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Construction of a local and global Lyapunov function for discrete dynamical systems using radial basis functions

Peter Giesl

Department of Mathematics, University of Sussex, Falmer, Brighton, BN1 9RF, UK

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Abstract

The basin of attraction of an asymptotically stable fixed point of the discrete dynamical system given by the iteration $x_{n+1} = g(x_n)$ can be determined through sublevel sets of a Lyapunov function. In Giesl [On the determination of the basin of attraction of discrete dynamical systems. J. Difference Equ. Appl. 13(6) (2007) 523–546] a Lyapunov function is constructed by approximating the solution of a difference equation using radial basis functions. However, the resulting Lyapunov function is non-local, i.e. it has no negative discrete orbital derivative in a neighborhood of the fixed point. In this paper we modify the construction method by using the Taylor polynomial and thus obtain a Lyapunov function with negative discrete orbital derivative both locally and globally.

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1. Introduction

In this paper we study the basin of attraction of a fixed point of the general discrete dynamical system given by the iteration $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geqslant 1$ and $d \in \mathbb{N}$. While existence and exponential asymptotic stability of a fixed point $\overline{x}$ can be checked straightforward, the determination of its basin of attraction $A(\overline{x})$ consisting of all initial points such that their

E-mail address: p.a.giesl@sussex.ac.uk.

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iterates converge to $\overline{x}$, is a difficult problem, since it involves global information as opposed to local information for the stability. A common way to analyze the basin of attraction is the method of a Lyapunov function.

A Lyapunov function $L \in C^0(\mathbb{R}^d, \mathbb{R})$ is a function with negative discrete orbital derivative $L'(x) < 0$ for all $x \in B\setminus\{\overline{x}\}$, where B is an open neighborhood of the fixed point. The discrete orbital derivative is defined by $L'(x) := L(g(x)) - L(x)$. Then connected and bounded sublevel sets O_R, i.e. $L(x) < R$ for all $x \in O_R$ and $L(x) = R$ for all $x \in \partial O_R$, are subsets of the basin of attraction of $\overline{x}$ supposed that $L'(x) < 0$ holds for all $x \in \overline{O_R}\setminus\{\overline{x}\}$.

The existence of Lyapunov functions has been established by converse theorems which, however, offer no construction method for Lyapunov functions. One of these converse theorems proves the existence of the Lyapunov function $V \in C^\sigma(A(\overline{x}), \mathbb{R})$ satisfying $V'(x) = -\|x - \overline{x}\|^2$, cf. [11,4, Theorem 2.8]. In [4], this function V was approximated as the solution of a linear difference equation using radial basis functions. For the approximation v one chooses the coefficients of a certain ansatz function such that the difference equation is satisfied for all points of a grid X_N, i.e. $v'(x_j) = -\|x_j - \overline{x}\|^2$ for all $x_j \in X_N$. The approximation then satisfies the error estimate $|V'(x) - v'(x)| \leqslant \varepsilon^2$, i.e. $v'(x) \leqslant -\|x - \overline{x}\|^2 + \varepsilon^2$, where ε depends on the density of the grid. This ensures $v'(x) < 0$ if $x \notin \overline{B_\varepsilon(\overline{x})}$, i.e. we obtain a non-local Lyapunov function. The local part was solved in [4] by using a local Lyapunov function, which can be obtained by a Lyapunov function of the linearized equation $x_{n+1} = Dg(\overline{x})x_n$, where Dg denotes the Jacobian matrix of first derivatives.

Other methods for the construction of Lyapunov functions for discrete and continuous dynamical systems consider special Lyapunov functions like quadratic, polynomial, piecewise linear, piecewise quadratic or polyhedral. Julián [13] approximated the differential equation by a piecewise linear right-hand side and constructed a piecewise linear Lyapunov function using linear programming (linear optimization). Hafstein [9], improved this ansatz and constructed a piecewise linear Lyapunov function for the original nonlinear system also using linear programming. The resulting Lyapunov functions are not smooth since they are piecewise linear or quadratic, on the other hand they require less smoothness than our approach and can also be used for control problems. For these methods a triangulation of the space is necessary which gets more complicated for higher dimensions. Johansen [12] used a family of smooth basis functions and determines the parameters by convex optimization.

A different method deals with the Zubov equation and computes a solution of this partial differential equation. In a similar approach to Zubov's method, Vannelli and Vidyasagar [17] use a rational function as Lyapunov function candidate and present an algorithm to obtain a maximal Lyapunov function in the case that f is analytic. In Camilli et al. [2], Zubov's method was extended to control problems in order to determine the robust domain of attraction. The corresponding generalized Zubov equation is a Hamilton–Jacobi–Bellmann equation. This equation has a viscosity solution which can be approximated using standard techniques after regularization at the equilibrium, e.g. one can use piecewise affine approximating functions and adaptive grid techniques, cf. Grüne [7]. The error estimate here is given in terms of $|V(x) - v(x)|$, where V denotes the regularized Lyapunov function and v its approximation, and not in terms of the orbital derivative as in our approach. In [8] an ISDS Lyapunov function for control problems is approximated via set orientated methods.

We use radial basis functions to solve the difference equation $V'(x) = -\|x - \overline{x}\|^2$. The main advantage of this method is that it is meshless, i.e. no triangulation of the space $\mathbb{R}^d$ is needed. Other methods first generate a triangulation of the space, use functions on each part of the triangulation, e.g. affine functions as in some examples discussed above, and then patch them together obtaining a global function. The resulting function is not very smooth and the method is not very effective in higher space dimensions. Moreover, the interpolation problem stated by radial basis functions is always uniquely solvable and we can choose scattered grid points. Radial basis functions give smooth approximations, but at the same time require smooth functions that are approximated.

In this paper, we seek to construct a Lyapunov function v which has negative discrete orbital derivative also near $\overline{x}$ and thus is local and global. To achieve this goal, we calculate a Taylor polynomial-like function $\mathrm{n}(x)$ of V, such that the function $W(x) := \frac{V(x)}{\mathrm{n}(x)}$ for $x \in A(\overline{x})\setminus\{\overline{x}\}$ and $W(\overline{x}) := 1$ is a smooth function. Then we approximate the function $W(x)$ satisfying the linear difference equation $\mathrm{n}'(x)W(x) + \mathrm{n}(x)W'(x) = -\|x - \overline{x}\|^2$ by an approximation w, which satisfies this equation for all points of a grid and, additionally, $w(\overline{x}) = 1$. The radial basis function chosen for this mixed approximation will be of the class of Wendland's compactly supported radial basis functions. Local and global error estimates of the approximation ensure that $v(x) = \mathrm{n}(x)w(x)$ has negative discrete orbital derivative in particular near $\overline{x}$, i.e. v is a local and global Lyapunov function. Hence, connected sublevel sets of v are subsets of the basin of attraction of $\overline{x}$. We prove in this paper that each open, bounded and connected subset of the basin of attraction can be covered by a sublevel set of v, when choosing the grid points properly. The method can be extended to more complicated attractors such as periodic orbits $\overline{x}_1, \ldots, \overline{x}_m$ using the local information given by $Dg(\overline{x}_1), \ldots, Dg(\overline{x}_m)$.

For continuous dynamical systems given by an autonomous ordinary differential equation $\dot{x} = f(x)$ a similar method of constructing Lyapunov functions was established in [3]; improved error estimates were obtained in [6]. The construction of a global and local Lyapunov function similar to this paper was presented in [5].

The paper is organized as follows: in Section 2 we recall the properties of the Lyapunov function V and prove properties of its Taylor polynomial-like function $\mathrm{n}(x)$ and the function $W(x) = \frac{V(x)}{\mathrm{n}(x)}$. Section 3 deals with the approximation w of W using radial basis functions. In Section 4 we derive local and global error estimates for w, which are used in Section 5 to prove the main results: setting $v(x) = \mathrm{n}(x)w(x)$, the function v is a local and global Lyapunov function, i.e. $v'(x) < 0$ holds for all $x \in B\setminus\{\overline{x}\}$, cf. Theorem 5.1, and each open, bounded and connected subset of the basin of attraction can be covered by a sublevel set of v, cf. Theorem 5.2. In Section 6 we give two examples: a comparison with the method in [4] (two-dimensional example) and a model for the demand for education, cf. [15] (three-dimensional example).

