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On certain (block) Toeplitz matrices related to radial functions

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Abstract

Interpolation of smooth functions and the discretization of elliptic PDEs by means of radial functions lead to structured linear systems which, for equidistant grid points, have almost the (block) Toeplitz structure. We prove upper bounds for the condition numbers of the $n \times n$ Toeplitz matrices which discretize the model problem $u''(x) = f(x)$, $x \in (0, 1)$, $u(0) = a$, $u(1) = b$ over an equally spaced grid of $n + 2$ points in $[0, 1]$ by means of the collocation method based on radial functions of the multiquadric, inverse multiquadric and Gaussian type. These bounds are asymptotically sharp.

Keywords: Radial functions; Toeplitz matrices; Condition number; Multiquadric; Inverse multiquadric

1. Introduction

Radial functions have been introduced in 1980s as an effective tool for solving interpolation problems and have been used by several authors for discretizing PDE's by means of collocation techniques [5,6,8,14,16]. Fast and effective evaluation algorithms for radial functions have been developed in the literature [4,10,17,20]. Informally, a radial function $\phi(x): \mathbb{R}^n \to \mathbb{R}$ is a function of the Euclidean norm $\|x\|$ of x, i.e., $\phi(x) = \eta(\|x\|)$, for $\eta(t): \mathbb{R} \to \mathbb{R}$. A special interest has been addressed to radial functions of the following type:

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$$\begin{array}{ll} \sqrt{t^2+c^2}, & \text{multiquadric (MQ)}, \\ 1/\sqrt{t^2+c^2}, & \text{inverse multiquadric (IMQ)}, \\ \mathrm{e}^{-\frac{t^2}{c^2}}, & \text{Gaussian}, \end{array} \tag{1}$$

where c is a parameter called *shape parameter*, for their particular suitability in modeling different problems.

Although for the interpolation with multiquadric radial functions of the kind (1) some explicit error bounds have been formally proved, the results concerning the numerical solution of PDE's are mainly experimental. In particular, in the many numerical experiments performed by several authors, it has been observed that, using a discretization grid with maximum step size h, the discretization error behaves like $\mathrm{O}(\lambda^{c/h})$ for the MQ and like $\mathrm{O}(\lambda^{\sqrt{c}/h})$ for the IMQ and Gaussian, where $0 < \lambda < 1$; while the condition number of the matrix which discretizes the differential equation grows like $\mathrm{O}(\mathrm{e}^{\theta c/h})$ as function of c/h for some positive constant θ [6,9,16]. So far, theoretical estimates of the condition numbers of the radial function matrices which discretize differential operators are missing.

In this paper, we provide explicit asymptotic estimates, as function of c/h to the condition number $\mu(T_n)$ of the Toeplitz matrix T_n related to the discretization of the one-dimensional model problem:

$$\begin{cases} u''(x) = f(x), & x \in (0,1), \\ u(0) = a, \quad u(1) = b \end{cases} \tag{2}$$

with the collocation technique over a grid of equally spaced points $x_i = ih$, $i = 0, 1, \ldots, n+1$ for $h = 1/(n+1)$, based on the MQ, IMQ and Gaussian radial functions, respectively.

In fact, the collocation matrix A_n associated with the linear system which discretizes (2) can be viewed as the sum of a symmetric Toeplitz matrix T_n and a rank-2 correction, where the matrix T_n represents a discretized version of the differential operator which does not include the boundary conditions.

Denoting $g = c/h$, we prove that there exists a function $\rho(g)$ of g such that for any n it holds $\mu(T_n) \leqslant \rho(g)$, where equality is reached for $n \to \infty$, moreover, we provide an asymptotical estimate $\gamma(g)$ of $\rho(g)$ for $g \to \infty$, such that

$$\rho(g) \approx \gamma(g) = \begin{cases} (\mathrm{e}^{\pi})^g & \text{for the MQ}, \\ (\mathrm{e}^{2\pi})^g/(2\mathrm{e}^2\pi^{3/2}g^{3/2}) & \text{for the IMQ}, \\ (\mathrm{e}^{\pi^2})^{g^2}/(2e\pi^2 g^2) & \text{for the Gaussian}, \end{cases}$$

where $\phi(x) \approx \psi(x)$ if $\lim_{x\to\infty} \phi(x)/\psi(x) = 1$.

