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Entire solutions to the Monge–Ampère equation

Lamia Bel Kenani Toukabri [a], Saoussen Kallel-Jallouli [b,*]

[a] *Departement de Mathématiques, Institut Préparatoire des Études d'Ingénieurs de Tunis, Tunisia*
[b] *Departement de Mathématiques, Faculté des Sciences de Tunis, Tunisia*

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ABSTRACT

We consider the Monge–Ampère equation $\det(D^2u) = \Psi(x, u, Du)$ in $\mathbb{R}^n$, $n \geqslant 3$, where Ψ is a positive function in $C^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$. We prove the existence of convex solutions, provided there exist a subsolution of the form $\underline{u} = a|x|^2$ and a superharmonic bounded positive function φ satisfying: $\Psi > (2a + \frac{\Delta\varphi}{n})^n$.

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1. Introduction

In this paper we study the existence of convex solutions to the Monge–Ampère equation

$$\det\left(D^2u\right) = \Psi(x, u, Du) \quad \text{in } \mathbb{R}^n, \tag{1}$$

where $\Psi(x, z, p)$ is a positive function in $C^2(\mathbb{R}^n \times \mathbb{R} \times \mathbb{R}^n)$, $Du = (u_1, \dots, u_n)$ denotes the gradient of u and $D^2u = \{u_{ij}\}$ denotes the hessian of u ($u_i = \frac{\partial u}{\partial x_i}$, $u_{ij} = \frac{\partial^2 u}{\partial x_i \partial x_j}$).

The Monge–Ampère equation on bounded domains has been studied by many authors (see for instance [1–5,8,9,11]) but very little is known when the domain is unbounded (see for instance [2,6,7]). When Ψ depends only on x, the problem was solved by K.S. Chou and X.J. Wang [6]. Here we generalize the latter work and prove an existence result of entire convex solutions provided there exist a subsolution of the form $\underline{u} = a|x|^2$ and a superharmonic bounded positive function φ satisfying: $\Psi > (2a + \frac{\Delta\varphi}{n})^n$. Since no entire bounded positive superharmonic function exists for $n \leqslant 2$ (see [12]), we assume that $n \geqslant 3$ in all this note. For $n \geqslant 4$, $\varphi(x) = \frac{1}{1+|x|^{n-2}}$ is an example of superharmonic bounded positive function given in [10]. So let

$$\psi^{\frac{1}{n}} = e^{\frac{2}{\Pi}\frac{\Delta\varphi}{n}\operatorname{arctg}(u^2+|p|^2)},$$

then we can easily verify that the assumptions above on ψ are all satisfied with $a = 1$.

* Corresponding author.
E-mail addresses: Lamia.Toukabri@ipeit.rnu.tn (L. Bel Kenani Toukabri), Saoussen.Kallel@fst.rnu.tn (S. Kallel-Jallouli).

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2. Main result

Using the C^2 estimates of the solution up to the boundary (see [1,7]) we prove the following theorem.

Theorem 2.1. *Suppose that the function $\underline{u} = a|x|^2$ is a subsolution of (1), that is*

$$\det D^2\underline{u} \geqslant \Psi(x, \underline{u}, D\underline{u}) \tag{2}$$

with a a positive large constant, and assume that Ψ is a C^2 function satisfying:

$$\Psi > \left(2a + \frac{\Delta\varphi}{n}\right)^n, \qquad \Psi_u \geqslant 0, \qquad \frac{|D\Psi| + |D^2\Psi|}{\Psi} < \infty, \tag{3}$$

where φ is a superharmonic bounded positive function. Then Eq. (1) admits at least one convex solution u satisfying:

$$a|x|^2 \leqslant u \leqslant a|x|^2 + \varphi, \quad u \in C^{2,\alpha}(K);\ \forall K \Subset \mathbb{R}^n,\ \forall 0 < \alpha < 1. \tag{4}$$

To prove Theorem 2.1 we shall proceed as follows. Suppose there exists a subsolution $\underline{u} = a|x|^2$. For $k \geqslant 1$, denote by $B_n(0,k)$ the ball in $\mathbb{R}^n$ of center the origin and radius k. We know that (see [1,4]), for any $k \geqslant 0$, the Dirichlet problem:

$$\begin{cases} \det(D^2u^k) = \Psi(x, u^k, Du^k) & \text{in } B_n(0,k), \\ u^k = \underline{u} & \text{on } \partial B_n(0,k) \end{cases} \tag{5}$$

has a unique solution $u^k \in C^{2,\alpha}(B_n(0,k))$, $\forall 0 < \alpha < 1$. Using the barrier constructions (see [1,7]) for estimating the second tangential and mixed derivatives at the boundary, we prove that bounds of the second derivatives of u^k are independent of k in all compact set of $\mathbb{R}^n$. Finally, standard Calabi's interior estimates for the third derivatives (see [4]) yield local uniform bounds of D^3u^k. Using a diagonal sequence argument, we get a subsequence $\{u^{k_i}\}_{i\geqslant 1}$, that converges locally in $C^{2,\alpha}$ norm to a strictly convex solution of our original problem.