2. The Lyapunov function V and its Taylor polynomial

Throughout the paper we consider the discrete dynamical system given by the iteration

$$x_{n+1} = g(x_n),$$

where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geqslant 1$ and $n \in \mathbb{N}$. The flow of the dynamical system is denoted by $S_n x_0 := x_n$, $n \in \mathbb{N}_0$. Denote $B_\varepsilon(y) = \{x \in \mathbb{R}^d \mid \|x - y\| < \varepsilon\}$ for $\varepsilon > 0$ and $y \in \mathbb{R}^d$.

Let us recall some basic definitions from dynamical systems: A point $\overline{x} \in \mathbb{R}^d$ is called a **fixed point** if $g(\overline{x}) = \overline{x}$. A fixed point $\overline{x}$ is called

- **stable** if for all $\varepsilon > 0$ there is a $\delta > 0$ such that $S_n x \in B_\varepsilon(\overline{x})$ holds for all $n \in \mathbb{N}_0$ and for all $x \in B_\delta(\overline{x})$.
- **attractive** if there is a $\delta' > 0$ such that $\lim_{n\to\infty} \|S_n x - \overline{x}\| = 0$ holds for all $x \in B_{\delta'}(\overline{x})$.

- **exponentially attractive** if there is a $\delta' > 0$ and a $\mu > 0$ such that $\lim_{n\to\infty} \|S_n x - \overline{x}\| \exp(\mu n) = 0$ holds for all $x \in B_{\delta'}(\overline{x})$.
- **(exponentially) asymptotically stable**, if it is both stable and (exponentially) attractive.

If all eigenvalues λ of the Jacobian matrix of the first derivatives $Dg(\overline{x})$ satisfy $|\lambda| < 1$, then $\overline{x}$ is exponentially asymptotically stable.

A set $O \subset \mathbb{R}^d$ is called **positively invariant** if $S_n x \in O$ holds for all $n \in \mathbb{N}_0$ and all $x \in O$.

We assume throughout the paper that $\overline{x} \in \mathbb{R}^d$ is a fixed point and that all eigenvalues λ of $Dg(\overline{x})$ satisfy $|\lambda| < 1$.

In the following theorem a sufficient condition for the determination of a subset of the basin of attraction is given through the Lyapunov function L, cf. [10,11,4, Theorem 2.2].

Theorem 2.1. *Let $\overline{x}$ be an asymptotically stable fixed point of the discrete dynamical system $x_{n+1} = g(x_n)$, let $\overline{x} \in O \subset \mathbb{R}^d$ be an open, bounded and connected set. Let $L \in C^0(\mathbb{R}^d, \mathbb{R})$ be a function and $R^* \in \mathbb{R}$ such that*

1. *$L(x) < R^*$ holds for all $x \in O$ and $L(x) = R^*$ holds for all $x \in \partial O$.*
2. *$L'(x) < 0$ holds for all $x \in \overline{O} \setminus \{\overline{x}\}$, where $L'(x) := L(g(x)) - L(x)$ denotes the discrete orbital derivative.*

Then $\overline{O}$ is positively invariant and $\overline{O} \subset A(\overline{x})$ holds.

The following theorem ensures the existence and smoothness of a special Lyapunov function, cf. [11, Theorem 49.3] for the existence and [4, Theorem 2.8] with $p(x) = -\|x - \overline{x}\|^2$ for the smoothness. The theorem holds true for other right-hand sides $p(x)$, but for the Taylor polynomial, cf. Definition 2.3, and its calculation, cf. Remark 2.5, we need a quadratic form. Hence, instead of $-\|x - \overline{x}\|^2$ we could choose $-(x - \overline{x})^T C (x - \overline{x})$ with any positive definite matrix C, but for simplicity we restrict ourselves to $C = I$.

Theorem 2.2 (*Existence of V*). *Let $\overline{x}$ be a fixed point of $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geqslant 1$. Let $|\lambda| < 1$ hold for all eigenvalues of $Dg(\overline{x})$.*

Then there exists a function $V \in C^\sigma(A(\overline{x}), \mathbb{R})$ with $V(\overline{x}) = 0$ such that

$$V'(x) = -\|x - \overline{x}\|^2$$

holds for all $x \in A(\overline{x})$. The set $K_R := \{x \in A(\overline{x}) \mid V(x) \leqslant R\}$ is compact in $\mathbb{R}^d$ for all $R \geqslant 0$.

In the following Definition 2.3 we define the function $\mathfrak{h}$ which turns out to be the Taylor polynomial of V of order P, cf. Lemma 2.4.

Definition 2.3 (*Definition of* $\mathfrak{h}$). Let $\sigma \geqslant P \geqslant 2$. Let $\mathfrak{h}$ be the function

$$\mathfrak{h}(x) := \sum_{2 \leqslant |\alpha| \leqslant P} c_\alpha (x - \overline{x})^\alpha \tag{2.1}$$

$$\text{such that } \mathfrak{h}'(x) = \mathfrak{h}(g(x)) - \mathfrak{h}(x) = -\|x - \overline{x}\|^2 + o(\|x - \overline{x}\|^P). \tag{2.2}$$

Lemma 2.4. *There is one and only one function $\mathfrak{h}$ of the form* (2.1) *which satisfies* (2.2). *The function $\mathfrak{h}$ is the Taylor polynomial of V, cf. Theorem* 2.2, *of order P.*

The proof of the lemma can be found in Appendix B. Let us explain how to calculate the function $\mathfrak{h}$.

Remark 2.5. Eq. (2.2) can be solved by plugging the ansatz (2.1) into (2.2) and replacing g by its Taylor polynomial of order $P-1$. Then (2.2) becomes

$$\sum_{2\leqslant|\alpha|\leqslant P} c_\alpha \left[\left(Dg(\overline{x})(x-\overline{x}) + \sum_{2\leqslant|\beta|\leqslant P-1} \frac{\partial^\beta g(\overline{x})}{\beta!}(x-\overline{x})^\beta \right)^\alpha - (x-\overline{x})^\alpha \right]$$
$$= -\|x-\overline{x}\|^2 + o(\|x-\overline{x}\|^P). \tag{2.3}$$

The difference of g to its Taylor polynomial is of order $o(\|x-\overline{x}\|^{P-1})$ and thus the difference of the whole left-hand side of (2.3) to $\mathfrak{h}'(x)$ is of order $o(\|x-\overline{x}\|^P)$.

We add a high order polynomial $M\|x-\overline{x}\|^{2H}$ to $\mathfrak{h}$ in order to obtain a function $\mathfrak{n}$ for which $\mathfrak{n}(x) > 0$ holds for all $x \neq \overline{x}$.

Definition 2.6 (*Definition of* $\mathfrak{n}$). Let $\sigma \geqslant P \geqslant 2$ and let $\mathfrak{h}$ be as in Definition 2.3. Let $H := \lfloor \frac{P}{2} \rfloor + 1$ and $M \geqslant 0$ and define

$$\mathfrak{n}(x) = \mathfrak{h}(x) + M\|x-\overline{x}\|^{2H}$$
$$= \sum_{2\leqslant|\alpha|\leqslant P} c_\alpha (x-\overline{x})^\alpha + M\|x-\overline{x}\|^{2H}. \tag{2.4}$$

Choose the constant M so large that $\mathfrak{n}(x) > 0$ holds for all $x \neq \overline{x}$. Note that $\mathfrak{n} \in C^\infty(\mathbb{R}^d, \mathbb{R})$.

This is possible since for $x \to \overline{x}$ the leading term is $\mathfrak{h}_2(x) = (x-\overline{x})^T B(x-\overline{x})$ which is a positive definite quadratic form, and for $x \to \infty$ the leading term is $M\|x-\overline{x}\|^{2H}$; for details cf. [3, Definition 2.56].

