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Face Size and the Maximum Genus of a Graph 1. Simple Graphs[1]

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This paper shows that a simple graph which can be cellularly embedded on some closed surface in such a way that the size of each face does not exceed 7 is upper embeddable. This settles one of two conjectures posed by Nedela and Škoviera (1990, *in* "Topics in Combinatorics and Graph Theory," pp. 519–529, Physica Verlag, Heidelberg). The other conjecture will be proved in a sequel to this paper.
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1. INTRODUCTION

All graphs considered in this paper are finite and undirected and, unless explicitly stated otherwise, they are also connected. In general, we allow graphs to have loops and multiple edges. Graphs which lack both loops and multiple edges will be called *simple*.

By a *surface* S we mean a compact connected 2-dimensional manifold without boundary (that is, a *closed surface*). We consider both orientable and nonorientable surfaces. It s well known that each orientable surface is homeomorphic to a sphere with h handles while every nonorientable one is homeomorphic to a sphere with k crosscaps. This number h or k is called the *genus*, denotes by $g(S)$, of the surface S when S is orientable or nonorientable, respectively. A *cellular embedding* $j: G \to S$ of a graph G on a

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surface S is a homeomorphism of G to a subspace of S such that each component of $S-j(G)$ is homeomorphic to an open disk; we usually identify $j(G)$ with G.

Each of the components of $S-G$ is called a *face* of G on S. The boundary of each face is a closed walk in G. The *size* of a face is the number of edges appearing on this walk, repeated edges being counted twice.

Every cellular embedding satisfies *Euler's formula*. Let G be a connected graph with p vertices and q edges embedded on a surface S with r faces. Then

$$p-q+r=\begin{cases} 2-2g(S), & \text{if } S \text{ is orientable,} \\ 2-g(S), & \text{if } S \text{ is nonorientable.} \end{cases}$$

Note that a disconnected graph does not admit a cellular embedding on any surface.

The *maximum genus* $\gamma_M(G)$ of a connected graph G is the largest integer k with the property that there exists a cellular embedding of G on the orientable surface S of genus k. Since any cellular embedding must have at least one face, Euler's formula implies that

$$\gamma_M(G) \leqslant \lfloor \beta(G)/2 \rfloor,$$

where $\beta(G)=q-p+1$ is known as the *Betti number* (or *cycle rank*) of G. A graph G is said to be *upper embeddable* if $\gamma_M(G)=\lfloor \beta(G)/2 \rfloor$. The quantity $\beta(G)-2\gamma_M(G)$ is called the *deficiency* of G (or *Betti deficiency* of G) and is denoted by $\xi(G)$. The importance of $\xi(G)$ is in that the maximum genus of a graph is usually determined by calculating ξ. In particular, a graph G is upper embeddable if and only if $\xi(G) \leqslant 1$ where $\xi(G)=0$ or 1 depending on whether $\beta(G)$ is even or odd, respectively. A graph G whose deficiency is 2 or larger will be called a *deficient* graph; in other words, a graph is deficient if and only if it is connected and not upper embeddable.

Since the introductory article on the maximum genus of graphs by Nordhaus *et al.* [8] in 1971, the maximum genus of graphs has received a considerable attention. Many authors (see, e.g., [6, 9–15, 17]) were dealing particularly with upper embeddability of graphs. For example, Xuong [17] and Jungerman [6] proved that every 4-edge-connected graph is upper embeddable, although a little weaker condition of 3-edge-connectivity is not enough for a graph to be upper embeddable (see [2] and [15] for more details). In 1990, Nedela ans Škoviera [7] proved that a loopless graph which has a cellular embedding on a closed surface such that the size of each face does not exceed 4 is upper embeddable. Moreover, they made two conjectures. First, they conjectured that every loopless graph admitting a cellular embedding on a closed surface with maximum face size at most

5 is upper embeddable. Second, that restricting to simple graphs the conclusion remains true when the condition is relaxed to requiring that the maximum face size does not exceed 7. In the present paper we establish the latter conjecture by proving the following result:

MAIN THEOREM. *Let G be a simple graph. If G admits a cellular embedding on a closed surface S (orientable or nonorientable) such that the size of each face does not exceed* 7, *then G is upper embeddable.*

An analogous result confirming the former conjecture will be published in as sequel to this paper [5].

A simple example (which will be given at the end of this paper) shows that the condition of the maximum face size not exceeding 7 in the above theorem is best possible.

The following interesting corollary is just a simple rephrasing of the theorem, nevertheless it reveals a surprising property of all cellular embeddings of a deficient simple graph.

