, yz, wx, wy\in A$ (so x and y have both a common prey and a common predator). Then the *double competition* number of a graph G, $\mathrm{dk}(G)$, is the smallest integer such that $G\cup I_{\mathrm{dk}}$ is the competition-common enemy graph of some acyclic digraph. Scott showed that the double competition number was well defined; surprisingly all of her examples had $\mathrm{dk}=2$. In 1987 Jones et al. [2] found a class of graphs having arbitrarily large double competition numbers; however they were only able to find one triangle-free graph having $\mathrm{dk}>2$. Here we find an infinite class of triangle-free graphs with $\mathrm{dk}>2$.

2. Preliminaries

Scott [4] showed that the double competition number of a graph with no isolated nodes is at least 2, since the nodes at the top and bottom of the food web will become isolated nodes in the competition-common enemy graph. She also found an upper bound in terms of the competition number:

Theorem 1. *For every graph* G, $\mathrm{dk}(G) \le k(G)+1$.

Roberts [3] had proved the following result on the competition number of triangle-free graphs:

Theorem 2. *If* $G=(V,E)$ *is connected and triangle-free with* $|V|>1$, *then* $k(G)=|E|-|V|+2$.

So it has been of interest to find a class of triangle-free graphs of large double competition number. Jones et al. [2] proved the following for the complete tripartite graphs:

Theorem 3. $\mathrm{dk}(K_{n,n,n}) \ge 2\sqrt{n}$.

These graphs are not triangle-free, but we will generalize them to get a class of triangle-free graphs.

3. The triangle-free graphs

Let $C(5,n)$ be the graph of $5n$ nodes and $5n^2$ edges which consists of 5 columns of n nodes each (labelled (a)–(e)) with the columns arranged in a 5-cycle (a)–(b)–(c)–(d)–(e) such that any two nodes in adjacent columns are adjacent (so $C(5,n)$ can be considered as a product of C_5 and I_n). Note that $K_{n,n,n}$ would be $C(3,n)$ in this notation. It is easy to construct a digraph to show that $\mathrm{dk}(C(5,1))=2$, and Jones et al. [2] showed that $\mathrm{dk}(C(5,2))=3$ by an exhaustive computer search. We now deal with the rest of this class.

Theorem 4. $\mathrm{dk}(C(5,n))>2$ *for all* $n>3$.

Proof. Suppose $\mathrm{dk}(C(5,n))=2$. Then there exists an acyclic digraph D having $C(5,n)\cup I_2$ as its competition-common enemy graph. Since D is acyclic there exists a numbering of the nodes of D (and hence of $C(5,n)\cup I_2$) with the numbers $0,1,2,\dots,5n+1$ such that each arc ij in D has $i<j$; i.e., a predator is always numbered less than its prey. Moreover, since $C(5,n)$ does not contain any isolated nodes, one of the nodes of I_2 must be numbered 0 and the other numbered $5n+1$. Then the structure of $C(5,n)$ imposes the following conditions on D:

(I) any two nodes in adjacent columns must have both a common predator and a common prey,

(II) any two nodes in the same column or in nonadjacent columns either do not have a common predator or do not have a common prey.

Since $C(5,n)$ is symmetric, we may assume without loss of generality that node

1 is in (a). Then 1 must have a common predator with each node of adjacent columns (b) and (e); but the only possible predator for 1 is 0. Thus 1 and all nodes of (b) and (e) have 0 as predator.

Now consider node $5n$. The only possible prey for $5n$ is $5n+1$, so $5n$ and all of its neighbours have $5n+1$ as prey. If $5n$ were in (a), (c) or (d) (and thus adjacent to (b) or (e)), then all of the nodes of (b) or (e) would have 0 as predator and $5n+1$ as prey, a contradiction to (II). Thus $5n$ must be in (b) or (e). By symmetry we may assume that $5n$ is in (e), and thus $5n$ and all nodes of (a) and (d) have $5n+1$ as prey. Let k be the first node (other than 0) that is *not* in (a), and let m be the last node (other than $5n+1$) that is not in (e); then $1<k<n+2$ and $4n-1<m<5n$. Now $1,2,\dots,k-1$ are all in (a) and they all have $5n+1$ as prey, so no two of them can have a common predator. But each of them must have at least one predator from $0,1,\dots,k-2$. Thus for each $i=1,2,\dots,k-1$, node i has $i-1$ as predator, and each node of (b) and (e) has $i-1$ as a common predator with i. It follows that each of $m+1,m+2,\dots,5n$ in (e) has $0,1,\dots,k-2$ as predators. In particular since $m+1$ has $0,1,\dots,k-2$ as predators, $0,1,\dots,k-2$ all have $m+1$ as prey.