In Proposition 2.7 we summarize some properties of $\mathfrak{n}(x)$ and $W(x) := \frac{V(x)}{\mathfrak{n}(x)}$.

Proposition 2.7. *Let* $\sigma \geqslant P \geqslant 2$ *and let* $\mathfrak{n}$ *be as in Definition* 2.6. *Then*

1. $\mathfrak{n}(x) > 0$ *holds for all* $x \neq \overline{x}$.
2. *For each compact set* K, *there is a constant* $C > 0$ *such that* $\mathfrak{n}(x) \leqslant C\|x-\overline{x}\|^2$ *holds for all* $x \in K$.
3. $W(x) = \frac{V(x)}{\mathfrak{n}(x)} \in C^{P-2}(A(\overline{x}), \mathbb{R})$, *where* $W(\overline{x}) = 1$, *and* $\partial_x^\alpha W(\overline{x}) = 0$ *for all* $1 \leqslant |\alpha| \leqslant P-2$.

Proof. The proposition can be proved as in the continuous case, cf. [3, Proposition 2.58]. □

Example 2.8. Consider the difference equation

$$\begin{cases} x_{n+1} = \frac{1}{2}x_n + x_n^2 - y_n^2, \\ y_{n+1} = -\frac{1}{2}y_n + x_n^2 \end{cases} \tag{2.5}$$

with fixed point $\overline{x} = (0, 0)$. This is Example 1 of [4] and will serve as an example in Section 6.1. We calculate the polynomial $\mathfrak{h}(x, y)$ and start with the terms $\mathfrak{h}_2(x, y)$ of second order. By the proof of Lemma 2.4, $\mathfrak{h}_2(x, y) = (x, y)B\begin{pmatrix} x \\ y \end{pmatrix}$, where B is the solution of $Dg(0,0)^T B Dg(0,0) - B = -I$. In our case $B = \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}$, i.e. $\mathfrak{h}_2(x, y) = \frac{4}{3}x^2 + \frac{4}{3}y^2$, cf. [4].

For the terms of order three, $\mathfrak{h}_3(x,y) = ax^3 + bx^2y + cxy^2 + dy^3$, we consider (2.2) with $P = 3$

$$\begin{aligned}
&-x^2 - y^2 + o(\|(x,y)\|^3)\\
&= (\mathfrak{h}_2 \circ g)(x,y) + (\mathfrak{h}_3 \circ g)(x,y) - \mathfrak{h}_2(x,y) - \mathfrak{h}_3(x,y)\\
&= \tfrac{4}{3}\left(\tfrac{1}{2}x + x^2 - y^2\right)^2 + \tfrac{4}{3}\left(-\tfrac{1}{2}y + x^2\right)^2 + a\left(\tfrac{1}{2}x + x^2 - y^2\right)^3\\
&\quad + b\left(\tfrac{1}{2}x + x^2 - y^2\right)^2\left(-\tfrac{1}{2}y + x^2\right) + c\left(\tfrac{1}{2}x + x^2 - y^2\right)\left(-\tfrac{1}{2}y + x^2\right)^2\\
&\quad + d\left(-\tfrac{1}{2}y + x^2\right)^3 - \tfrac{4}{3}x^2 - \tfrac{4}{3}y^2 - ax^3 - bx^2y - cxy^2 - dy^3\\
&= -x^2 - y^2 + x^3\left(\tfrac{4}{3} + \tfrac{1}{8}a - a\right) + x^2y\left(-\tfrac{4}{3} - \tfrac{1}{8}b - b\right) + xy^2\left(-\tfrac{4}{3} + \tfrac{1}{8}c - c\right)\\
&\quad + y^3\left(-\tfrac{1}{8}d - d\right) + o(\|(x,y)\|^3),
\end{aligned}$$

i.e. $a = \frac{32}{21}$, $b = -\frac{32}{27}$, $c = -\frac{32}{21}$ and $d = 0$. Thus, $\mathfrak{h}_3(x,y) = \frac{32}{21}x^3 - \frac{32}{27}x^2y - \frac{32}{21}xy^2$.

In a similar way we calculate the terms of order 4 and 5 and obtain for $P = 5$

$$\begin{aligned}
\mathfrak{h}(x,y) &= \tfrac{4}{3}x^2 + \tfrac{4}{3}y^2 + \tfrac{32}{21}x^3 - \tfrac{32}{27}x^2y - \tfrac{32}{21}xy^2\\
&\quad + \tfrac{10\,624}{2835}x^4 + \tfrac{4096}{3213}x^3y - \tfrac{1408}{315}x^2y^2 - \tfrac{256}{459}xy^3 + \tfrac{64}{35}y^4\\
&\quad + \tfrac{3\,657\,728}{874\,045}x^5 + \tfrac{1\,416\,704}{530\,145}x^4y - \tfrac{65\,536}{9639}x^3y^2 - \tfrac{1\,581\,056}{530\,145}x^2y^3 + \tfrac{34\,304}{9765}xy^4 + \tfrac{2560}{5049}y^5.
\end{aligned} \tag{2.6}$$

A function $\mathfrak{n}$ such that $\mathfrak{n}(x,y) > 0$ holds for all $(x,y) \neq (0,0)$ is given by

$$\mathfrak{n}(x,y) = \mathfrak{h}(x,y) + 2(x^2 + y^2)^3. \tag{2.7}$$

3. Approximation of W using radial basis functions

In this section we describe the approximation of the function W by w using radial basis functions. For an overview on radial basis function, cf. [19,1].

Since the 1970s radial basis functions have been used to interpolate scattered data. Given points $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$ and values $f_1, \ldots, f_N$ the goal is to find a smooth function $f\colon \mathbb{R}^d \to \mathbb{R}$ interpolating the data, i.e. $f(x_j) = f_j$ for all $j = 1, \ldots, N$. Using the following ansatz for the interpolating function $f(x) = \sum_{k=1}^N \beta_k \Psi(x - x_k)$, where Ψ is a fixed radial basis function, the interpolating conditions become the system of linear equations $A\beta = \alpha$, where $\alpha = (f_1, \ldots, f_N)$ and $A = (a_{jk})_{j,k=1,\ldots,N}$ with $a_{jk} = \Psi(x_j - x_k)$. The choice of the radial basis function $\Psi(x) = \psi(\|x\|)$ is such that the matrix A is positive definite for a general choice of the grid points. In this article we use Wendland's compactly supported radial basis functions.

Radial basis functions can also be used to interpolate the values of general linear operators, e.g. linear differential operators; in the above example of the interpolation of function values the linear operator was the identity. Hence, they can be used to solve partial differential equations, also with boundary conditions, cf. [6]. In this article, however, we will use the linear operator $D_{\mathfrak{n}}$ defined in (3.1) at the points $x_1, \ldots, x_N$ and the identity at $\overline{x}_0$.

Recall that the Lyapunov function $V(x) = \mathfrak{n}(x)W(x)$ satisfies $V'(x) = -\|x - \overline{x}\|^2$. Hence, the function W satisfies the equation $\mathfrak{n}(g(x))W(g(x)) - \mathfrak{n}(x)W(x) = -\|x - \overline{x}\|^2$, or in other words

$$D_{\mathfrak{n}}W(x) = -\|x - \overline{x}\|^2 \tag{3.1}$$

for all $x \in A(\overline{x})$, where $D_{\mathfrak{n}}W(x) := \mathfrak{n}(g(x))W(g(x)) - \mathfrak{n}(x)W(x)$ is a linear operator.

For the approximating function $w\colon \mathbb{R}^d \to \mathbb{R}$ we choose a radial basis function $\Psi(x) := \psi(\|x\|)$ and a grid $X_N = \{x_1, \ldots, x_N\} \subset \mathbb{R}^d$, and we make the following ansatz:

$$w(x) = \beta_0 \Psi(x - \overline{x}) + \sum_{k=1}^{N} \beta_k (\delta_{x_k} \circ D_{\text{n}})^y \Psi(x - y), \tag{3.2}$$

where $\beta_k \in \mathbb{R}$ and δ denotes Dirac's δ-distribution, i.e. $\delta_{x^*} f(x) = f(x^*)$. The ansatz is a mixed approximation including the linear operator D_{n} for the grid points $x_1, \ldots, x_N$ and the linear operator $D^0 = \text{id}$ for the grid point $\overline{x}$.