COROLLARY. *Every cellular embedding of a simple deficient graph on a closed surface (orientable or nonorientable) contains a face with size at least* 8.

The method of proof of our main theorem is based on Nebeský's maximum genus theorem combined with an induction on the genus employing surface surgery.

Our notation and terminology can be found in [1] or [4] except the following. Let G be a graph and $A \subseteq E(G)$. Denote by $G-A$ the graph obtained from G by deleting all edges in A. Let $c(G-A)$ and $b(G-A)$ denote the number of components of $G-A$ and the number of components of $G-A$ with odd Betti number, respectively. For two subgraphs F and K of G denote by $E_G(F, K)$ the set of all edges whose two end-vertices are respectively in F and K. A *circuit* C is a connected 2-regular graph, and the number of edges in C is called the *length* of C. A circuit with length k is also said to be a k-*circuit*. The degree $\deg_G(v)$ of a vertex v in G is the number of edges incident with v, loops being counted twice. When we talk about an edge e, we understand that the edge e does not include its two end-vertices. *Contracting* and edge e (not a loop) in G means to delete e from G and identify its two end-vertices.

Let G be a graph cellularly embedded on a surface S. Throughout this paper, a face with size not exceeding 7 will be said to be *short*. We say that G is *short-face embedded* on S if each face of G on S is a short face; in this case we also say that G has a *short-face embedding* on S.

The cardinality of a set X will be denoted by $|X|$.

2. DEFICIENT GRAPHS

This section together with the next one will be devoted to preparations for the main proof. Here we will deal with the maximum genus of a graph in a greater detail with emphasis on deficient graphs. The next section will be devoted to a surgery on surfaces. As we have already mentioned, the maximum genus of a graph G is usually determined by means of its deficiency. Nebeský [9] gave a completely combinatorial characterization of this invariant as follows:

LEMMA 2.1 [9]. *Let G be a connected graph. Then*:

(1) $\xi(G)=\max_{A\subseteq E(G)}\{c(G-A)+b(G-A)-|A|-1\}$;

(2) *G is upper embeddable if and only if* $c(G-A)+b(G-A)-|A| \leqslant 2$ *for any subset* $A\subseteq E(G)$.

Let us call a subset $A\subseteq E(G)$ a *Nebeský set* in G if $\xi(G)=c(G-A)+b(G-A)-|A|-1$. A *minimal Nebeský set* is a Nebeský set that is minimal under inclusion. The next lemma, which is obtained by combining results of Nebeský [11] and Fu and Tsai [3] provides structural information about a deficient graph G (i.e., a graph G with $\xi(G)\geqslant 2$) in terms of minimal Nebeský sets.

LEMMA 2.2. *Let G be a deficient graph, and let $A\subseteq E(G)$ be a minimal Nebeský set in G. Then*:

(i) $c(G-A)\geqslant 2$, *and each component F of $G-A$ has an odd Betti number, that is*, $\beta(F)=1 \pmod 2$;

(ii) *each component F of $G-A$ is an induced subgraph of G*;

(iii) $|A|\leqslant 2c(G-A)-3$;

(iv) *for any two distinct components F and K of $G-A$ one has* $|E_G(F,K)|\leqslant 1$.

Proof. The details of the proofs of the properties (i) and (ii) can be found in [11, Proof of Theorem 1] while the proofs of the properties (iv) and also (i) are in [3, Lemma 2.1]. Clearly, the property (iii) is immediate from the choice of A, the property (i), and the condition $\xi(G)\geqslant 2$. ∎

Let G be a connected graph and let F be a connected vertex-induced subgraph of G. Denote by $E(G,F)$ the set of all edges of G which do not belong to $E(F)$ but are incident with vertices in $V(F)$. We now introduce an operation whose purpose is to spit the graph G into a family of smaller graphs with suitable properties. In order to do this, we first form a graph $\overline{G}$ by the following process:

(a) we remove $V(F)$ and $E(F)$ from G but do not remove $E(G, F)$;

(b) we take a family $\mathscr{C} = \{C_1, C_2, ..., C_n\}$, $n \geqslant 1$ of pairwise disjoint circuits of arbitrary lengths $\geqslant 3$;

(c) finally, we attach each edge of $E(G, F)$ formerly incident with a vertex of F to an arbitrary vertex of some C_j $(1 \leqslant j \leqslant n)$.