The proof of these bounds is based on classical spectral properties of Toeplitz matrices and on some tools of the theory of special functions like the modified Bessel functions of the first kind of order 0 and 1. The key step of our approach is to provide an explicit expression of the symbol associated with the matrix T_n, so that the well-known asymptotic spectral properties of symmetric Toeplitz matrices provide a tight bound to the condition number of T_n.

From the numerical experiments performed, the above asymptotic bounds are very strict even for small values of g, moreover, there does not seem to be a large difference between the condition number of T_n and of A_n.

Related results concerning the analysis of spectral properties of Toeplitz matrices encountered in interpolation with radial functions and the design of preconditioners can be found in [2,3].

Effective techniques for controlling the growth of the condition number in the case of meshless grids and general linear operators are analyzed in [18].

Although our bounds are proved only for equally spaced grid, we believe that this is a first step to prove exponential-type bounds for almost any kind of discretization grids. We recall that for scattered grids x_i for which there exists a "sufficiently regular" function $\beta(t)$ such that $x_i = \beta(ih)$, $i = 0, 1, \dots, n+1$ where $\beta(0) = 0$, $\beta(1) = 1$, the asymptotic spectral properties of the finite difference discretization of differential operators are preserved to a certain extent even though their Toeplitz structure is lost (see [19] for more details). It is our opinion that the techniques introduced in this paper may be combined with the approach of [19] to obtain more general results on the asymptotic condition number.

Our interest is mainly theoretical, even though information about the condition number may provide insights for the choice of the value of the discretization step h, for the design of preconditioners and for the control of ill conditioning of the collocation martix.

The paper is organized as follows. In Section 2, we recall the main tools used to prove our result, more specifically, the collocation technique and the main results on spectral properties of symmetric Toeplitz matrices.

In Section 3, we prove our main result. More precisely, Section 3.1 concerns the MQ function while Section 3.2 concerns the IMQ and the Gaussian functions.

Section 4 contains some comments related to multidimensional problems. Section 5 reports the results of some numerical experiments which confirm the asymptotic bounds.

2. Tools

In this section, we recall the collocation technique for solving the boundary problem (2) and the main results concerning the spectral properties of Toeplitz matrices used to prove the asymptotic bounds. For more details on these techniques we refer the reader to the papers [3,5,6,8–10,12,14,16,17,20,13], respectively.

2.1. Collocation with radial functions

For n positive integer, let $x_0 = 0 < x_1 < \cdots < x_n < x_{n+1} = 1$ and define $\phi(x) = \eta(|x|)$ where $\eta(t)$ is any function in the class (1). We are looking for an approximation to the solution $u(x)$ of (2) in the vector space spanned by the functions $\phi(x - x_i)$, $i = 0, 1, \dots, n+1$. Set $v = \sum_{j=0}^{n} v_j \phi(x - x_j)$, replace u with v in (2) and get

$$\begin{cases} \sum_{j=0}^{n} v_j \phi''(x - x_j) = f(x), & x \in (0, 1), \\ v(0) = a, \quad v(1) = b. \end{cases}$$

Setting $x = x_i$ in the latter equation for $i = 1, 2, \dots, n$ yields the linear system

$$\begin{cases} \sum_{j=0}^{n} \phi(x_0 - x_j) v_j = a, \\ \sum_{j=0}^{n} \phi''(x_i - x_j) v_j = f(x_i), \quad i = 1, 2, \dots, n, \\ \sum_{j=0}^{n} \phi(x_{n+1} - x_j) v_j = b, \end{cases}$$

whose matrix $A_{n+2} = (a_{i,j})_{i,j=0,n+1} \in \mathbb{R}^{(n+2)\times(n+2)}$ is such that $a_{0,j} = \phi(x_0 - x_j)$, $a_{n+1,j} = \phi(x_{n+1} - x_j)$ for $j = 0, \dots, n+1$ and $a_{i,j} = \phi''(x_i - x_j)$ for $i = 1, \dots, n$, $j = 0, \dots, n+1$. Let us denote by $T_n = (\phi''(x_i - x_j))_{i,j=1,n}$ the submatrix of A_{n+2} obtained by removing its first and last row and column.