The coefficients β_k are determined by the claim that

$$w(\overline{x}) = W(\overline{x}) = 1 \tag{3.3}$$

$$\text{and} \quad (\delta_{x_j} \circ D_{\text{n}})^x w(x) = (\delta_{x_j} \circ D_{\text{n}})^x W(x) \tag{3.4}$$

holds for all $j = 1, \ldots, N$, i.e. $D_{\text{n}} W(x_j) = D_{\text{n}} w(x_j)$. Plugging the ansatz (3.2) into (3.3) and (3.4) one obtains with (3.1)

$$1 = \beta_0 \Psi(0) + \sum_{k=1}^{N} \beta_k (\delta_{x_k} \circ D_{\text{n}})^y \Psi(\overline{x} - y) \quad \text{and}$$

$$\begin{aligned} -\|x_j - \overline{x}\|^2 &= (\delta_{x_j} \circ D_{\text{n}})^x W(x) \\ &= (\delta_{x_j} \circ D_{\text{n}})^x w(x) \\ &= (\delta_{x_j} \circ D_{\text{n}})^x \Psi(x - \overline{x}) \beta_0 + \sum_{k=1}^{N} (\delta_{x_j} \circ D_{\text{n}})^x (\delta_{x_k} \circ D_{\text{n}})^y \Psi(x - y) \beta_k \end{aligned}$$

for all $j = 1, \ldots, N$, since D_{n} is a linear operator. This is equivalent to the following system of linear equations with interpolation matrix $A = (a_{jk})_{j,k=0,\ldots,N}$ and vector $\alpha = (\alpha_j)_{j=0,\ldots,N}$, cf. also Proposition 3.5:

$$A\beta = \alpha,$$

$$\text{where} \quad a_{jk} = (\delta_{x_j} \circ D_{\text{n}})^x (\delta_{x_k} \circ D_{\text{n}})^y \Psi(x - y) \quad \text{for } j, k \geqslant 1, \tag{3.5}$$

$$a_{0k} = a_{k0} = (\delta_{x_k} \circ D_{\text{n}})^y \Psi(\overline{x} - y) \quad \text{for } k \geqslant 1, \tag{3.6}$$

$$a_{00} = \Psi(0) \tag{3.7}$$

$$\text{and} \quad \alpha_j = -\|x_j - \overline{x}\|^2 \quad \text{for } j \geqslant 1,$$

$$\alpha_0 = 1.$$

A clearly is a symmetric matrix. For existence and uniqueness of the solution β, the interpolation matrix A must have full rank. We will even obtain a positive definite matrix A for grids which include no fixed or periodic point, cf. Proposition 3.4.

The radial basis functions Ψ used in this paper will be from the class of Wendland functions which were introduced by Wendland [18]. They have compact support and are polynomials on their support.

Definition 3.1 (*Wendland functions [18]*). Let $l \in \mathbb{N}$, $k \in \mathbb{N}_0$. We define by recursion $\psi_{l,0}(r) = (1 - r)_+^l$ and $\psi_{l,k+1}(r) = \int_r^1 t \psi_{l,k}(t)\, dt$ for $r \in \mathbb{R}_0^+$. Here we set $x_+ = x$ for $x \geqslant 0$ and $x_+ = 0$ for $x < 0$.

For the rest of the paper we define Ψ by a Wendland function $\psi_{l,k}$ with $k \geqslant 1$.

Assumptions. We define the radial basis function Ψ by

$$\Psi(x) := \psi_{l,k}(c\|x\|),$$

where $\psi_{l,k}$ is a Wendland function, $k \in \mathbb{N}$, $l := \left\lfloor \frac{d}{2} \right\rfloor + k + 1$ and $c > 0$. Denote $\psi(r) := \psi_{l,k}(cr)$. Note that the support of Ψ is given by $\operatorname{supp} \Psi = \{x \in \mathbb{R}^d \mid \|x\| \leqslant \frac{1}{c}\}$.

In Proposition 3.2 we summarize important properties of $\psi_{l,k}(r)$ and $\Psi(x)$. For a proof, cf. [18,3].

Proposition 3.2. *Let $k \in \mathbb{N}$ and $l := \left\lfloor \frac{d}{2} \right\rfloor + k + 1$. Let $\psi(r) := \psi_{l,k}(cr)$ with $c > 0$ and $\Psi(x) := \psi(\|x\|)$. Moreover, define $\psi_1(r) := \frac{\frac{d}{dr}\psi(r)}{r}$ for $r > 0$ and $\psi_2(r) := \frac{\frac{d}{dr}\psi_1(r)}{r}$ for $r > 0$ and set $\psi_2(0) = 0$. Set*

$$\mathcal{F}^* = H^{-\left(\frac{d+1}{2}+k\right)}(\mathbb{R}^d) \quad \text{and} \tag{3.8}$$

$$\mathcal{F} = H^{\frac{d+1}{2}+k}(\mathbb{R}^d), \tag{3.9}$$

where H denotes the Sobolev space. Then

1. *$\Psi \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ and Ψ has compact support.*
2. *For $\psi(r) := \psi_{l,k}(cr)$ we have the following asymptotics for the functions ψ_1 and ψ_2, respectively:*
   - *$\frac{d}{dr}\psi(r) = O(r)$ for $r \to 0$.*
   - *$\psi_1(r) = O(1)$ for $r \to 0$ and $\lim_{r\to 0} \psi_1(r) =: \psi_1(0)$ exists.*
   - *$\frac{d}{dr}\psi_1(r) = O(1)$ for $r \to 0$.*
   - *$\psi_2(r) = O\left(\frac{1}{r}\right)$ for $r \to 0$.*
3. *For the Fourier transform $\hat{\Psi}(\omega) = \int_{\mathbb{R}^d} \Psi(x) e^{-ix^T\omega}\, dx$ we have*

$$C_1 \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)} \leqslant \hat{\Psi}(\omega) \leqslant C_2 \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)} \tag{3.10}$$

   with positive constants C_1, C_2.
4. *Let $\lambda \in \mathcal{F}^* \subset \mathcal{S}'(\mathbb{R}^d)$, where $\mathcal{S}'(\mathbb{R}^d)$ denotes the dual of the Schwartz space $\mathcal{S}(\mathbb{R}^d)$ of rapidly decreasing functions. Then we define the norm $\|\lambda\|^2_{\mathcal{F}^*} = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\hat{\lambda}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega$, which is equivalent to the usual Sobolev norm in $\mathcal{F}^* = H^{-((d+1)/2+k)}(\mathbb{R}^d)$. Moreover, $\|\lambda\|^2_{\mathcal{F}^*} = \lambda^x \overline{\lambda^y} \Psi(x - y)$.*
5. *If $\lambda \in \mathcal{F}^*$, then $\lambda * \Psi \in \mathcal{F}$.*

In the following lemma we show that several operators are in $\mathcal{F}^*$ and, moreover, the approximating function w is an element of the function space $\mathcal{F}$.

Lemma 3.3. *Let the linear operator $D_\mathfrak{n}$ be given by $D_\mathfrak{n} w(x) := \mathfrak{n}(g(x))w(g(x)) - \mathfrak{n}(x)w(x)$ and let the linear operator D of the discrete orbital derivative be given by $Dw(x) := w(g(x)) - w(x)$. Denote by $\mathcal{E}'(\mathbb{R}^d)$ the distributions with compact support.*

For all $x \in \mathbb{R}^d$ we have $\delta_x, \delta_x \circ D, \delta_x \circ D_\mathfrak{n} \in \mathcal{F}^* \cap \mathcal{E}'(\mathbb{R}^d)$. Moreover, for each grid X_N and any $\beta_0, \beta_1, \ldots, \beta_N \in \mathbb{R}$ the ansatz function w of (3.2) *satisfies $w(x) = \beta_0 \Psi(x - \overline{x}) + \sum_{k=1}^N \beta_k (\delta_{x_k} \circ D_\mathfrak{n})^y \Psi(x - y) \in \mathcal{F}$.*

Proof. To prove that a linear operator λ belongs to $\mathcal{F}^*$ we show that $\int_{\mathbb{R}^d} |\hat{\hat{\lambda}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega < \infty$, cf. 4 of Proposition 3.2. Note that $\hat{\Psi}(\omega) \leqslant C_2 \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)}$ by (3.10).