The graph $\bar{G}$ is thus obtained from G by replacing F with a collection $\mathscr{C}$ a pairwise disjoint circuits. Let $\{G_1, G_2, ..., G_m\}$ $(m \geqslant 1)$ be the set of all components of $\bar{G}$. We shall call the family $\{G_1, G_2, ..., G_m\}$ an *F-resolution* of G by the family of circuits $\mathscr{C}$ or simply and *F-resolution* of G.

Observe that number and length of the chosen circuits as well as their incidence with the edges of $E(G, F)$ in the definition of an F-resolution are irrelevant. Therefore the resulting graph $\bar{G}$ and the corresponding F-resolution of G are not uniquely determined. With these notions we can prove the following lemmas.

LEMMA 2.3. *Let G be a connected graph and let $\{G_1, G_2, ..., G_m\}$ be an F-resolution of G by a family of circuits $\mathscr{C}$. Then each G_i contains at least one of the circuits $C_j \in \mathscr{C}$ (it may occur that $G_i = C_j$).*

Proof. Without loss of generality assume that G_i does not contain any circuit $C_j \in \mathscr{C}$. By the definition of an F-resolution, G_i has no vertex incident with any edge of $E(G, F)$. Furthermore, it follows that G_i is a component of $G - (V(F) \cup E(F) \cup E(G, F))$. This implies that in G the vertices of G_i are not joined to any vertex in F, contradicting the connectivity of G. ∎

The following lemma is crucial for the proof of our main result.

LEMMA 2.4. *Let $\{G_1, G_2, ..., G_m\}$ be an F-resolution of a deficient graph G where F is a component of $G - A$ for some minimal Nebeský set $A \subseteq E(G)$. Then at least one of the graphs G_1, G_2, ..., G_m is a deficient graph.*

Proof. By Lemma 2.2., F is a vertex-induced subgraph of G. Set $A_i = A \cap E(G_i)$. Note that A_i may happen to be empty because G_i can possibly be one C_j. It is easy to see that each component of $G_i - A_i$ is either one of the circuits C_j or one of the components of $G - A$ other than F.

Note that for each $1 \leqslant i \leqslant m$ we have

$$c(G_i - A_i) = b(G_i - A_i). \tag{1}$$

Furthermore,

$$\sum_{i=1}^{m} |A_i| = |A| \tag{2}$$

and

$$\sum_{i=1}^{m} c(G_i - A_i) = c(G - A) - 1 + n. \tag{3}$$

Now assume, to the contrary, that each $G_i (1 \leqslant i \leqslant m)$ is upper embeddable. By combining Lemmas 2.1 and 2.2 and (1) we obtain that

$$|A_i| \geqslant c(G_i - A_i) + b(G_i - A_i) - 2 = 2c(G_i - A_i) - 2.$$

If we sum these inequalities for $1 \leqslant i \leqslant m$ and use (2) and (3) we get

$$|A| = \sum_{i=1}^{m} |A_i| \geqslant 2 \sum_{i=1}^{m} c(G_i - A_i) - 2m = 2c(G - A) - 2 + 2n - 2m.$$

Note that Lemma 2.3 implies that $n \geqslant m$. Therefore we have that $|A| \geqslant 2c(G - A) - 2$, which contradicts the property (iii) of Lemma 2.2. This completes the proof. ∎

Recall that in the definition of an F-resolution the length of each circuit C_j $(1 \leqslant j \leqslant n)$ may be arbitrary. With this in mind we obtain:

LEMMA 2.5. *Let $\{G_1, G_2, ..., G_m\}$ be an F-resolution of a graph G by a family of circuits $\mathscr{C} = \{C_1, C_2, ..., C_n\}$ and assume that for each i the graph G'_i is obtained from G_i by contracting some edges of circuits in $\mathscr{C}$ contained in G_i, $1 \leqslant i \leqslant m$. Then $\{G'_1, G'_2, ..., G'_m\}$ is an F-resolution of G.*

Proof. Trivial. ∎

3. SURGERY ON A SURFACE

Let $j: G \to S$ be a 2-cell embedding of a graph G on a surface S, and let F be a connected subgraph of G. For any positive and arbitrarily small real number ε let $N(F, \varepsilon)$ denote the open collaring of $j(F)$ in S in which the distance of all points from $j(F)$ is smaller than ε. Analogously, for each vertex $v \in V(F)$, let $N(v, \varepsilon)$ denote the open ε-neighborhood of $j(v)$ in S, that is, the distance of all points of $N(v, \varepsilon)$ from $j(v)$ is less than ε. Notice that ε can be chosen arbitrarily small, and therefore we can assume its complement $S - N(F, \varepsilon)$ on S to be a bordered surface, possibly disconnected.