In Section 3 we shall give some technical lemmas. In Section 4, we give the proof of Theorem 2.1.

3. Some technical lemmas

To prove uniform bound of the second derivative of u we shall return to work of L.A. Caffarelli, L. Niremberg and J. Spruck in [4], Bo Guan in [1], F. Finster and O.C. Schnürer in [7] and adapt to the situation of the theorem above that we prove in the next paragraph. By c we denote a constant independent of k which may change its value from line to line throughout the text.

Lemma 3.1. *Let $k \geqslant 1$. For any $x \in \partial B_n(0,k)$, set $\nu_k(x)$ the inner unit normal to $\partial B_n(0,k)$ at x and write any $y \in \mathbb{R}^n$ as*

$$y - x = y_\nu \nu_k(x) + y', \quad (y_\nu, y') \in \mathbb{R} \times \nu_k(x)^\perp,$$

then $\partial B_n(0,k) \cap B_n(x, \frac{1}{2})$ is given explicitly by an equation of the type

$$y_\nu = \rho_k(y'), \tag{6}$$

where $\rho_k \in C^\infty(B_n(0, \frac{1}{2}) \cap \nu_k(x)^\perp)$ and satisfies

$$\rho_k(0) = 0, \qquad D\rho_k(0) = 0, \qquad \left|D^2\rho_k(0)\right| \leqslant \frac{c}{k}, \qquad \left|D^3\rho_k\right|_{0,1,B_{n-1}(0,\frac{1}{2})} \leqslant \frac{c}{k^2} \tag{7}$$

with c a positive constant independent of k.

Proof. Let $x \in \partial B_n(0,k)$. Without loss of generality we may suppose that $x = (0,\dots,0,-k)$ and then $\nu_k(x) = -\frac{x}{|x|} = e_n = (0,\dots,0,1)$, and $\nu_k(x)^\perp = \mathbb{R}^{n-1}$. We write any $y \in \mathbb{R}^n$ as

$$y - x = y_{n,k}e_n + y'.$$

Set

$$\rho(y') = 1 - \sqrt{1 - |y'|^2}, \quad y' \in B_{n-1}(0,1),$$

and

$$\rho_k(y') = k\rho\left(\frac{y'}{k}\right),$$

where $B_{n-1}(0,1)$ is the unit open ball of $\mathbb{R}^{n-1}$. For $k \geqslant 1$, we have $B_n(x, \frac{1}{2}) \cap \partial B_n(0,k)$ is given by

$$y_{n,k} = \rho_k(y'), \quad y' \in B_n\left(0, \frac{1}{2}\right) \cap \mathbb{R}^{n-1} = B_{n-1}\left(0, \frac{1}{2}\right).$$

We have

$$\rho_k(0) = 0, \qquad D\rho_k(0) = 0, \qquad D^2\rho_k(0) = \left\{\frac{\delta_{ij}}{k}\right\}$$

and since

$$D^i\rho_k(y') = \frac{1}{k^{i-1}} D^i\rho\left(\frac{y'}{k}\right), \quad \forall i \geqslant 1,$$

it follows that

$$\left|D^2\rho_k(0)\right| \leqslant \frac{c}{k}, \qquad \left|D^3\rho_k\right|_{0,1,B_{n-1}(0,\frac{1}{2})} \leqslant \frac{c}{k^2}$$

with c is uniform in k. This completes the proof of Lemma 3.2. □

We shall use, in addition, the following lemmas. Let u^k be a solution of the Dirichlet problem (5).

Lemma 3.2 *(Estimation of u^k). Set $\overline{u}(x) = a|x|^2 + \varphi$. As $k \to \infty$, the function u^k converges locally uniformly to a convex function u. Moreover,*

$$\underline{u} \leqslant u^k \leqslant \overline{u} \quad \text{in } B_n(0,k),\ \forall k \geqslant 1.$$

Proof. Applying the arithmetic–geometric mean to the convex function u^k we deduce that

$$\Delta u^k \geqslant n\Psi^{\frac{1}{n}} > \Delta\overline{u},$$

and then using the maximum principle we obtain

$$\underline{u} \leqslant u^k \leqslant \overline{u} \quad \text{in } B_n(0,k).$$

Hence for $k_1 < k_2$,

$$u^{k_1} \leqslant u^{k_2} \quad \text{on } \partial B_n(0,k_1),$$

and again from the maximum principle,

$$u^{k_1} \leqslant u^{k_2} \quad \text{in } B_n(0,k_1).$$

We conclude that the sequence $\{u^k\}_{k\geqslant 1}$ is monotone. Its pointwise limit is convex and thus continuous. So it converges locally uniformly according to Dini's theorem. □

From now on we omit the index k and assume that u is a solution of (5).

Lemma 3.3 *(Estimation of Du^k in $\overline{B_n(0,k)}$). Let $u \in C^2(\overline{B_n(0,k)})$ be a locally convex solution to the Dirichlet problem* (5)*. Then*

$$|Du|_{0,\overline{B_n(0,k)}} \leqslant ck, \tag{8}$$

with c is uniform in k, and in all compact subset K of $\mathbb{R}^n$, we have $|Du|_{0,K}$ is uniformly bounded in k for k sufficiently large.