In the case where the set $x_i = ih$, $i = 0, \dots, n+1$, for $h = 1/(n+1)$, forms a grid of equally spaced points in the interval [0, 1] the matrix $T_n = (\phi''((i-j)h))$ is a symmetric Toeplitz matrix, i.e., its entries are function of $i - j$. Moreover, A_{n+2} is a rank-2 correction to a symmetric Toeplitz matrix.

We recall that this collocation technique can be slightly modified in order to obtain a symmetric matrix which, for equally spaced knots, is Toeplitz up to a low rank correction.

Analyzing the condition number of a symmetric Toeplitz matrix T_n can be performed by means of the symbol associated with T_n.

2.2. *Toeplitz matrices and their spectral properties*

Let $s(x) = c_0 + 2\sum_{k=1}^{+\infty} c_k \cos(2\pi kx)$ be uniformly convergent over [0, 1] and for any positive integer n associate with $s(x)$ the symmetric Toeplitz matrix $T_n = (t_{i,j})_{i,j=1,n} \in \mathbb{R}^{n\times n}$ such that $t_{i,j} = c_{|i-j|}$. The function $s(x)$ is called *symbol*. We recall the following properties [13]:

Theorem 1. *If* $s(x) > 0$ *over* [0, 1] *then* T_n *is positive definite for any* n *and its eigenvalues belong to* $(\min s(x), \max s(x))$. *Moreover,* $\lambda_1^{(n)}$ *is a decreasing sequence converging to* $\min s(x)$, *and* $\lambda_n^{(n)}$ *is an increasing sequence converging to* $\max s(x)$, *where* $\lambda_1^{(n)}$ *and* $\lambda_n^{(n)}$ *are the minimum and the maximum eigenvalues of* T_n, *respectively. Therefore, the spectral condition number* $\mu(T_n) = \lambda_n^{(n)}/\lambda_1^{(n)}$ *is an increasing sequence converging to* $\max s(x)/\min s(x)$.

The above result holds true also for $n \times n$ block Toeplitz matrices $T_{n,m}$ with $m \times m$ Toeplitz blocks associated with a symbol $s(x, y)\colon [0, 1] \times [0, 1] \to \mathbb{R}$. In particular, if $s(x, y) > 0$ for $(x, y) \in [0, 1] \times [0, 1]$ then $\mu(T_{n,m})$ is an increasing sequence converging to $\max s(x, y)/\min s(x, y)$ as $m, n \to \infty$.

3. The main result

Let us define $g = c/h$ and consider the radial function $\phi(x) = \eta(|x|)$ for $\eta(t)$ in the class (1) defining the MQ, IMQ and Gaussian radial functions, respectively. A direct inspection shows that

$$\phi''(x_i - x_j) = \begin{cases} \frac{1}{h}\frac{g^2}{(g^2+(i-j)^2)^{3/2}} & \text{for the MQ,} \\ -\frac{1}{h^3}\frac{g^2-2(i-j)^2}{(g^2+(i-j)^2)^{5/2}} & \text{for the IMQ,} \\ \frac{-2}{h^2}\mathrm{e}^{-\frac{(i-j)^2}{g^2}}\frac{g^2-2(i-j)^2}{g^4} & \text{for the Gaussian.} \end{cases}$$

In this way, the Toeplitz matrix T_n is associated with the symbol

$$s(x) = c_0 + 2\sum_{k=0}^{+\infty} c_k \cos(2\pi kx),$$

which, up to a multiplicative term dependent at most on h, is formally defined by

$$c_k = \begin{cases} \frac{g^2}{(g^2+k^2)^{3/2}} & \text{for the MQ,} \\ \frac{g^2-2k^2}{(g^2+k^2)^{5/2}} & \text{for the IMQ,} \\ \mathrm{e}^{-\frac{k^2}{g^2}}(g^2-2k^2)/g^4 & \text{for the Gaussian.} \end{cases}$$

Observe that $s(x)$ is symmetric with respect to $x = 1/2$ so that its analysis can be restricted to the interval [0, 1/2].

3.1. The case of MQ

Observe that in the case of MQ the summation $\sum_k c_k^2$ is finite since $c_k^2 = g^4(k^2 + g^2)^{-3} < g^2 k^{-6}$ for $k \geqslant 1$. Therefore, the Fourier series $s(x)$ is convergent over $[0, 1/2]$ and is in L^2. Moreover, since $\sum_k |c_k|$ is finite, by the Weierstrass criterion uniform convergence holds true. In this way, we may apply the results of Section 2.2.