For any connected component M of $S - N(F, \varepsilon)$, we take an appropriate number of closed disks and identify each boundary circuit of M with that of a closed disk, thereby obtaining a new surface denoted by $S(M)$. Observe that the orientability of $S(M)$ may happen to be different from that of the original embedding surface S. We will call M the *connected component* and $S(M)$ the *surface* obtained by *removing* $N(F, \varepsilon)$ from S.

If C is a circuit of G, then $S - N(C, \varepsilon)$ has clearly at most two components. We will call C a *contractible* circuit on S if $S - N(C, \varepsilon)$ has precisely two components and at least one homeomorphic to a closed disk, otherwise, C will be said to be a *noncontractible* circuit. Equivalently, a contractible circuit on a surface is one that can be continuously contracted to a point in the surface. (Purely combinatorial definitions of contractible and noncontractible circuits were given by Thomassen in [16].)

The first result of this section easily follows from Euler's formula. It can also be found in [7].

LEMMA 3.1. *Let C be a circuit of a graph G that is cellularly embedded on a surface S, and let M be a connected component of $S - N(C, \varepsilon)$. Then*:

(1) *If C is a contractible circuit on S, then either $g(S_M) = g(S)$ or $g(S_M) = 0$.*

(2) *If C is a noncontractible circuit on S, then $g(S_M) < g(S)$.*

When applying the previous lemma repeatedly, we obtain the following result:

LEMMA 3.2. *Let F be a connected subgraph of a graph G that is cellularly embedded on a surface S, and let M be a connected component of $S - N(F, \varepsilon)$. Then*:

(1) *If each circuit in F is contractible on S, then either $g(S_M) = g(S)$ or $g(S_M) = 0$, and furthermore M has exactly one boundary circuit.*

(2) *If F has at least one noncontractible circuit on S, then $g(S_M) < g(S)$.*

Now let G be a deficient simple graph, and let $j: G \to S$ be a short-face embedding of G on S. Throughout, we will identify $j(G)$ with G. Let F be a component of $G - A$ where A is a minimal Nebeský subset of $E(G)$. By Lemma 2.2(i), F is not a tree and hence contains some circuits; by Lemma 2.2(ii), F is a vertex-induced subgraph of G.

Intuitively, we want to cut the surface S along F and leave a copy of each edge on each side of the resulting (possibly disconnected) bordered surface. We perform this, roughly, as follows. First, we make the graph F "fat"—let L denote such a fat F. The main property of L is that this fat graph (in particular, its fat vertices and fat edges) can be continuously shrunk onto the original graph F. Then we take another fat copy of F, denoted by L', on the same surface S, but one which is only "half as fat."

Now we excise L' from S and take any connected component M_i $(i = 1, ..., m)$ of the resulting (possibly disconnected) bordered surface. Clearly, some parts of $L - L'$ are contained in M_i. We shrink them in the same manner as we would do L, but only halfway, because the other half was excised. The part of G lying in the interior of M_i together with these shrunken parts of $L - L'$ lying in M_i (in fact, in its boundary) form a graph F_i embedded in M_i. Finally we take closed disks and cap each boundary component by one of them thereby obtaining a closed surface $S(F_i) = S(M_i)$ with F_i embedded in it. Note that the orientability character of $S(F_i)$ may be different from that of S.

Formally, the construction of F_i and its embedding on $S(F_i)$ will be performed as follows. For a sufficiently small positive real number ε, let us take the open ε-collaring $L = N(F, \varepsilon)$ of F, that is, the set of all points on S whose distance from F in S is smaller than ε. Let us recall that, in general, an open collaring of F in the surface S is an open set $L \subseteq S$ such that $F \subseteq L$ and F is a *deformation retract* of L. This means that there exists a continuous mapping, called a *deformation retraction*, Φ: $L \times [0, 1] \to L$ such that Φ_0 is the identity mapping on L, $\Phi_1(L) = F$, and $\Phi_1 \mid F = \mathrm{id}_F$. In this particular case where $L = N(F, \varepsilon)$ we clearly can and will adopt the following useful "regularity" assumption: for each $t > 0$ we let $\Phi_t(N(F, \varepsilon)) = N(F, \varepsilon(1 - t))$. Now we remove $L' = N(F, \varepsilon/2)$ from S, thereby getting a bordered, possibly disconnected, surface $S - N(F, \varepsilon/2)$. Observe that its boundary consists of all elements of $N(F, \varepsilon)$ whose distance from F equals $\varepsilon/2$. Let M_i be any component of $S - N(F, \varepsilon/2)$ $(i = 1, ..., m)$ and let $L_i = L \cap M_i$. Define the graph F_i to be $\Phi_{\varepsilon/2}(L_i) \cup (j(G) \cap \mathrm{int}(M_i))$.