Moreover, for $k^2 - \frac{1}{4} \leqslant |x|^2 \leqslant k^2$, let $\nu = \frac{x}{|x|}$ and τ be a unit vector orthogonal to x then

$$\left|u_\nu(x) - \underline{u}_\nu(x)\right| \leqslant c, \tag{9}$$

$$\left|u_\tau(x)\right| \leqslant c. \tag{10}$$

Proof. Since u is locally strictly convex, $|Du|$ takes its maximum on the boundary. It suffices then to estimate $|Du|$ at the boundary. Tangential derivatives vanish there in view of Dirichlet boundary conditions. It suffices then to estimate u_ν the exterior normal derivative of u on $\partial B_n(0,k)$. Letting $x \in \partial B_n(0,k)$ we have

$$u_\nu(x) = \lim_{t\to 0^-} \frac{u(x+t\nu) - u(x)}{t}.$$

As $u(x) = \underline{u}(x)$, we have

$$\forall t < 0, \quad \frac{u(x+t\nu) - u(x)}{t} \leqslant \frac{\underline{u}(x+t\nu) - \underline{u}(x)}{t}.$$

Then

$$u_\nu(x) \leqslant \underline{u}_\nu(x) \quad \text{on } \partial B_n(0,k). \tag{11}$$

To estimate $u_\nu(x)$ from below we simply make use of the convexity of u. The exterior unit normal to $\partial B_n(0,k)$ at x being $\nu = \frac{x}{|x|}$. Using the convexity of u as well as the fact that $\underline{u}$ lies below u and $\underline{u}(x = k\nu) = u(x = k\nu)$, $\underline{u}(y = -k\nu) = u(y = -k\nu)$ we obtain

$$-\underline{u}_{-\nu}(y) \leqslant -u_{-\nu}(y) = u_\nu(y) \leqslant u_\nu(x) \leqslant \underline{u}_\nu(x),$$

we deduce that

$$\left|u_\nu(x)\right| \leqslant 2ak.$$

Now, for $\theta > 0$, denote

$$\tilde{U}_\theta = \left\{x/\bar{u}(x) < \theta\right\}, \qquad U_\theta = \left\{x/u(x) < \theta\right\}, \qquad \underline{U}_\theta = \left\{x/\underline{u}(x) < \theta\right\}.$$

These domains are all open, bounded and U_θ, $\underline{U}_\theta$ are convex subsets of $\mathbb{R}^n$. Moreover, according to the C^0 estimates in Lemma 3.2, there exists a sufficiently large k forthwith they satisfy

$$\tilde{U}_\theta \subset U_\theta \subset \underline{U}_\theta \subset B_n(0,k).$$

Let K be a compact subset of $\mathbb{R}^n$. We can find $\theta > 0$ and $k \geqslant 1$ such that $K \subset \tilde{U}_\theta$ and $\underline{U}_{2\theta} \subset B_n(0,k)$. As

$$\tilde{U}_\theta \subset U_\theta \Subset \underline{U}_{2\theta},$$

and u is convex, it is not difficult to deduce, using C^0 estimates in Lemma 3.2, that

$$\max_K |Du| \leqslant \max_{\partial U_\theta} |Du| \leqslant \max_{\partial \underline{U}_{2\theta}} \frac{u - \theta}{d(\partial \underline{U}_\theta, \partial \underline{U}_{2\theta})} \leqslant c.$$

Now, let $k^2 - \frac{1}{4} \leqslant |x|^2 \leqslant k^2$ and $\nu = \frac{x}{|x|}$. Using the convexity of u we have

$$\begin{aligned} \left(u(x) - a|x|^2\right) - \left(u\left(\left(|x|-1\right)\nu\right) - a\left|\left(|x|-1\right)\nu\right|^2\right) &\leqslant u_\nu(x) - 2a|x| + a \\ &= u_\nu(x) - \underline{u}_\nu(x) + a \\ &\leqslant \underline{u}_\nu(k\nu) - 2a\left(k - \frac{1}{2}\right) + a = 2a. \end{aligned}$$

The C^0 estimates of Lemma 3.2 imply that $|u(y) - a|y|^2| < |\varphi|_0$ in $\overline{B_n(0,k)}$ and then

$$\left|u_\nu(x) - \underline{u}_\nu(x)\right| \leqslant 2a + |\varphi|_0.$$

In order to derive (10), we consider u along the line segment $x + \lambda\tau$ parametrized by $\lambda \in [-\lambda_0, \lambda_0]$, $\lambda_0 = \sqrt{k^2 - |x|^2}$. The boundary values of u are $u(\pm\lambda_0) = \underline{u}(\pm\lambda_0)$. Thus using that u lies above $\underline{u}$ and is convex, we obtain the estimate

$$\underline{u}'(-\lambda_0) \leqslant u'(-\lambda_0) \leqslant u'(\lambda = 0) \leqslant u'(\lambda_0) \leqslant \underline{u}'(\lambda_0),$$

and thus

$$\left|u_\tau(x)\right| = \left|u'(\lambda = 0)\right| \leqslant \max\left\{\left|\underline{u}'(-\lambda_0)\right|, \left|\underline{u}'(\lambda_0)\right|\right\}.$$