We have the following:

Proposition 1. *The function $s(x)$ is such that*

1. $s'(x) < 0$ *for* $x \in [0, 1/2]$;
2. $s(x) > 0$;
3. $\max s(x)/\min s(x) = s(0)/s(1/2) \approx \gamma(g) = (\mathrm{e}^{\pi})^g/(\pi\sqrt{2g})$.

Moreover, the matrix T_n is positive definite for any $n > 0$ and its condition number $\mu(T_n)$ is bounded from above by a function of g independent of n

$$\mu(T_n) \leqslant \theta(\mathrm{e}^{\pi})^g/(\pi\sqrt{2g}).$$

The graph of the function $s(x)$ is shown in Fig. 1 for $g = 1, 2, 3, 4$.

In order to prove the above result we need some auxiliary lemmas.

Lemma 1 (Poisson summation formula [15]). *Let $F(x)\colon \mathbb{R} \to \mathbb{R}$ be such that*:

1. *$F(x)$ is continuous*;
2. *the series $\sum_{k=-\infty}^{+\infty} F(x + k)$ is uniformly convergent over any finite interval*;
3. *the integral $\int_{-\infty}^{+\infty} |F(x)| \mathrm{d}x$ is convergent*;
4. *the series $\sum_{k=-\infty}^{+\infty} |f_k|$ is convergent, where $f_k = \int_{-\infty}^{+\infty} F(t)\mathrm{e}^{-2\pi \mathrm{i} k t}\, \mathrm{d}t$ are the Fourier coefficients.*

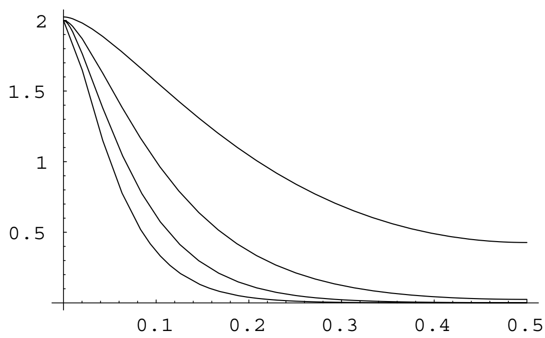

Fig. 1. Graph of $s(x)$ for $g = 1, 2, 3, 4$.

Then

$$\sum_{k=-\infty}^{+\infty} F(x+k) = \sum_{k=-\infty}^{+\infty} \mathrm{e}^{2\pi \mathrm{i} kx} \int_{-\infty}^{+\infty} F(t)\mathrm{e}^{-2\pi \mathrm{i} kt}\,\mathrm{d}t. \tag{3}$$

Relying on (3) we are ready to prove the following:

Lemma 2. *The first derivative*

$$s'(x) = -2\pi \sum_{k=1}^{\infty} kc_k \sin(2\pi kx)$$

of $s(x)$ is such that

$$s'(x) = -(2\pi g)^2 \sum_{k=-\infty}^{+\infty} (x+k)K_0(2\pi g|x+k|), \tag{4}$$

where $K_0(x)$ is the modified Bessel function of order 0.

Proof. Let $F(x) = 2\pi gxK_0(2\pi g|x|)$. A direct inspection shows that the hypotheses 1–3 of Lemma 1 are satisfied. Computing the Fourier coefficients of $F(x)$ yields

$$\int_{-\infty}^{+\infty} F(t)\mathrm{e}^{-2\pi \mathrm{i} kt}\,\mathrm{d}t = -4g\mathrm{i}\pi \int_0^{\infty} tK_0(2\pi gt)\sin(2\pi kt)\mathrm{d}t = -\mathrm{i}g\frac{k}{(g^2+k^2)^{3/2}},$$

where we used Eq. (47) on p. 105 of [7], so that also the assumption 4 is satisfied. Replacing the above expression in the right-hand side of (3) yields (4) and the proof is complete. □