Since δ_x has compact support, $\hat{\hat{\delta}}_x(\omega) = \delta_x e^{ix^T\omega} = e^{ix^T\omega}$. Thus,

$$\int_{\mathbb{R}^d} |\hat{\hat{\delta}}_x(\omega)|^2 \hat{\Psi}(\omega)\, d\omega \leqslant C_2 \int_{\mathbb{R}^d} \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)} d\omega < \infty.$$

Since $\delta_x \circ D$ has compact support, $(\delta_x \circ D)^{\hat{\hat{}}}(\omega) = (\delta_x \circ D) e^{ix^T\omega} = e^{ig(x)^T\omega} - e^{ix^T\omega}$. Thus,

$$\int_{\mathbb{R}^d} |(\delta_x \circ D)^{\hat{\hat{}}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega \leqslant 4C_2 \int_{\mathbb{R}^d} \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)} d\omega < \infty.$$

Since $\delta_x \circ D_\mathfrak{n}$ has compact support, $(\delta_x \circ D_\mathfrak{n})^{\hat{\hat{}}}(\omega) = (\delta_x \circ D_\mathfrak{n}) e^{ix^T\omega} = \mathfrak{n}(g(x)) e^{ig(x)^T\omega} - \mathfrak{n}(x) e^{ix^T\omega}$. Thus,

$$\begin{aligned}
&\int_{\mathbb{R}^d} |(\delta_x \circ D_\mathfrak{n})^{\hat{\hat{}}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega \\
&\quad \leqslant 4C_2 \max(\mathfrak{n}(g(x)), \mathfrak{n}(x))^2 \int_{\mathbb{R}^d} \left(1 + \|\omega\|^2\right)^{-((d+1)/2+k)} d\omega \\
&\quad < \infty.
\end{aligned}$$

Let $\lambda = \beta_0 \delta_{\overline{x}} + \sum_{j=1}^N \beta_j (\delta_{x_j} \circ D_\mathfrak{n})$. Since $\lambda \in \mathcal{S}'(\mathbb{R}^n)$ has the compact support $\operatorname{supp}(\lambda) = \bigcup_{j=1}^N \{x_j\} \cup \bigcup_{j=1}^N \{g(x_j)\} \cup \{\overline{x}\} =: K$, we have $(\lambda * \Psi)(x) = \lambda^y \Psi(x - y) = w(x)$. We conclude $\lambda \in \mathcal{F}^* \cap \mathcal{E}'(\mathbb{R}^d)$ in a similar way as above. This implies $w = \lambda * \Psi \in \mathcal{F}$ by 5 of Proposition 3.2. □

Now we show that the collocation matrix A is positive definite.

Proposition 3.4. *Let $\Psi(x)$ be as in the Assumptions and let $X_N = \{x_1, x_2, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Then the matrix A defined in* (3.5)–(3.7) *is positive definite.*

Proof. For $\lambda = \beta_0 \delta_{\overline{x}} + \sum_{j=1}^N \beta_j (\delta_{x_j} \circ D_\mathfrak{n}) \in \mathcal{F}^* \cap \mathcal{E}'(\mathbb{R}^d)$, cf. Lemma 3.3, we have with 4 of Proposition 3.2 $\beta^T A \beta = \lambda^x \overline{\lambda}^y \Psi(x - y) = \|\lambda\|_{\mathcal{F}^*}^2 = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\hat{\lambda}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega$. Since $\hat{\Psi}(\omega) > 0$ holds for all $\omega \in \mathbb{R}^d$, the matrix A is positive semidefinite.

Now we show that $\beta^T A \beta = 0$ implies $\beta = 0$. We assume that $\beta^T A \beta = (2\pi)^{-d} \int_{\mathbb{R}^d} |\hat{\hat{\lambda}}(\omega)|^2 \hat{\Psi}(\omega)\, d\omega = 0$. Then the analytic function satisfies $\hat{\hat{\lambda}}(\omega) = 0$ for all $\omega \in \mathbb{R}^d$. By Fourier transformation in $\mathcal{S}'(\mathbb{R}^d)$ we have $\mathcal{S}'(\mathbb{R}^d) \ni \lambda = 0$, i.e.

$$\lambda(h) = \beta_0 h(\overline{x}) + \sum_{j=1}^N \beta_j \left[\mathfrak{n}(g(x_j)) h(g(x_j)) - \mathfrak{n}(x_j) h(x_j)\right] = 0 \tag{3.11}$$

for all test functions $h \in \mathcal{S}(\mathbb{R}^d)$.

We show that $\beta_0 = 0$. The points x_j, $j = 1, \ldots, N$ are distinct from the fixed point $\overline{x}$. Denote $I := \{j \in \{1, \ldots, N\} | g(x_j) \neq \overline{x}\}$. There is a neighborhood $B_\delta(\overline{x})$ such that $x_j \notin B_\delta(\overline{x})$ holds for all $j = 1, \ldots, N$ and $g(x_i) \notin B_\delta(\overline{x})$ holds for all $i \in I$. Define the function $h(x) = 1$ for $x \in B_{\delta/2}(\overline{x})$ and $h(\overline{x}) = 0$ for $x \notin B_\delta(\overline{x})$, and extend it smoothly such that $h \in \mathcal{S}(\mathbb{R}^d)$. Then (3.11) yields $0 = \lambda(h) = \beta_0$ since for $i \notin I$ we have $\text{n}(g(x_i))h(g(x_i)) = \text{n}(\overline{x})h(\overline{x}) = 0$.

Now we show $\beta_i = 0$ for $i = 1, \ldots, N$. We use the notation $g^{\circ k}$ for $\underbrace{g \circ g \circ \cdots \circ g}_{k \text{ times}}$. Fix an $i \in \{1, \ldots, N\}$. Denote $J := \{j \in \{1, \ldots, N\} \setminus \{i\} | \exists k \in \mathbb{N} : g^{\circ k} x_j = x_i \text{ and } \forall l \in \{1, \ldots, k-1\} : g^{\circ l} x_j \in X_N\}$. Note that $g(x_k) \in \{x_j | j \in J \cup \{i\}\}$ for all $k \in J$ by definition of J and $g(x_i) \notin \{x_j | j \in J \cup \{i\}\}$, since x_i is no fixed point and no periodic point. Moreover, by definition of J we have

$$g(x_k) \notin \{x_j | j \in J \cup \{i\}\} \text{ for all } k \in \{1, \ldots, N\} \setminus (J \cup \{i\}). \tag{3.12}$$

Define the function $h \in \mathcal{S}(\mathbb{R}^d)$ in the following way:

$$h(x_j) = \begin{cases} 1/\text{n}(x_j) & \text{for } j \in J \cup \{i\}, \\ 0 & \text{for } j \in \{1, \ldots, N\} \setminus (J \cup \{i\}), \end{cases}$$

$$h(g(x_j)) = 0 \quad \text{for } j \in \{1, \ldots, N\} \setminus J,$$

$$h(\overline{x}) = 0.$$

Note that $\text{n}(x) \neq 0$ for $x \neq \overline{x}$. Extend the function smoothly and prolongate it by zero such that $h \in \mathcal{S}(\mathbb{R}^d)$. Note that this is possible by (3.12).