As $\lambda_0 = \sqrt{k^2 - |x|^2} \leqslant \frac{1}{2}$ we obtain

$$\left|\underline{u}'(\pm\lambda_0)\right| = \left|2(x \pm \lambda_0\tau).\tau\right| = 2\lambda_0 \leqslant 1$$

and the proof of Lemma 3.3 is complete. □

Let x be a point on $\partial B_n(0,k)$, set $\nu_k(x)$ the inner unit normal to $\partial B_n(0,k)$ at x and write any $y \in \mathbb{R}^n$ as

$$y - x = y_\nu \nu_k(x) + y', \quad (y_\nu, y') \in \mathbb{R} \times \nu_k(x)^\perp.$$

Using Lemma 3.1, choosing $(\tau_1, \dots, \tau_{n-1})$ an orthonormal basis in $\nu_k(x)^\perp$ and writing $y' = \sum_{\alpha=1}^{n-1} y_\alpha \tau_\alpha$, we get that $\partial B_n(0,k) \cap B_n(x, \frac{1}{2})$ is given explicitly by an equation of the type

$$y_\nu = \rho_k(y') = \frac{1}{2} \sum_{1 \leqslant \alpha, \beta \leqslant n-1} B_{\alpha\beta} y_\alpha y_\beta + \text{cubic of } y' + O\left(|y'|^4\right), \tag{12}$$

where $O(|y'|^4) \leqslant \frac{c}{k^3}$ with c uniform in k.

Lemma 3.4 *(Tangential strict convexity of u^k). Let $u \in C^2(\overline{B_n(0,k)})$ be a locally convex solution to* (5)*. Then for k sufficiently large we have*

$$c_0 \leqslant \sum_{\alpha,\beta=1}^{n-1} u_{\alpha\beta}(x) \xi_\alpha \xi_\beta \leqslant c, \tag{13}$$

for any unit vector $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Where c_0 and c are positive constants uniform in k.

Proof. Since $u = \underline{u}$ on $\partial B_n(0,k)$ we have for $1 \leqslant \alpha, \beta \leqslant n-1$:

$$u_{\alpha\beta}(x) = \underline{u}_{\alpha\beta}(x) + (\underline{u}_\nu - u_\nu)(x) \rho_{\alpha\beta}(x). \tag{14}$$

Using Lemma 3.1, (9) in Lemma 3.3 and $|D^2 \underline{u}| = |2a\{\delta_{ij}\}| \leqslant c$, we obtain $|u_{\alpha\beta}(x)| \leqslant c$ with c a positive constant uniform in k.

Next we shall establish:

$$\sum_{\alpha,\beta<n} u_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geqslant c_0,$$

for any unit vector $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. Since

$$(u - \underline{u})_{\alpha\beta}(x) + (u - \underline{u})_\nu(x) \rho_{\alpha\beta}(x) = 0,$$

and according to (9) as well as

$$D^2 \underline{u} = 2a\{\delta_{ij}\}, \qquad \rho_{\alpha\beta}(x) = \frac{\delta_{\alpha\beta}}{k},$$

we obtain for k sufficiently large

$$\sum_{\alpha,\beta<n} u_{\alpha\beta}(x) \xi_\alpha \xi_\beta \geqslant 2a - \frac{c}{k} \geqslant 2a - 1,$$

for any unit vector $\xi = (\xi_1, \dots, \xi_{n-1}) \in \mathbb{R}^{n-1}$. The proof of Lemma 3.4 is then complete. □

Lemma 3.5 *(Estimates of the mixed second derivatives of u^k on $\partial B_n(0,k)$). If $u \in C^2(\overline{B_n(0,k)})$ is the locally strictly convex solution of* (5)*, then for $x \in \partial B_n(0,k)$,*

$$\left|u_{\alpha\nu}(x)\right| \leqslant c, \quad 1 \leqslant \alpha \leqslant n-1,$$

where the constant c is uniform in k.

Proof. Rewrite Eq. (1) in the form

$$\log\det\left(D^2 u\right) = \log \Psi(y, u, Du) \equiv f(y, u, Du),$$

and let $\mathcal{L}$ denote the linear operator defined by

$$\mathcal{L}\omega = u^{ij} \omega_{ij} - f_{p_i}(y, u, Du) \omega_i \quad \text{for } \omega \in C^2\left(\overline{B_n(0,k)}\right),$$

where $\{u^{ij}\}$ is the inverse matrix of $\{u_{ij}\}$ and $f_{p_i}(y,z,p) = \frac{\partial f}{\partial p_i}(y,z,p)$. For a fixed $\alpha < n$, consider the differential operator

$$T = \partial_\alpha + \sum_{\beta<n} B_{\alpha\beta}(y_\beta \partial_\nu - y_\nu \partial_\beta).$$