Similarly, for each $j=m+1,m+2,\dots,5n$ node j has $j+1$ as prey, each node of (a) and (d) has $j+1$ as a common prey with j, each of $1,2,\dots,k-1$ has $m+2$, $m+3,\dots,5n,5n+1$ as prey, and $m+2,m+3,\dots,5n,5n+1$ all have $k-1$ as predator. To summarize:

$$i \text{ has } i-1 \text{ as predator for } i=1,2,\dots,k-1, \tag{1}$$

$$1,2,\dots,k-2 \text{ each has } m+1,m+2,\dots,5n,5n+1 \text{ as prey}, \tag{2}$$

$$k-1 \text{ has } m+2,m+3,\dots,5n,5n+1 \text{ as prey}, \tag{3}$$

$$m+1 \text{ has } 0,1,\dots,k-2 \text{ as predators}, \tag{4}$$

$$m+2,m+3,\dots,5n \text{ each has } 0,1,\dots,k-1 \text{ as predators}, \tag{5}$$

$$j \text{ has } j+1 \text{ as prey for } j=m+1,m+2,\dots,5n. \tag{6}$$

Now consider the location of k, in (b), (c), (d) or (e). Suppose first that k is in (c) or (e), and thus is adjacent to (d). Since each node of (d) and (a) has $5n+1$ as prey, and these two columns are not adjacent, (II) ensures that the predators of each of these $2n$ nodes must be distinct. By (1), $0,1,\dots,k-2$ all occur as predators in (a), so the nodes of (d) cannot have any node less than $k-1$ as predator. But by (I) each of the n nodes of (d) must have a common predator with k. Since the only possible common predator is $k-1$, and we need n of them, we get a contradiction.

Next suppose k is in (d). As above, since k is in (d) the only predator k can have is $k-1$. But then every node of (c) and (e) must have $k-1$ as a common predator with k. So consider the nodes of (c). Since they all have $k-1$ as predator, they must all have distinct prey (by (II)). But each must have a common prey with each node of (b). Since the nodes of (b) all have 0 as predator they must all have distinct prey. Thus each of the n nodes of (c) must have n distinct common prey with (b), which means (c) needs at least n^2 prey. Since the least possible numbering of a node occurring in (c) is $k+1$, these n^2 prey must be chosen from $k+2, \ldots, 5n+1$; i.e., from $5n-k$ possibilities. Thus $n^2 \leq 5n-k \leq 5n-2$. Moreover, each node of (e) also has $k-1$ as predator and must have at least one prey distinct from those of the rest of (e). Since (c) and (e) are not adjacent, the n^2 prey of (c) cannot include the (at least) n prey of (e), so $n^2 \leq 4n-2$, a contradiction since $n>3$.

Thus k is in (b). Since each node of (a) has $5n+1$ as prey, k must have a distinct predator in common with each node of (a), and so k must have at least n predators. Thus $k=n$ or $k=n+1$. By symmetry (reversing the roles of predators and prey) m is in (d) and $m=4n$ or $m=4n+1$.

So consider (c). Each node of (c) has (at least) one predator in common with k and (at least) one prey in common with m. Identify with each node of (c) a pair (x, y), where x is a common predator of the node with k and y is a common prey with m, so $0 \leq x \leq k-1$ and $m+1 \leq y \leq 5n+1$. Since (c) is not adjacent to (a) nor to (e), (x, y) cannot occur as a predator/prey pair at any node of (a) or (e). By (1) and (2), $(x, y) \neq (i, j)$ for $i=0, 1, \ldots, k-3$ and $j=m+1, m+2, \ldots, 5n+1$. By (5) and (6), $(x, y) \neq (i, j)$ for $i=0, 1, \ldots, k-1$ and $j=m+3, m+4, \ldots, 5n+1$. By (1) and (3), $(x, y) \neq (k-2, m+2)$. Thus the only possibilities for (x, y) are $(k-1, m+1)$, $(k-1, m+2)$ and $(k-2, m+1)$. But we need n pairs, with $n>3$, a contradiction.

Thus $\mathrm{dk}(C(5, n))>2$. □

An exhaustive computer search has shown that $\mathrm{dk}(C(5,3))=3$, which completes the class. Moreover a minor modification of the above proof to allow one additional isolated node can be used to show the following theorem:

Theorem 5. $\mathrm{dk}(C(5, n))>3$ *for* $n>9$.

Unfortunately this method of proof breaks down for $\mathrm{dk}>4$. However we conjecture that the double competition numbers of this class are arbitrarily large. We have also studied $C(m, n)$ for values of m other than 5, without finding any large double competition numbers. For all even m, we have found digraphs which give $\mathrm{dk}(C(m, n))=2$ for all n (and in fact $\mathrm{dk}=2$ for every bipartite graph we have looked at). For all odd m greater than 5, we have found digraphs which give $\mathrm{dk}(C(m, 2))=2$. However our study has led to the following conjecture:

Conjecture. $\mathrm{dk}(C(m, n)) \rightarrow \infty$ *as* $n \rightarrow \infty$ *for all odd* $m \geq 3$.

Acknowledgement

I would like to thank Dr. Janice Jeffs for her helpful comments on this proof, and Dr. Kathryn Jones for her encouragement in writing it up.

References

[1] J.E. Cohen, Interval graphs and food webs: A finding and a problem, RAND Corporation Document 17696-PR, Santa Monica, CA (1968).

[2] K.F. Jones, J. Richard Lundgren, F.S. Roberts and S. Seager, Some remarks on the double competition number of a graph, Congr. Numer. 60 (1987) 17–24.

[3] F.S. Roberts, Food webs, competition graphs, and the boxicity of ecological phase space, in: Y. Alavi and D. Lick, eds., Theory and Applications of Graphs (Springer, New York, 1978) 477–490.

[4] D.D. Scott, The competition-common enemy graph of a digraph, Discrete Appl. Math. 17 (1987) 269–280.