Then (3.11) yields

$$0 = \lambda(h) = 0 + \sum_{j \in J} \beta_j [1 - 1] + \beta_i [0 - 1] + \sum_{k \in \{1, \ldots, N\} \setminus (J \cup \{i\})} \beta_k [0 - 0] = -\beta_i.$$

This argumentation holds for all $i = 1, \ldots, N$ and thus $\beta = 0$. Hence, A is positive definite. □

In the following proposition we calculate the matrix elements of the collocation matrix A, cf. (3.5)–(3.7).

Proposition 3.5. *Let $X_N = \{x_1, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Let $\Psi(x) := \psi(\|x\|)$ be defined by a Wendland function as in the Assumptions.*

The matrix elements a_{jk} of the collocation matrix A are then given by

$$a_{jk} = \text{n}(g(x_j))\text{n}(g(x_k))\psi(\|g(x_j) - g(x_k)\|) - \text{n}(g(x_j))\text{n}(x_k)\psi(\|g(x_j) - x_k\|)$$
$$-\text{n}(x_j)\text{n}(g(x_k))\psi(\|x_j - g(x_k)\|) + \text{n}(x_j)\text{n}(x_k)\psi(\|x_j - x_k\|), \tag{3.13}$$

$$a_{k0} = a_{0k} = \text{n}(g(x_k))\psi(\|\overline{x} - g(x_k)\|) - \text{n}(x_k)\psi(\|\overline{x} - x_k\|), \tag{3.14}$$

$$a_{00} = \psi(0) \tag{3.15}$$

for $j, k = 1, \ldots, N$.

The vector α is given by

$$\alpha_j = -\|x_j - \overline{x}\|^2 \quad \textit{for } j \geqslant 1,$$

$$\alpha_0 = 1.$$

Let the vector $\beta = (\beta_0, \beta_1, \ldots, \beta_N)^T$ be the unique solution of the system of linear equations $A\beta = \alpha$.

Then the approximant $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ is given by

$$w(x) = \beta_0 \psi(\|x - \overline{x}\|) + \sum_{k=1}^{N} \beta_k \left[\mathfrak{n}(g(x_k))\psi(\|x - g(x_k)\|) - \mathfrak{n}(x_k)\psi(\|x - x_k\|) \right].$$

Proof. By (3.5) we have for $j, k \geqslant 1$

$$\begin{aligned} a_{jk} &= (\delta_{x_j} \circ D_\mathfrak{n})^x (\delta_{x_k} \circ D_\mathfrak{n})^y \psi(\|x - y\|) \\ &= (\delta_{x_j} \circ D_\mathfrak{n})^x \left[\mathfrak{n}(g(x_k))\psi(\|x - g(x_k)\|) - \mathfrak{n}(x_k)\psi(\|x - x_k\|) \right] \\ &= \mathfrak{n}(g(x_j))\mathfrak{n}(g(x_k))\psi(\|g(x_j) - g(x_k)\|) - \mathfrak{n}(g(x_j))\mathfrak{n}(x_k)\psi(\|g(x_j) - x_k\|) \\ &\quad - \mathfrak{n}(x_j)\mathfrak{n}(g(x_k))\psi(\|x_j - g(x_k)\|) + \mathfrak{n}(x_j)\mathfrak{n}(x_k)\psi(\|x_j - x_k\|). \end{aligned}$$

The formulas for $j = 0$, $k = 0$ and $w(x)$ follow by similar calculations, cf. also (3.2). □

The following proposition is the minimality property of the approximation w; it can be proved in a similar way as e.g. [3, Proposition 3.34].

Proposition 3.6. *Let $\Psi(x)$ be defined as in the Assumptions. Let $X_N = \{x_1, x_2, \ldots, x_N\}$ be a set of pairwise distinct points, which are no fixed or periodic points. Let $W \in \mathcal{F}$ and let $w \in \mathcal{F}$, cf. Lemma* 3.3, *be the approximation of W with respect to the grid X_N in the sense of Proposition* 3.5. *Then*

$$\|W - w\|_{\mathcal{F}} \leqslant \|W\|_{\mathcal{F}}.$$

4. Error estimates

In this section we prove three error estimates. More precisely, we estimate $|W'(x) - w'(x)|$ (Proposition 4.1) and $|W(x) - w(x)|$ (Proposition 4.2) in a neighborhood of $\overline{x}$. Moreover, we estimate $|D_\mathfrak{n} W(x) - D_\mathfrak{n} w(x)|$ for all points in the area where the grid points X_N are set, cf. Theorem 4.4.

Proposition 4.1. *Let $\Psi(x)$ be defined as in the Assumptions. Let $g \in C^1(\mathbb{R}^d, \mathbb{R}^d)$ and let $W \in \mathcal{F}$.*

For all $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left| W'(x) - w'(x) \right| \leqslant \varepsilon \quad \text{for } x \in \overline{B_\delta(\overline{x})} \tag{4.1}$$

holds for all approximations w of W with respect to any grid X_N in the sense of Proposition 3.5.

Proof. There is a constant c_1 such that $\|Dg(x) - I\| \leqslant c_1$ holds for all $x \in \overline{B_1(\overline{x})}$. There is a constant c_2 such that $|\psi_1(\|g(x) - x\|)| \leqslant c_2$ holds for all $x \in \overline{B_1(\overline{x})}$, since ψ_1 and g are continuous functions; for the definition of ψ_1 cf. Proposition 3.2. Set

$$\delta := \min\left(1, \frac{\varepsilon}{c_1 \|W\|_{\mathcal{F}} \sqrt{2c_2}}\right). \tag{4.2}$$

Denote the linear operator of the discrete orbital derivative by $Dw(x) := w(g(x)) - w(x)$. Let $x^* \in \overline{B_\delta(\overline{x})}$, and set $\lambda = \delta_{x^*} \circ D \in \mathcal{F}^*$ and $\mu = \delta_{\overline{x}} \circ D \in \mathcal{F}^*$, cf. Lemma 3.3. We have $\mu(W - w) = 0$, since $W'(\overline{x}) = 0 = w'(\overline{x})$ due to $g(\overline{x}) = \overline{x}$.

Hence,

$$\begin{aligned} |\lambda(W) - \lambda(w)| &= |(\lambda - \mu)(W - w)| \leqslant \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W - w\|_{\mathcal{F}} \\ &\leqslant \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W\|_{\mathcal{F}} \end{aligned} \tag{4.3}$$

by Proposition 3.6. Then we have with a similar calculation as in [4]—note that $g(\overline{x}) = \overline{x}$:

$$\begin{aligned} \|\lambda - \mu\|^2_{\mathcal{F}^*} &= (\lambda - \mu)^x (\lambda - \mu)^y \Psi(x - y) \\ &= (\delta_{x^*} \circ D - \delta_{\overline{x}} \circ D)^x (\delta_{x^*} \circ D - \delta_{\overline{x}} \circ D)^y \Psi(x - y) \\ &= 2\psi(0) - 2\psi(\|g(x^*) - x^*\|). \end{aligned}$$

Note that by Taylor's Theorem there is a $\xi = \theta x^* + (1 - \theta)\overline{x}$ where $\theta \in [0, 1]$ such that

$$\psi(\|g(x^*) - x^*\|) = \psi(0) + \psi_1(\|g(\xi) - \xi\|)(g(\xi) - \xi)^T (Dg(\xi) - I)(x^* - \overline{x}).$$

Again by Taylor's Theorem there is a $\xi' = \theta'\xi + (1 - \theta')\overline{x}$ where $\theta' \in [0, 1]$ such that

$$\begin{aligned} g(\xi) - \xi &= g(\overline{x}) + Dg(\xi')(\xi - \overline{x}) - \xi \\ &= (Dg(\xi') - I)(\xi - \overline{x}). \end{aligned}$$

Altogether, we obtain

$$\begin{aligned} \|\lambda - \mu\|^2_{\mathcal{F}^*} &\leqslant 2|\psi_1(\|g(\xi) - \xi\|)| \cdot \|g(\xi) - \xi\| \cdot \|Dg(\xi) - I\| \cdot \|x^* - \overline{x}\| \\ &\leqslant 2c_2 \cdot \|Dg(\xi') - I\| \cdot \|Dg(\xi) - I\| \|x^* - \overline{x}\|^2 \\ &\leqslant 2c_1^2 c_2 \delta^2 \\ &\leqslant \frac{\varepsilon^2}{\|W\|^2_{\mathcal{F}}} \end{aligned}$$

by (4.2) since $\|x^* - \overline{x}\| \leqslant \delta$. Hence, we have $\|\lambda - \mu\|_{\mathcal{F}^*} \leqslant \frac{\varepsilon}{\|W\|_{\mathcal{F}}}$ and the proposition follows by (4.3). □

Proposition 4.2. *Let $\Psi(x)$ be defined as in the Assumptions. Let $g \in C^0(\mathbb{R}^d, \mathbb{R}^d)$ and let $W \in \mathcal{F}$. For all $\varepsilon > 0$ there is a $\delta > 0$ such that*

$$|W(x) - w(x)| \leqslant \varepsilon \quad \text{for } x \in \overline{B_\delta(\overline{x})} \tag{4.4}$$

holds for all approximations w of W with respect to any grid X_N in the sense of Proposition 3.5.