On $\partial B_n(0,k) \cap B_n(x,\sigma)$, $\sigma < k - \sqrt{k^2 - \frac{1}{4}}$, we have

$$\left|T(u-\underline{u})\right| \leqslant \left|(u-\underline{u})_\alpha + \left(\sum_{\beta<n} B_{\alpha\beta} y_\beta\right)(u-\underline{u})_\nu\right| + \left|y_\nu \sum_{\beta<n} B_{\alpha\beta}(u-\underline{u})_\beta\right|.$$

To estimate the first term of the last inequality we use

$$(u-\underline{u})_\alpha + \left(\sum_{\beta<n} B_{\alpha\beta} y_\beta\right)(u-\underline{u})_\nu = (u-\underline{u})_\alpha + \rho_\alpha(y')(u-\underline{u})_\nu + O\left(|y'|^2\right)(u-\underline{u})_\nu,$$

where, by (7), $O(|y'|^2) \leqslant \frac{c}{k^2}|y'|^2$.

Since $(u-\underline{u})_\alpha + \rho_\alpha(y')(u-\underline{u})_\nu = 0$ on $\partial B_n(0,k) \cap B_n(x,\sigma)$, and according to (9), (10) it follows

$$\left|(u-\underline{u})_\alpha + \left(\sum_{\beta<n} B_{\alpha\beta} y_\beta\right)(u-\underline{u})_\nu\right| \leqslant \frac{c}{k}|y'|^2.$$

By Lemma 3.3, the second term verifies

$$\left|y_\nu \sum_{\beta<n} B_{\alpha\beta}(u-\underline{u})_\beta\right| = \left|\rho(y') \sum_{\beta<n} B_{\alpha\beta}(u-\underline{u})_\beta\right| \leqslant c|y'|^2.$$

Consequently, we have

$$\left|T(u-\underline{u})\right| \leqslant c|y|^2 \quad \text{on } \partial B_n(0,k) \cap B_n(x,\sigma).$$

Following [4], we shall prove that

$$\left|\mathcal{L}T(u-\underline{u})\right| \leqslant c\left(1 + \sum u^{ii}\right) \quad \text{in } \overline{B_n(0,k) \cap B_n(x,\sigma)}.$$

Let $(\xi_1, \ldots, \xi_{n-1}, \xi_n = \nu)$ be an orthonormal basis in $\mathbb{R}^n$, then we have, using Einstein summation convention

$$\begin{aligned}
\mathcal{L}(Tu) &= u^{ij}(Tu)_{ij} - f_{p_i}(Tu)_i \\
&= T\left[\log\det\left(D^2u\right)\right] - f_{p_i}\left[u_{\alpha i} + \sum_{\beta<n} B_{\alpha\beta}(\delta_{\beta i}u_\nu + y_\beta u_{\nu i} - \delta_{i\nu}u_\beta - y_\nu u_{\beta i})\right] \\
&= T\left[f(y,u,Du)\right] - \left[f_{p_i}u_{\alpha i} + \sum_{\beta<n} B_{\alpha\beta}(y_\beta f_{p_i}u_{\nu i} - y_\nu f_{p_i}u_{\beta i})\right] \\
&\quad + \delta_{i\nu} f_{p_i} \sum_{\beta<n} B_{\alpha\beta}u_\beta - u_\nu f_{p_i} \sum_{\beta<n} B_{\alpha\beta}\delta_{\beta i} \\
&= f_\alpha + \sum_{\beta<n} B_{\alpha\beta}(y_\beta f_\nu - y_\nu f_\beta) + f_z u_\alpha + \sum_{\beta<n} B_{\alpha\beta}(y_\beta f_z u_\nu - y_\nu f_z u_\beta) \\
&\quad + f_{p_n} \sum_{\beta<n} B_{\alpha\beta}u_\beta - u_\nu \sum_{1\leqslant i<n} B_{\alpha i} f_{p_i}.
\end{aligned}$$

As

$$|u_\nu| \leqslant ck, \qquad B_{\alpha\beta} = \frac{\delta_{\alpha\beta}}{k}, \qquad |y-x| = y_\nu^2 + \sum_{\alpha=1}^{n-1} y_\alpha^2 < \frac{1}{2},$$

and using (9), (10) we obtain

$$\left|\mathcal{L}(Tu)\right| \leqslant c.$$

In another hand, we have

$$\left|\mathcal{L}T(u-\underline{u})\right| \leqslant |\mathcal{L}Tu| + |\mathcal{L}T\underline{u}| \leqslant c + |\mathcal{L}T\underline{u}|.$$

Since $T\underline{u} = \underline{u}_\alpha = 2ay_\alpha$,

$$\mathcal{L}T\underline{u} = -2af_{p_\alpha}.$$

Thus

$$|\mathcal{L}T\underline{u}| \leqslant c.$$

So,

$$\left|\mathcal{L}T(u-\underline{u})\right| \leqslant c \quad \text{in } \overline{B_n(0,k)\cap B_n(x,\sigma)}.$$

We shall employ a barrier function of the form

$$v = (u-\underline{u}) + td - Nd^2,$$

where d is the distance function to $\partial B_n(0,k)$, and t, N are positive constants to be determined. We have $d(y) = k - |y|$ is C^∞ smooth in $B_n(0,k)\cap B_n(x,\sigma)$.