Lemma 3. *It holds $s'(x) < 0$ for $x \in (0, 1/2)$.*

Proof. Let

$$F(x) = xK_0(2\pi g|x|) \tag{5}$$

and define $N = 1 + \left\lfloor \frac{x_0}{2\pi g} \right\rfloor$, where x_0 is the only value in $(0,\infty)$ where $F[x]$ takes its relative maximum. Let

$$N < x < N + 1/2. \tag{6}$$

Then, $F(x)$ is decreasing in (N,∞), moreover

$$k \leqslant |x-N+k| \leqslant k+\frac{1}{2}, \quad k+\frac{1}{2} \leqslant |x-N-k-1| \leqslant k+1, \qquad k=0,1,\ldots$$

The monotonicity of $F(x)$ implies that

$$F(x-N+k) + F(x-N-k-1) > 0, \quad k=0,1,\ldots$$

In view of (5) and (6), summing up the latter inequalities yields $-s'(x) > 0$ for $N < x < N+1/2$, whence, for the periodicity of $s'(x)$ it follows that $-s'(x) > 0$ for $0 < x < 1/2$. □

Lemma 4. *One has*

$$s(x) = 2\sum_{k=-\infty}^{+\infty} |2\pi g(x+k)|K_1(|2\pi g(x+k)|), \tag{7}$$

where $K_1(t)$ is the modified Bessel function of the first order.

Proof. Applying Lemma 1 with $F(x) = |2\pi gx|K_1(|2\pi gx|)$, Eq. (4) turns into (7) in view of the equation

$$\frac{1}{(k^2+g^2)^{3/2}} = \frac{2}{\pi g}\int_0^{\infty} xK_1(gx)\cos(kx)\mathrm{d}x$$

given in [11, p. 749, n. 6.71.12]. □

Proof of Proposition 1. Now we are ready to prove Proposition 1. Part 1, i.e., $s'(x) < 0$, follows from Lemma 3.

Concerning part 2, we observe that $K_1(x)$ is real and positive when $x > 0$. This and Lemma 4 prove that $s(x) > 0$. We will prove part 3 giving asymptotic estimates of $s(0), s(1/2)$ which are needed in order to bound the ratio $\max s(x)/\min s(x)$. To do this let us recall the following asymptotic estimates [1, p. 375, Eqs. (9.6.9) and (9.7.2)]:

$$\begin{aligned} &K_1(x) \approx 1/x, \quad x \to 0, \\ &K_1(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left(1 - \frac{3}{8x} + \mathrm{O}\left(\frac{1}{x^2}\right)\right), \quad x \to \infty. \end{aligned} \tag{8}$$

By using (8) together with (7) one has

$$\begin{aligned} s(0) &= \sum_{k=-\infty}^{+\infty} |2\pi gk|K_1(|2\pi gk|) \\ &= 1 + 4\pi g\sum_{k=1}^{\infty} K_1(2\pi gk)k \\ &= 1 + 2\pi\sqrt{g}\mathrm{e}^{-2\pi g}(1+\mathrm{O}(1/g)), \quad g \to \infty. \end{aligned}$$

In similar way one obtains

$$\begin{aligned} s(1/2) &= 2\pi g\sum_{k=0}^{\infty} K_1(\pi g(2k+1))(2k+1) \\ &= \pi\sqrt{2g}\mathrm{e}^{-\pi g}(1+\mathrm{O}(1/g)), \quad g \to \infty. \end{aligned}$$

Thus it follows

$$\frac{s(0)}{s(1/2)} \approx \frac{1}{\pi\sqrt{2g}}\mathrm{e}^{\pi g}, \quad g \to \infty.$$

This completes the proof of Part 3 of Proposition 1. The remaining part of the proposition directly follows from Theorem 1. □

3.2. The case of IMQ and Gaussian functions

In the case of IMQ and Gaussian the symbols associated with the Toeplitz matrix T_n are given, up to a constant factor, by

$$s_1(x) = \frac{1}{g^3} + 2\sum_{k=1}^{\infty}\frac{g^2-2k^2}{(g^2+k^2)^{5/2}}\cos(2\pi kx),$$

$$s_2(x) = \frac{2}{g^2} + \frac{4}{g^4}\sum_{k=1}^{\infty} \mathrm{e}^{-k^2/g^2}(g^2 - 2k^2)\cos(2\pi kx),$$

respectively. We have the following result:

Lemma 5. *It holds*

$$s_1(x) = 8\pi^2 \sum_{k=-\infty}^{+\infty} (x+k)^2 K_0(2\pi g|g+k|),$$

$$s_2(x) = \frac{\sqrt{\pi}}{g} \sum_{k=-\infty}^{+\infty} (2\pi g(x+k))^2 \mathrm{e}^{-(\pi g)^2(x+k)^2}. \tag{9}$$