Proof. Let $m := \max_{r \in [0,1]} \left|\frac{d}{dr}\psi(r)\right|$ which exists by Proposition 3.2. Define

$$\delta := \min\left(1, \frac{\varepsilon^2}{2m\|W\|^2_{\mathcal{F}}}\right). \tag{4.5}$$

Let $x^* \in \overline{B_\delta(\overline{x})}$, and set $\lambda = \delta_{x^*} \in \mathcal{F}^*$ and $\mu = \delta_{\overline{x}} \in \mathcal{F}^*$, cf. Lemma 3.3. We have $\mu(W - w) = 0$, since $W(\overline{x}) = 1 = w(\overline{x})$ by construction of w, cf. (3.3). Hence,

$$\begin{aligned} |\lambda(W) - \lambda(w)| &= |(\lambda - \mu)(W - w)| \leqslant \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W - w\|_{\mathcal{F}} \\ &\leqslant \|\lambda - \mu\|_{\mathcal{F}^*} \cdot \|W\|_{\mathcal{F}} \end{aligned} \tag{4.6}$$

by Proposition 3.6. Denoting $r := \|x^* - \overline{x}\| \leqslant \delta$ and using Taylor's Theorem there is a $\tilde{r} \in [0, r]$ such that

$$\begin{aligned}\|\lambda - \mu\|^2_{\mathcal{F}^*} &= (\lambda-\mu)^x(\lambda-\mu)^y\Psi(x-y)\\ &= (\delta_{x^*}-\delta_{\overline{x}})^x(\delta_{x^*}-\delta_{\overline{x}})^y\Psi(x-y)\\ &= 2\psi(0)-2\psi(r)\\ &= -2\frac{d}{dr}\psi(\tilde{r})r\\ &\leqslant 2m\delta\\ &\leqslant \frac{\varepsilon^2}{\|W\|^2_{\mathcal{F}}}\end{aligned}$$

holds by (4.5). Hence, we have by (4.6) $|\lambda(W) - \lambda(w)| \leqslant \varepsilon$, which shows the proposition. □

For Theorem 4.4 we define the fill distance of a grid.

Definition 4.3 (*Fill distance*). Let $K \subset \mathbb{R}^d$ be a compact set. Furthermore, let $X_N := \{x_1, \ldots, x_N\} \subset K$ be a grid (set of pairwise distinct points). The positive real number

$$h := h_{K,X_N} = \max_{y\in K}\min_{x\in X_N}\|x-y\|$$

is called the *fill distance of* X_N in *K*. In particular, for all $y \in K$ there is a grid point $x_k \in X_N$ such that $\|y - x_k\| \leqslant h$.

Theorem 4.4. *Let* $\Psi(x)$ *be defined as in the Assumptions. Let* $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$, $\sigma \geqslant 1$ *and let* $W \in \mathcal{F}$.

Let K be a compact set. For all $H > 0$ *there is a constant* $C^* = C^*(K)$ *such that*: *Let* $X_N := \{x_1, \ldots, x_N\} \subset K$ *be a grid with pairwise distinct points which are no fixed or periodic points and with fill distance* $0 < h \leqslant H$ *in K, and let* $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ *be the approximation of W in the sense of Proposition* 3.5.

Then

$$|D_{\mathfrak{n}}W(x) - D_{\mathfrak{n}}w(x)| \leqslant C^*h \quad \text{holds for all } x \in K. \tag{4.7}$$

Proof. Let $x^* \in K$ and choose a grid point $\tilde{x} \in X_N$ such that $r := \|x^* - \tilde{x}\| \leqslant h$. Set $\lambda = \delta_{x^*} \circ D_{\mathfrak{n}} \in \mathcal{F}^*$ and $\mu = \delta_{\tilde{x}} \circ D_{\mathfrak{n}} \in \mathcal{F}^*$, cf. Lemma 3.3. We have $\mu(W - w) = 0$ by definition of w, cf. (3.4). Hence,

$$\begin{aligned}|\lambda(W)-\lambda(w)| &= |(\lambda-\mu)(W-w)| \leqslant \|\lambda-\mu\|_{\mathcal{F}^*}\cdot\|W-w\|_{\mathcal{F}}\\ &\leqslant \|\lambda-\mu\|_{\mathcal{F}^*}\cdot\|W\|_{\mathcal{F}}\end{aligned} \tag{4.8}$$

by Proposition 3.6. We obtain with a similar calculation as in Proposition 3.5

$$\begin{aligned}\|\lambda-\mu\|^2_{\mathcal{F}^*} &= (\lambda-\mu)^x(\lambda-\mu)^y\Psi(x-y)\\ &= (\delta_{x^*}\circ D_{\mathfrak{n}} - \delta_{\tilde{x}}\circ D_{\mathfrak{n}})^x(\delta_{x^*}\circ D_{\mathfrak{n}} - \delta_{\tilde{x}}\circ D_{\mathfrak{n}})^y\Psi(x-y)\\ &= \psi(0)[\mathfrak{n}(\tilde{x})^2 + \mathfrak{n}(x^*)^2 + \mathfrak{n}(g(\tilde{x}))^2 + \mathfrak{n}(g(x^*))^2]\\ &\quad -2\psi(\|x^*-\tilde{x}\|)\mathfrak{n}(x^*)\mathfrak{n}(\tilde{x}) - 2\psi(\|g(x^*)-g(\tilde{x})\|)\mathfrak{n}(g(x^*))\mathfrak{n}(g(\tilde{x}))\\ &\quad +2\psi(\|g(\tilde{x})-x^*\|)\mathfrak{n}(g(\tilde{x}))\mathfrak{n}(x^*) - 2\psi(\|g(\tilde{x})-\tilde{x}\|)\mathfrak{n}(g(\tilde{x}))\mathfrak{n}(\tilde{x})\\ &\quad +2\psi(\|g(x^*)-\tilde{x}\|)\mathfrak{n}(\tilde{x})\mathfrak{n}(g(x^*)) - 2\psi(\|g(x^*)-x^*\|)\mathfrak{n}(x^*)\mathfrak{n}(g(x^*)).\end{aligned}$$

Due to the continuity of g, Dg, $\mathfrak{n}$, $\nabla\mathfrak{n}$ and $\text{Hess}\,\mathfrak{n}$, there are constants such that $\|g(x)\| \leqslant c_0$, $\|Dg(x)\| \leqslant c_1$, $\mathfrak{n}(x) \leqslant c_2$, $\|\nabla\mathfrak{n}(x)\| \leqslant c_3$ and $\|\text{Hess}\,\mathfrak{n}(x)\| \leqslant c_4$ hold for all $x \in \overline{\text{conv}(K \cup g(K))}$.