The key ingredient is the following:

Lemma 3.6. *For N sufficiently large and t sufficiently small,*

$$\begin{aligned}&\mathcal{L}v \leqslant -\frac{a}{2}\Bigl(1+\sum u^{ii}\Bigr) \quad \textit{in } B_n(0,k)\cap B_n(x,\sigma),\\ &v \geqslant 0 \quad \textit{on } \partial\bigl(B_n(0,k)\cap B_n(x,\sigma)\bigr).\end{aligned} \tag{15}$$

Proof. We have

$$u^{ij}(u_{ij}-\underline{u}_{ij}) = tr\bigl(\{\delta_{ij}\}\bigr) - 2au^{ij}\delta_{ij} = n - 2a\sum_{i=1}^{n} u^{ii}.$$

Using (9), (10) it follows that

$$\mathcal{L}(u-\underline{u}) = u^{ij}(u_{ij}-\underline{u}_{ij}) - f_{p_i}(x,u,Du)(u_i-\underline{u}_i) \leqslant c - 2a\sum u^{ii},$$

where c is uniform in k.

Moreover it is easy to see that

$$\bigl|\mathcal{L}(d)\bigr| \leqslant c\Bigl(1+\sum u^{ii}\Bigr),$$

for some $c>0$ uniform in k. Thus

$$\mathcal{L}v \leqslant c + tc + (tc-2a)\sum u^{ii} - N\bigl(\mathcal{L}d^2\bigr) \quad \text{in } B_n(0,k)\cap B_n(x,\sigma).$$

Since

$$\mathcal{L}d^2 = 2d\mathcal{L}d + 2u^{ij}d_i d_j$$

it follows, in $B_n(0,k)\cap B_n(x,\sigma)$

$$\mathcal{L}v \leqslant c + tc + (tc-2a)\sum u^{ii} - 2N\bigl(d\mathcal{L}d + u^{ij}d_i d_j\bigr). \tag{16}$$

Furthermore, since $\{u^{ij}\}$ is positive definite,

$$\begin{aligned}u^{ij}d_i d_j &= \sum_{i=1}^{n} u^{ii}d_i^2 + 2\sum_{i<j} u^{ij}d_i d_j\\ &= u^{nn}d_n^2 + 2\sum_{\beta<n} u^{n\beta}d_n d_\beta + \sum_{1\leqslant i\leqslant n-1} u^{ij}d_i d_j\\ &\geqslant u^{nn}d_n^2 + 2\sum_{\beta<n} u^{n\beta}d_n d_\beta.\end{aligned}$$

Since $d_\nu(x)=1$, $d_\beta(x)=0$ for all $\beta<n$, we can find, for any $\delta>0$ a sufficiently small $\sigma<\delta$ such that $1+\frac{1}{\sqrt{2}} \geqslant d_\nu(y) \geqslant \frac{1}{\sqrt{2}}$ and $|d_\beta(y)| < \frac{\delta}{\sqrt{2}}$, $\forall y \in B_n(0,k)\cap B_n(x,\sigma)$. Then

$$\Bigl|\sum_{\beta<n} u^{n\beta}d_n d_\beta\Bigr| \leqslant \frac{\delta}{\sqrt{2}}\sum u^{ii},$$

and

$$u^{ij}d_id_j \geqslant u^{nn}d_n^2 + 2\sum_{\beta<n} u^{n\beta}d_nd_\beta \geqslant \frac{u^{nn}}{2} - c\delta\sum u^{ii} \quad \text{in } B_n(0,k)\cap B_n(x,\sigma),$$

with c is uniform in k.

Now, letting $\lambda_1 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of $\{u^{ij}\}$ we have $\sum u^{ii} = \sum \lambda_i^{-1}$, $u^{nn} \geqslant \lambda_n^{-1}$, and, by arithmetic–geometric mean,

$$\begin{aligned}\frac{a}{2}\sum u^{ii} + Nu^{nn} &\geqslant \frac{a}{2}\left(\sum_{i=1}^{n-1}\lambda_i^{-1} + N\lambda_n^{-1}\right)\\ &\geqslant \frac{na}{2}\left(N\lambda_1^{-1}\cdots\lambda_n^{-1}\right)^{\frac{1}{n}}\\ &\geqslant \frac{na}{2(|\Psi|_{0,\mathbb{R}^n}^{\frac{1}{n}})}N^{\frac{1}{n}} \equiv c_1N^{\frac{1}{n}}.\end{aligned}$$