Proof. This result follows from Lemma 1 by replacing $F(x)$ with $x^2 K_0(gx)$ for $s_1(x)$, and with $(2\pi gx)^2 \mathrm{e}^{-(2\pi g)^2}$ for $s_2(x)$, respectively. In the application of Lemma 1 the integral [11, p. 749, n. 6.699.11]

$$\frac{2}{\pi}\int_0^{\infty} x^2 K_0(gx)\mathrm{d}x = \frac{g^2 - 2k^2}{(g^2+k^2)^{5/2}}$$

and [11, p. 480, n. 3.896.4]

$$\int_0^{\infty} \mathrm{e}^{-g^2x^2}\cos(kx)\mathrm{d}x = \frac{\sqrt{\pi}}{2g}\mathrm{e}^{-k^2/(4g^2)}$$

have to be used. □

The asymptotic values of $\min s_i(x)$, $\max s_i(x)$, $i = 1, 2$, as $g \to \infty$ can be established by following the same argument of Section 3.1. In case of $s_1(x)$ the estimate [1, p. 375, 9.6.8]

$$K_0(x) \approx -\log x, \quad x \to 0 \tag{10}$$

and [1, p. 378, 9.7.2]

$$K_0(x) = \sqrt{\frac{\pi}{2x}}\mathrm{e}^{-x}\left(1 - \frac{3}{8x} + \mathrm{O}\left(\frac{1}{x^2}\right)\right), \quad x \to \infty \tag{11}$$

are required.

We note that in both cases the relative maximum of $s_1(x), s_2(x)$ is attained at a value, say x_{M1}, x_{M2}, respectively, which lies in the interval $(0, 1/2)$. An asymptotic estimate of x_{M1} and x_{M2} can be obtained by differentiating equations (9) and by equating a few terms of the resulting series for $s_1'(x), s_2'(x)$ to zero. In such a way one finds

$$x_{M1} \approx \frac{1}{\pi g}, \quad g \to \infty,$$

$$x_{M2} \approx \frac{1}{2\pi g}, \quad g \to \infty.$$

By using the latter estimates in Eq. (9), and by taking into account (11) in the case of $s_1(x)$, it follows:

$$s_1(x_{M1}) \approx 4\mathrm{e}^{-2}\sqrt{\pi}\frac{1}{g^2}, \quad g \to \infty,$$

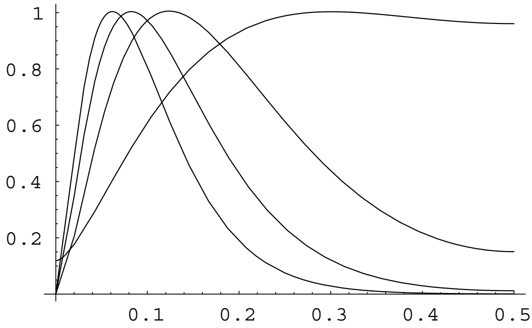

Fig. 2. Graph of $s_1(x)$ for $g = 1, 2, 3, 4$ suitably normalized.

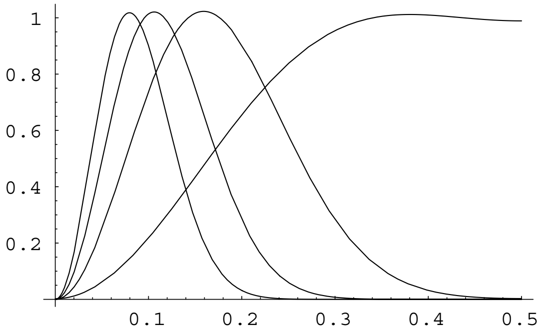

Fig. 3. Graph of $s_2(x)$ for $g = 1, 2, 3, 4$, suitably normalized.

$$s_2(x_{M2}) \approx 4\mathrm{e}^{-1}\sqrt{\pi}\frac{1}{g}, \quad g \to \infty.$$