It is obvious that F_i is embedded in M_i. Moreover, $B(F_i) = \Phi_{\varepsilon/2}(L_i)$ is the part of F_i which lies on the boundary of M_i whereas $j(G) \cap \mathrm{int}(M_i)$ which lies in the interior of M_i.

We observe that for each edge e' of F_i lying on the boundary $B(F_i)$ of M_i, there exists an original edge e of F such that e' consists of all elements of $N(F, \varepsilon)$ whose distance from e equals $\varepsilon/2$. Analogously, for each vertex v' of F_i lying on $B(F_i)$, there exists an original vertex v of F such that the distance between v' and v equals $\varepsilon/2$. In this sense we can say that e' and v' correspond to e and v, respectively. We must note that for each edge e, there exist exactly two such edges e' and e'' (not necessarily belonging to the same $B(F_i)$) corresponding to e, while for each vertex f of F the number of such vertices corresponding to v depends on the number of corners on S formed by the edges of F incident with v. Let $B(F_i)$, the boundary of M_i, be composed of n_i disjoint circuits of F_i, say C_i^1, C_i^2, ..., $C_i^{n_i}$. We now cap each boundary circuit C_i^j $(1 \leqslant j \leqslant n_i)$ by a closed 2-cell D_i^j and thus obtain a new surface $S(F_i)$, together with an embedding of the graph F_i.

Now we are in a position to establish several useful properties of the graph F_i and its embedding on the surface $S(F_i)$ $(1 \leqslant i \leqslant m)$.

Let us denote by $f(C_i^j)$ the face of F_i in $S(F_i)$ which is bounded by C_i^j and obtained from D_i^j by removing its boundary. Clearly, the face $f(C_i^j)$ is an open 2-cellular face, and the size of $f(C_i^j)$ is the length of C_i^j.

CLAIM 1. *For each i $(1 \leqslant i \leqslant m)$, F_i is a connected simple graph which is cellularly embedded in $S(F_i)$. Moreover, each face of the embedding of F_i in $S(F_i)$ is a short face except possibly the faces $f(C_i^j)$, $1 \leqslant j \leqslant n_i$.*

Proof. The simplicity of F_i follows directly from the definitions of F_i and M_i and from the simplicity of G. In order to prove that F_i is connected, it suffices to prove that F_i is cellularly embedded in $S(F_i)$, since a cellular embedding of a graph immediately implies its connectivity. To do this, let us first analyze the possible position of each face of F_i with respect to $B(F_i)$. For any face f of the embedding of F_i in $S(F_i)$, one of the following three cases occurs.

Case (a). f, together with its boundary, lies entirely in the interior of M_i. It follows that f may be viewed as a face of the original embedding of G in S.

Case (b). f lies entirely in the interior of M_i; however, its boundary intersects the boundary of M_i at some vertices or edges of $B(F_i)$. In this case the face f is homeomorphic to a face f' of G in S, where the boundary of f' can be obtained from that of f by replacing the vertices or the edges belonging to $B(F_i)$ with their corresponding vertices or edges of F. Therefore f has the same size as f'.

Case (c). f is a face $f(C_i^j)$, $1 \leqslant j \leqslant n_i$.

The above three cases show that each face of F_i on $S(F_i)$ is an open 2-cell and the embedding is thus cellular. In particular, F_i is connected. Furthermore, each face f of the embedding is a short face except possibly the faces $f(C_i^j)$, $1 \leqslant j \leqslant n_i$. ∎

Observe that there are three kinds of edges in F_i. We will say that an edge e of F_i is of *type* k, $k = 0, 1, 2$, if k end-vertices of e belong to $B(F_i)$. An edge of type 0 can be identified with the corresponding original edge of F. As far as the other two types are concerned we have the following:

CLAIM 2. *For each F_i the following holds true:*

(1) *An edge e of F_i is of type* 2 *if and only if it lies entirely in $B(F_i)$.*

(2) *If $e = uv'$ and $e'' = uv''$ are adjacent type* 1 *edges such that the vertices v' and v'' belong to $B(F_i)$ but u does not, then $e' = e''$.*

Proof. First we prove the statement (1). If e belongs to $B(F_i)$, then by the definitions of F_i and M_i the two end-vertices of e must belong to $B(F_i)$. Thus e is of type 2. Conversely, let e be of type 2. Assume to contrary that e does not belong to $B(F_i)$. Then we easily see rom the definitions of F_i and M_i that, in the graph G, the two end-vertices of e are contained in $V(F)$ while $e \in E(G) - E(F)$. This contradicts the statement of Lemma 2.2(ii) that F is a vertex-induced subgraph of G.