Now, we fix $t>0$ sufficiently small so that the constant tc in (16) satisfies: $tc \leqslant \frac{a}{2}$ and fix N so that $c_1N^{\frac{1}{n}} \geqslant c+2a$. We obtain

$$\begin{aligned}\mathcal{L}v &\leqslant c + \frac{a}{2} - \frac{a}{2}\sum u^{ii} - Nu^{nn} + 2N(c\delta + dc)\sum u^{ii} + 2Ndc - a\sum u^{ii}\\ &\leqslant c + \frac{a}{2} - c_1N^{\frac{1}{n}} + 2N(c\delta + dc)\sum u^{ii} + 2Ndc - a\sum u^{ii}\\ &\leqslant -\frac{3a}{2} - a\sum u^{ii} + 4N\delta c\sum u^{ii} + 2N\delta c,\end{aligned}$$

if we require δ to satisfy $4Nc\delta \leqslant \frac{a}{2}$, we get

$$\mathcal{L}v \leqslant -\frac{a}{2}\Bigl(1+\sum u^{ii}\Bigr) \quad \text{in } B_n(0,k)\cap B_n(x,\sigma).$$

It remains to examine the value of v on $\partial(B_n(0,k)\cap B_n(x,\sigma))$.

On $\partial B_n(0,k)\cap B_n(x,\sigma)$ we have $v=0$. If we require, in addition, $N\sigma \leqslant t$, we get

$$v \geqslant td - Nd^2 \geqslant (t-N\sigma)d \geqslant 0 \quad \text{on } B_n(0,k)\cap\partial B_n(x,\sigma).$$

Now we fix σ sufficiently small and the proof of Lemma 3.6 is complete. □

We can now complete the proof of Lemma 3.5. Using Lemma 3.6, we have

$$\begin{aligned}\mathcal{L}\bigl(Av + B|y|^2 \pm T(u-\underline{u})\bigr) &= A\mathcal{L}(v) + 2B\sum u^{ii} - 2f_{p_i}y_i \pm \mathcal{L}\bigl(T(u-\underline{u})\bigr)\\ &\leqslant -\frac{a}{2}A + c - 2f_{p_i}y_i + \left(-\frac{a}{2}A + 2B\right)\sum u^{ii}.\end{aligned}$$

Consequently,

$$\mathcal{L}\bigl(Av + B|y|^2 \pm T(u-\underline{u})\bigr) \leqslant 0 \quad \text{in } B_n(0,k)\cap B_n(x,\sigma),$$

for A sufficiently large (depending on c, B, $\frac{|\Psi|_1}{\Psi_0}$).

Since $v \geqslant 0$ on $\partial(B_n(0,k)\cap B_n(x,\sigma))$ and

$$\bigl|T(u-\underline{u})\bigr| \leqslant c|y|^2 \quad \text{on } \partial\bigl(B_n(0,k)\cap B_n(x,\sigma)\bigr),$$

we can choose $A \gg B \gg 1$ so that

$$Av + B|x|^2 \pm T(u-\underline{u}) \geqslant 0 \quad \text{on } \partial\bigl(B_n(0,k)\cap B_n(x,\sigma)\bigr).$$

It follows from the maximum principle that

$$\bigl|T(u-\underline{u})\bigr| \leqslant Av + B|y|^2 \quad \text{in } B_n(0,k)\cap B_n(x,\sigma),$$

and according to (9)

$$\bigl|\partial_\nu T(u-\underline{u})(x)\bigr| \leqslant \partial_\nu\bigl(Av + B|y|^2\bigr)(x) = A\partial_\nu v(x) = A(u-\underline{u})_\nu(x) + tAd_\nu(x) \leqslant Ac,$$

with c is uniform in k. But

$$\partial_\nu T(u-\underline{u})(x)=u_{\nu\alpha}(x)-\underline{u}_{\nu\alpha}(x).$$

So,

$$\left|u_{\alpha\nu}(x)\right|\leqslant c.$$

This completes the proof of Lemma 3.5. □

Lemma 3.7 *(Estimation of $u^k_{\nu\nu}$ on $\partial B_n(0,k)$). Let $x\in\partial B_n(0,k)$ and $\nu=-\frac{x}{|x|}$. We have*

$$u_{\nu\nu}(x)<c.$$

Proof. We choose an orthonormal basis such that the submatrix $\{u_{\alpha\beta}\}$ is diagonal. We expand the determinant,

$$\begin{aligned}\Psi(x,u,Du)=\det\bigl(D^2u(x)\bigr)&=u_{\nu\nu}(x)\prod_{1\leqslant\alpha\leqslant n-1}u_{\alpha\alpha}(x)-\sum_{1\leqslant\gamma\leqslant n-1}u^2_{\gamma\nu}(x)\prod_{\alpha\neq\gamma<n}u_{\alpha\alpha}(x)\\&=\prod_{1\leqslant\alpha\leqslant n-1}u_{\alpha\alpha}(x)\left(u_{\nu\nu}(x)-\sum_{1\leqslant\gamma\leqslant n-1}u^2_{\gamma\nu}(x)\frac{1}{u_{\gamma\gamma}(x)}\right).\end{aligned}$$

Now we substitute in the estimates of Lemmas 3.4 and 3.5 to obtain

$$0\leqslant u_{\nu\nu}(x)\leqslant\frac{1}{\prod_{1\leqslant\alpha\leqslant n-1}u_{\alpha\alpha}}\left(\Psi(x,u,Du)+\sum_{1\leqslant\gamma\leqslant n-1}\frac{u^2_{\gamma\nu}(x)}{u_{\gamma\gamma}(x)}\right)\leqslant c.\qquad\square$$