Moreover, a similar procedure which makes use of (10) and (11) in case of $s_1(x)$, shows that

$$\min s_1(x) = s_1(0) \approx 8\pi^2\mathrm{e}^{-2\pi g}\frac{1}{\sqrt{g}}, \quad g \to \infty,$$
$$\min s_2(x) = s_2(0) \approx 8\pi^{5/2}g\mathrm{e}^{-\pi^2 g^2}, \quad g \to \infty.$$

Whence we conclude with the bound $\mu(T_n) \leqslant \rho(g) = \max s(x)/\min s(x) \approx \gamma(g)$ where

$$\gamma(g) = \begin{cases} \frac{\mathrm{e}^{2\pi g}}{2\mathrm{e}^2\pi^{3/2}g^{3/2}}, & g \to \infty \quad \text{for the IMQ}, \\ \frac{\mathrm{e}^{\pi^2 g^2}}{2e\pi^2 g^2}, & g \to \infty \quad \text{for the Gaussian}. \end{cases} \tag{12}$$

In Figs. 2 and 3, we show the graphics of the functions $s_1(x)$ and $s_2(x)$, respectively, for $g = 1, 2, 3, 4$ normalized so that their maximum is 1.

4. The bivariate case

In the bivariate case the second derivative is replaced by the Laplace operator and the model problem (2) turns into the following Poisson equation:

$$\begin{cases} \frac{\partial^2}{\partial x^2}u + \frac{\partial^2}{\partial y^2}u = f(x,y) & \text{for } (x,y) \in \Omega = (0,1) \times (0,1), \\ u(x,y) = g(x,y) & \text{for } (x,y) \in \partial\Omega. \end{cases}$$

By discretizing the square Ω with a grid of knots $z_{i,j} = (x_i, y_j) = (hi, hj)$, $i, j = 0, \ldots, n+1$, $h = 1/(n+1)$, and by imposing the collocation conditions:

$$\left(\frac{\partial^2}{\partial x^2}\phi(x,y) + \frac{\partial^2}{\partial y^2}\phi(x,y)\right)_{x=x_i, y=y_j} = f(x_i, y_j),$$

one arrives at a system of linear equations where the structured part of the matrix is a block Toeplitz matrix with Toeplitz blocks defined by the symbol

$$\begin{aligned} s(x,y) = c_{0,0} &+ 2\sum_{k=1}^{\infty}(c_{k,0}\cos(2\pi kx) + c_{0,k}\cos(2\pi ky)) \\ &+ 4\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} c_{k,j}\cos(2\pi kx)\cos(2\pi jy). \end{aligned} \tag{13}$$

The values of the coefficients $c_{i,j}$ are determined by the radial function $\phi(x,y)$ used to approximate the solution. In particular, with $g = c/h$ it holds

$$c_{i,j} = \begin{cases} \frac{1}{h}\frac{1}{(i^2+j^2+g^2)^{1/2}} + \frac{1}{h}\frac{g^2}{(i^2+j^2+g^2)^{3/2}} & \text{for the MQ,} \\ \frac{1}{h^3}\frac{i^2+j^2-2g^2}{(i^2+j^2+g^2)^{5/2}} & \text{for the IMQ,} \\ \frac{4}{g^4h^2}(i^2+j^2-g^2)\mathrm{e}^{-\frac{i^2+j^2}{g^2}} & \text{for the Gaussian.} \end{cases}$$

It is important to remark that the collocation matrix is the sum of a symmetric block Toeplitz matrix with Toeplitz blocks and a matrix of rank $2n$ which collects the boundary conditions.

Unlike in the one-dimensional case, the symbol $s(x,y)$ is not defined for the MQ: in fact, the series (13) is not convergent for $x = 1$ and $y = 1$. It can be easily verified that the series associated

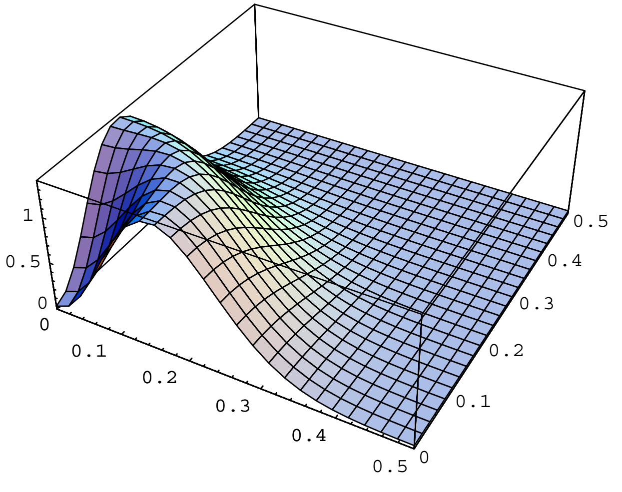

Fig. 4. Graph of the symbol $s_1(x,y)$ associated with the IMQ function for $g = 2$.