Now we prove (2). Assume that $e' = uv'$ and $e'' = uv''$ is a pair of adjacent edges where v' and v'' belong to $B(F_i)$ but the common vertex u of e' and e'' does not belong to $B(F_i)$. In the graph G, the vertex u is in $V(G) - V(F)$. So v is necessarily a vertex of some component of $G - A$ different from F, say K. Again, since the other end-vertices of e' and e'' belong to $B(F_i)$, it follows that in G, the other end-vertices of e' and e'' are in F. Therefore we have that both e' and e'' are in $E_G(F, K)$. Since $|E_G(F, K)| \leqslant 1$ by Lemma 2.2(iv), we have $e' = e''$. ∎

Claim 3. *The set* $\{F_1, F_2, ..., F_m\}$ *is an F-resolution of G.*

Proof. From the definitions of F_i and M_i it is clear that the union $\bigcup_{i=1}^m F_i$ can be regarded as being obtained from G by replacing the subgraph F with a collection of pairwise disjoint circuits $\bigcup_{i=1}^m (\bigcup_{j=1}^{n_i} C_i^j) = \bigcup_{i=1}^m B(F_i)$. Thus it is straightforward to check that $\{F_1, F_2, ..., F_m\}$ is an F-resolution of G. ∎

Now we turn each of the circuits C_i^j into a 3-circuit by contracting (if necessary) some of its edges along the surface $S(F_i)$. We let F'_i and $S(F'_i)$ be respectively the resulting graph and the resulting surface obtained by the edge contraction process. Obviously $S(F'_i)$ is homeomorphic to $S(F_i)$.

Claims 4 and 5 describe the graph F'_i and its embedding in $S(F''_i)$.

Claim 4. *Each* F'_i *is a connected simple graph* $(1 \leqslant i \leqslant m)$, *and the set* $\{F'_1, F'_2, ..., F'_m\}$ *is an F-resolution of G.*

Proof. Since F_i is connected (Claim 1), the connectivity of F'_i follows immediately. Next we prove that F'_i is simple. Let B_i^j $1 \leqslant j \leqslant n_i$ be the 3-circuit obtained from C_i^j by the contraction process.

Suppose F'_i has a loop e. Then e must be incident with a vertex in some B_i^j. In F_i, the edge e does not belong to C_i^j while its two end-vertices do. Thus in F_i, the edge e does not entirely lie in $B(F_i)$ while being of type 2. This contradicts Claim 2, part (i).

Now suppose that F'_i has a pair of parallel edges e' and e''. Let x and y be their two common end-vertices. Then one of the following three cases has to occur:

(a) both x and y belong to some B_i^j;

(b) x and y belong to two distinct circuits $B_i^{j_1}$ and $B_i^{j_2}$, $j_1 \neq j_2$, $1 \leqslant j_1, j_2 \leqslant n_i$; and

(c) one of x and y, say x, belongs to some B_i^j but y does not belong to any B_i^j.

If the case (a) occurs, then in the graph F_i at least one of e' and e'', say e'; does not belong to $B(F_i)$. However e' is of type 2, contradicting Claim 2, part (1). In the case (b) neither e' nor e'' belongs to $B(F_i)$ although they are of type 2. This contradicts Claim 2, part (1) again. Finally, the occurrence of the case (c) would imply a contradiction to Claim 2, part (2). Summing up the above considerations, F'_i $(1 \leqslant i \leqslant m)$ is simple.