4. Proof of Theorem 2.1

For each $k\geqslant 1$, we consider the Dirichlet problem (5). Using the fact that $\underline{u}$ is a locally strictly convex subsolution of (5), the C^∞ smoothness of boundary data in (5) allows us to deduce the existence of a unique solution to (5) satisfying

$$u_k\in C^{2,\alpha}\bigl(\overline{B_n(0,k)}\bigr)\quad\forall\alpha\in]0,1[$$

(see [1]). Furthermore, using Lemmas 3.2, 3.3, 3.4, 3.5 and 3.7 we deduce that:

$$\forall k\geqslant 1,\quad\left|D^2u_k\right|_{0,\overline{B_n(0,k)}}\leqslant c,\tag{17}$$

with c a positive constant independent of k.

Using Calabi's interior estimates for the third derivatives (see [4]) we deduce that

$$\left|D^3u_k\right|\leqslant\frac{\tilde{c}}{d(x,\partial B(0,k))},\quad k\geqslant 1,\ \text{in } B(0,k),$$

where $\tilde{c}$ is a positive constant depending only on the constant c given by (17).

Step 1. In $B_n(0,1)$ we have

$$\left|D^3u_k\right|\leqslant\frac{\tilde{c}}{d(\partial B_n(0,1),\partial B_n(0,2))},\quad\forall k\geqslant 2,$$

where $\tilde{c}$ is a positive constant independent of k.

Then, according to the C^0 and C^1 estimates of u in Lemmas 3.2 and 3.3 and using Lemma 6.36 in [8] we deduce that there exist a subsequent $(u_{\eta_1(k)})_{k\geqslant 1}$ of $(u_k)_{k\geqslant 1}$ and $v_1\in C^{2,\alpha}(\overline{B_n(0,1)})$ such that:

$$\lim_{k\to+\infty}|u_{\eta_1(k)}-v_1|_{2,\alpha,\overline{B_n(0,1)}}=0.$$

Step 2. As previously, from the sequence $(u_{\eta_1(k)})_{k\geqslant 1}$ we can extract a subsequent $(u_{\eta_2(k)})_{k\geqslant 1}$ such that $u_{\eta_2(k)}$ converges to v_2 in $C^{2,\alpha}(\overline{B_n(0,2)})$.

By uniqueness of the limit we have

$$v_1=v_2\quad\text{in }\overline{B_n(0,1)}.$$

So we construct iteratively a sequence $(u_{\eta_s(k)})_{k\geqslant 1}$ for all $s \geqslant 1$ such that

$$u_{\eta_s(k)} \to v_s \quad \text{in } C^{2,\alpha}\big(\overline{B_n(0,s)}\big),$$

and

$$\forall s \geqslant 1, \quad v_s = v_k, \quad \forall 1 \leqslant k \leqslant s, \text{ in } \overline{B_n(0,k)}.$$

We consider the sequence $(u_{\eta_k(k)})_{k\geqslant 1}$ obtained from $(u_{\eta_s(k)})_{s,k\geqslant 1}$ by the diagonal process. $(u_{\eta_k(k)})_{k\geqslant 1}$ is a subsequent of $(u_{\eta_s(k)})_{k\geqslant 1}$ for all $s \geqslant 1$. Therefore,

$$u_{\eta_k(k)} \to v_s \quad \text{when } k \to \infty, \text{ in } C^{2,\alpha}\big(\overline{B_n(0,s)}\big), \ \forall s \geqslant 1. \tag{18}$$

Thus $u_{\eta_k(k)}$ converges locally to u in $C^{2,\alpha}$ norm, with $u = v_s$ in $\overline{B_n(0,s)}$ for $s \geqslant 1$. Since Ψ, det are continuous then when passing to the limit we obtain u satisfies

$$\det\big(D^2u\big) = \Psi(x,u,Du) \quad \text{in } \mathbb{R}^n, \tag{19}$$

that is u is a solution in $\mathbb{R}^n$ of (1). Moreover, from the fact that $D^2u_{\eta_k(k)}$ is positive on $\overline{B_n(0,s)}$, $\forall k \geqslant s$, when passing to the limit we deduce that $D^2u = D^2v_s$ is nonnegative in $\overline{B_n(0,s)}$, $\forall s \geqslant 1$. Using (19) and $\Psi > 0$ we conclude that D^2u is positive in $\mathbb{R}^n$. Thus u is strictly convex in $\mathbb{R}^n$. In another hand, since $\forall s \geqslant 1$, $\forall k \geqslant s$, $\overline{u} \geqslant u_{\eta_k(k)} \geqslant \underline{u}$ on $\overline{B_n(0,s)}$, we deduce

$$\overline{u} \geqslant u \geqslant \underline{u} \quad \text{in } \mathbb{R}^n. \tag{20}$$

That completes the proof of Theorem 2.1.

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