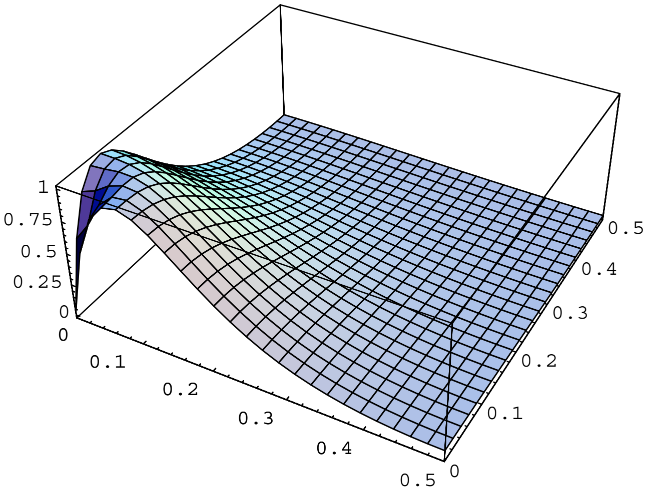

Fig. 5. Graph of the symbol $s_2(x, y)$ associated with the Gaussian function for $g = 2$.

with the IMQ and the Gaussian functions are convergent so that the Toeplitz matrix machinery can be in principle applied for these two classes of radial functions.

In Figs. 4 and 5, we report the graphs of the symbols $s_1(x, y)$, $s_2(x, y)$ for $g = 2$. We can see the same behavior of the one-dimensional analogs $s_1(x)$ and $s_2(x)$ of Figs. 2 and 3.

5. Numerical validation

We have numerically computed the ratio $\rho(g) = \max s(x)/\min s(x)$ and its asymptotic estimate $\gamma(g)$ given in Proposition 1 and in (12), for $g = 1, 2, 3, 4$, in the case of MQ, IMQ and Gaussian function, respectively. The results of this computation are reported in Table 1. We can see that the asymptotic estimates are very precise even for small values of g.

We have compared the values of $\gamma(g)$ with the actual condition numbers of the Toeplitz matrices T_n for several values of n. Table 2 reports the spectral condition number $\mu = \mu(T_n)$ of the Toeplitz matrix T_n, for different values of n in the case $g = 1$, $g = 2$, respectively. It is interesting to point out that in the MQ case with $g = 1, 2$ and IMQ case with $g = 1$, the asymptotic bounds are roughly reached for relatively small values of n whereas for the Gaussian case and for the IMQ with $g = 2$ the values of $\mu(T_n)$ are far from the asymptotic value even for moderately large values of n.

Table 1
Values of $\rho(g) = \max s(x)/\min s(x)$ and of its asymptotic estimate $\gamma(g)$ for the MQ, IMQ and Gaussian functions

g	MQ		IMQ		Gaussian	
	$\rho(g)$	$\gamma(g)$	$\rho(g)$	$\gamma(g)$	$\rho(g)$	$\gamma(g)$
1	4.73	5.21	8.4	6.5	4.2e2	3.6e2
2	8.1e1	8.5e1	1.2e3	1.2e3	6.5e14	6.5e14
3	1.5e3	1.6e3	3.4e5	3.6e5	7.8e35	7.8e35
4	3.1e4	3.2e4	1.2e8	1.2e8	4.4e65	4.4e65

Table 2
Values of the spectral condition number $\mu(T_n)$ for different values of n and g

n	MQ		IMQ		Gaussian	
	$g = 1$	$g = 2$	$g = 1$	$g = 2$	$g = 1$	$g = 2$
20	4.6	72.6	5.7	6.3	66	246
50	4.7	78.8	7.5	19.0	217	337
100	4.7	80.1	8.1	52.3	338	400
200	4.7	80.4	8.3	147	395	1543
400	4.7	81.0	8.3	375	413	6069

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