The rest of the claim follows easily from Lemma 2.5. ▮

Claim 5. (1) *If every circuit of F is contractible on S, then each F'_i $(1 \leqslant i \leqslant m)$ is short-face embedded on the surface $S(F'_i)$ with $g(S(F'_i)) = g(S)$ or $g(S(F'_i)) = 0$. Moreover, one has $|V(F'_i)| < |V(G)|$ unless F is a 3-circuit bounding a face.*

(2) *If F contains a noncontractible circuit on S, then each F'_i $(1 \leqslant i \leqslant m)$ is short-face embedded on the surface $S(F'_i)$ with $g(S(F'_i)) < g(S)$.*

Proof. We first prove the conclusion (1). Since each circuit of F is contractible on S, it is immediate from Lemma 3.2, part (1) that $g(S(F'_i)) = g(S(F_i)) = g(S)$ or $g(S(F'_i)) = g(S(F_i)) = 0$. The fact that F'_i is short-face embedded in $S(F'_i)$ follows immediately from Claim 2.

Now we deal with the rest of the conclusion (1). By Lemma 3.2, part (1), the bordered surface M_i has exactly one boundary circuit; in other words, $n_i = 1$. Let C_i denote the unique boundary circuit of M_i and let B_i be the circuit of F'_i obtained from C_i by the edge contracting process. Then B_i is a 3-circuit. Keeping in mind that F is simple and contains a circuit, we have the following two cases:

Case 1. F is not a 3-circuit. Then $|V(F)| > 3$, and thus we get that $|V(F'_i)| < |V(G)|$ since $|V(G)| - |V(F'_i)| \geqslant |V(F)| - |V(B_i)| > 0$.

Case 2. F is a 3-circuit but does not bound a face of G in S. By the hypothesis, F is a contractible 3-circuit on S. Thus $S - N(F, \varepsilon/2)$ has exactly two connected components M_1 and M_2, that is, $m = 2$. Furthermore, the unique boundary circuit C_i of M_i $(i = 1, 2)$ is a 3-circuit. Again, since F is a contractible 3-circuit but does not bound a face of G on S, we see that both M_1 and M_2 must contain some vertices of $V(G) - V(F)$ and thus so must both F_1 and F_2. Therefore $|V(F'_i)| < |V(G)|$ $(1 \leqslant i \leqslant m = 2)$.

The two cases above imply that $|V(F_i)| < |V(G)|$ unless F is a 3-circuit bounding a face of G. This finishes the proof of the conclusion (1).

If we apply the part (2) of Lemma 3.2, the proof of the conclusion (2) can be performed along the same lines as the proof of the conclusion (1) above. The difference on this case is that the number of the boundary circuits of each M_i is possibly larger than one ($1 \leqslant i \leqslant m$). However, here we need not consider the inequality between $|V(F'_i)|$ and $|V(G)|$. We leave the details to the reader. ▮

Summarizing the above claims, we can state the following result which is crucial for the proof of the main theorem in the next section.

LEMMA 3.3. *Let G be a deficient simple graph that is short-face embedded in a surface S. Let F be a component of $G - A$ where A is a minimal Nebeský set of $E(G)$. Then there exists an F-resolution $\{G_1, G_2, ..., G_m\}$ of G with the following properties*:

(a) *If every circuit of F is contractible on S, then each G_i ($1 \leqslant i \leqslant m$) is simple and has a short-face embedding on a surface S_i with $g(S_i) = g(S)$ or $g(S_i) = 0$. Moreover, one has $|V(G_i)| < |V(G)|$ unless F is a 3-circuit bounding a face.*

(b) *If F contains a noncontractible circuit on S, then each G_i ($1 \leqslant i \leqslant m$) is simple and has a short-face embedding on a surface S_i with $g(S_i) < g(S)$.*

4. PROOF OF MAIN THEOREM

In this section we give the proof of the main theorem. As we have already mentioned, the proof will be by induction on the genus. The next lemma verifies the basis of the induction.

LEMMA 4.1. *Let G be a simple graph. If G has a short-face embedding on the 2-sphere, then G is upper embeddable.*

Proof. Assume to contrary that the conclusion does not hold. Then there is a planar deficient graph with a short-face embedding on the 2-sphere. From among these choose G to have a minimum order. Fix a short-face embedding of G on the 2-sphere, and let A be a minimal Nebeský subset of $E(G)$.

Now we employ Lemma 3.3. Since G is embedded on the 2-sphere, each circuit in G is contractible, so part 1 of that lemma applies. Clearly, we only have to consider the following two cases.

Case 1. There exists a component F of $G-A$ that is not a 3-circuit bounding a face. Lemma 3.3 now implies that there exists an F-resolution $\{G_1, G_2, ..., G_m\}$ of G such that each G_i $(1 \leqslant i \leqslant m)$ is simple and has a short-face embedding on the 2-sphere, and furthermore $|V(G_i)| < |V(G)|$. By the choice of G, each G_i $(1 \leqslant i \leqslant m)$ is upper embeddable, contradicting Lemma 2.4.

Case 2. Each component F of $G-A$ is a 3-circuit bounding a face of G on the 2-sphere. Obviously, G has at least $c(G-A)$ faces with size three. Furthermore,

$$\begin{cases} |V(G)| = \sum_F |V(F)| = 3c(G-A) & (4) \\ |E(G)| = \sum_F |E(F)| + |A| = 3c(G-A) + |A|, & (5) \end{cases}$$

where the sum ranges over all the components F of $G-A$. Let $F(G)$ denote the set of faces of G. By (4), (5), and Euler's formula for the plane we have

$$|F(G)| = 2 + |E(G)| - |V(G)| - |V(G)| = |A| + 2. \tag{6}$$

Taking into account that G is short-face embedded on the 2-sphere with at least $c(G-A)$ faces with the size three, we thus have

$$2\,|E(G)| = \sum_{f \in F(G)} |f| \leqslant 3c(G-A) + 7(|F(G)| - c(G-A)). \tag{7}$$

Substituting (5) for $|E(G)|$ and (6) for $|F(G)|$ into (7), and simplifying, we obtain that

$$|A| \geqslant 2c(G-A) - \tfrac{14}{5} \geqslant 2c(G-A) - 3,$$

which contradicts the property (iv) of Lemma 2.2. This finishes the proof of the lemma. ∎

Now we proceed to the very proof of the main theorem.

Proof of Main Theorem. We employ induction on $g(S)$, the genus of the embedding surface S. The case when $g(S)=0$ has been covered by Lemma 4.1. We thus assume that the conclusion is true for any surface with genus smaller than $g(S) \geqslant 1$, but to the contrary that it is false for the surface S. We choose a graph G that is short-face embeddable on S, is not upper embeddable, and whose order is a small as possible. Let A be a minimal Nebeský set of $E(G)$. Then we can apply Lemma 3.3 and consider the following cases.

Case 1. For any component F of $G-A$, each circuit of F is contractible on S. In this case we shall distinguish two subcases according to part (1) of Lemma 3.3.

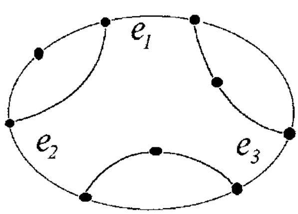
Subcase 1.1. There exists a component F of $G-A$ that is not a face-bounding 3-circuit. By Lemma 3.3, there exists an F-resolution $\{G_1, G_2, ..., G_m\}$ of G such that each G_i $(1 \leqslant i \leqslant m)$ is a simple graph that has a short-face embedding on a surface S_i with either $g(S_i) = g(S)$ or $g(S_i) = 0$, and furthermore $|V(G_i)| < |V(G)|$. If $g(S_i) = 0$, then G_i is upper embeddable by Lemma 4.1 (basis of induction). If $g(S_i) = g(S)$, then G_i is upper embeddable by the choice of G. In either case we get that each G_i $(1 \leqslant i \leqslant m)$ is upper embeddable. This contradicts Lemma 2.4.

Subcase 1.2. Each component F of $G-A$ is a 3-circuit bounding a face of G on S. By proceeding as in Case 2 of the proof of Lemma 4.1 we can induce a similar contradiction. The only difference is that the general form of Euler's formula yields the inequality $|F(G)| \leqslant 2 + |E(G)| - |V(G)|$.

Case 2. There exists a component F of $G-A$ that contains a noncontractible circuit on S. By part 2 of Lemma 3.3 there exists an F-resolution $\{G_1, G_2, ..., G_m\}$ of G such that each G_i $(1 \leqslant i \leqslant m)$ is simple and has a short-face embedding on a surface S_i with $g(S_i) < g(S)$. Therefore each $G_i (1 \leqslant i \leqslant m)$ is upper embeddable by the inductive hypothesis, a contradiction to Lemma 2.4 as well.

The contradictions in all the above cases show that G is upper embeddable. This completes the induction step and thereby establishes our theorem. ∎

Example. We now demonstrate that the condition requiring a short-face embedding of a simple graph to have faces of size at most 7 is best possible. Let G be the following graph embedded in the plane.



G

Clearly, G is simple and the size of each face does not exceed 8. However, taking $A = \{e_1, e_2, e_3\}$, we see that $c(G-A) = b(G-A) = 3$ and $c(G-A) + b(G-A) - |A| \nleqslant 2$, and therefore G is not upper embeddable by Lemma 2.1, part 2.

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