Recall that we have denoted $r := \|x^* - \tilde{x}\|$. Taylor's Theorem implies the existence of $r_0 \in [0, r]$ such that

$$\psi(0) - \psi(\|x^* - \tilde{x}\|) = -\frac{d}{dr}\psi(r_0)r.$$

Moreover, there is a $\xi = \theta x^* + (1-\theta)\tilde{x}$ where $\theta \in [0, 1]$ such that $\mathfrak{n}(x^*) - \mathfrak{n}(\tilde{x}) = \nabla\mathfrak{n}(\xi)^T(x^* - \tilde{x})$. Hence,

$$\begin{aligned}
&\psi(0)[\mathfrak{n}(\tilde{x})^2 + \mathfrak{n}(x^*)^2] - 2\psi(\|x^* - \tilde{x}\|)\mathfrak{n}(x^*)\mathfrak{n}(\tilde{x}) \\
&\quad = \psi(\|x^* - \tilde{x}\|)[\mathfrak{n}(x^*) - \mathfrak{n}(\tilde{x})]^2 + [\psi(0) - \psi(\|x^* - \tilde{x}\|)][\mathfrak{n}(x^*)^2 + \mathfrak{n}(\tilde{x})^2] \\
&\quad \leqslant |\psi(r)| c_3^2 r^2 + 2c_2^2 \left|\frac{d}{dr}\psi(r_0)\right| r \\
&\quad = O(r^2) \qquad (4.9)
\end{aligned}$$

for $r \to 0$ due to the properties of ψ, cf. 2 of Proposition 3.2.

In a similar way, Taylor's Theorem implies the existence of $\xi_1 = \theta_1 x^* + (1-\theta_1)\tilde{x}$ where $\theta_1 \in [0, 1]$ such that

$$\psi(0) - \psi(\|g(x^*) - g(\tilde{x})\|) = -\psi_1(\|\xi_1\|)(g(\xi_1) - g(\tilde{x}))^T Dg(\xi_1)(x^* - \tilde{x}).$$

Moreover, there are $\xi_2 = \theta_2\xi_1 + (1-\theta_2)\tilde{x}$ and $\xi_3 = \theta_3 x^* + (1-\theta_3)\tilde{x}$ where $\theta_2, \theta_3 \in [0, 1]$ such that $g(\xi_1) - g(\tilde{x}) = Dg(\xi_2)(\xi_1 - \tilde{x})$ and $\mathfrak{n}(g(x^*)) - \mathfrak{n}(g(\tilde{x})) = \nabla\mathfrak{n}(\xi_3)^T Dg(\xi_3)(x^* - \tilde{x})$. Hence,

$$\begin{aligned}
&\psi(0)[\mathfrak{n}(g(\tilde{x}))^2 + \mathfrak{n}(g(x^*))^2] - 2\psi(\|g(x^*) - g(\tilde{x})\|)\mathfrak{n}(g(x^*))\mathfrak{n}(g(\tilde{x})) \\
&\quad = \psi(\|g(x^*) - g(\tilde{x})\|)[\mathfrak{n}(g(x^*)) - \mathfrak{n}(g(\tilde{x}))]^2 \\
&\qquad + [\psi(0) - \psi(\|g(x^*) - g(\tilde{x})\|)][\mathfrak{n}(g(x^*))^2 + \mathfrak{n}(g(\tilde{x}))^2] \\
&\quad \leqslant c_1^2 c_3^2 |\psi(\|g(x^*) - g(\tilde{x})\|)| r^2 + 2c_1^2 c_2^2 |\psi_1(\|\xi_1\|)| r^2 \\
&\quad = O(r^2) \qquad (4.10)
\end{aligned}$$

due to the properties of ψ, cf. 2 of Proposition 3.2.

For $h(x) = \psi(\|x - g(\tilde{x})\|)\mathfrak{n}(x)$ we have $\nabla h(x) = \psi_1(\|x - g(\tilde{x})\|)\mathfrak{n}(x)(x - g(\tilde{x})) + \psi(\|x - g(\tilde{x})\|)\nabla\mathfrak{n}(x)$. Hence, Taylor's Theorem implies the existence of a $z_1 = \tau_1 x^* + (1-\tau_1)\tilde{x}$ where $\tau_1 \in [0, 1]$ such that

$$\begin{aligned}
&[\psi(\|\tilde{x} - g(\tilde{x})\|)\mathfrak{n}(\tilde{x}) - \psi(\|x^* - g(\tilde{x})\|)\mathfrak{n}(x^*)]\mathfrak{n}(g(\tilde{x})) \\
&\quad = [\psi_1(\|z_1 - g(\tilde{x})\|)\mathfrak{n}(z_1)(z_1 - g(\tilde{x}))^T \\
&\qquad + \psi(\|z_1 - g(\tilde{x})\|)\nabla\mathfrak{n}(z_1)^T](\tilde{x} - x^*)\mathfrak{n}(g(\tilde{x})). \qquad (4.11)
\end{aligned}$$

In a similar way we obtain

$$\begin{aligned}
&[\psi(\|\tilde{x} - g(x^*)\|)\mathfrak{n}(\tilde{x}) - \psi(\|x^* - g(x^*)\|)\mathfrak{n}(x^*)]\mathfrak{n}(g(x^*)) \\
&\quad = [\psi_1(\|z_2 - g(x^*)\|)\mathfrak{n}(z_2)(z_2 - g(x^*))^T \\
&\qquad + \psi(\|z_2 - g(x^*)\|)\nabla\mathfrak{n}(z_2)^T](\tilde{x} - x^*)\mathfrak{n}(g(x^*)) \qquad (4.12)
\end{aligned}$$

with $z_2 = \tau_2 x^* + (1-\tau_2)\tilde{x}$ where $\tau_2 \in [0, 1]$.

Subtracting (4.11) from (4.12) we obtain

$$\begin{aligned}
&\psi(\|g(\tilde{x})-x^*\|)\mathfrak{n}(g(\tilde{x}))\mathfrak{n}(x^*)-\psi(\|g(\tilde{x})-\tilde{x}\|)\mathfrak{n}(g(\tilde{x}))\mathfrak{n}(\tilde{x})\\
&\quad+\psi(\|g(x^*)-\tilde{x}\|)\mathfrak{n}(\tilde{x})\mathfrak{n}(g(x^*))-\psi(\|g(x^*)-x^*\|)\mathfrak{n}(x^*)\mathfrak{n}(g(x^*))\\
&\qquad=\mathfrak{n}(z_2)\mathfrak{n}(g(x^*))[\psi_1(\|z_2-g(x^*)\|)(z_2-g(x^*))^T\\
&\qquad\quad-\psi_1(\|z_1-g(\tilde{x})\|)(z_1-g(\tilde{x}))^T](\tilde{x}-x^*)\\
&\qquad\quad+[\mathfrak{n}(z_2)\mathfrak{n}(g(x^*))-\mathfrak{n}(z_1)\mathfrak{n}(g(\tilde{x}))]\psi_1(\|z_1-g(\tilde{x})\|)(z_1-g(\tilde{x}))^T(\tilde{x}-x^*)\\
&\qquad\quad+[\psi(\|z_2-g(x^*)\|)\nabla\mathfrak{n}(z_2)^T-\psi(\|z_1-g(\tilde{x})\|)\nabla\mathfrak{n}(z_1)^T](\tilde{x}-x^*)\mathfrak{n}(g(x^*))\\
&\qquad\quad+\psi(\|z_1-g(\tilde{x})\|)\nabla\mathfrak{n}(z_1)^T(\tilde{x}-x^*)[\mathfrak{n}(g(x^*))-\mathfrak{n}(g(\tilde{x}))]. \qquad (4.13)
\end{aligned}$$

We will discuss each term of (4.13). For the first term we consider with $z_3=\tau_3(z_1-g(\tilde{x}))+(1-\tau_3)(z_2-g(x^*))$ where $\tau_3\in[0,1]$

$$\begin{aligned}
&\|\psi_1(\|z_2-g(x^*)\|)(z_2-g(x^*))-\psi_1(\|z_1-g(\tilde{x})\|)(z_1-g(\tilde{x}))\|\\
&\quad\leqslant|\psi_1(\|z_2-g(x^*)\|)|\cdot\|(z_2-z_1)+(g(\tilde{x})-g(x^*))\|\\
&\qquad+|\psi_1(\|z_1-g(\tilde{x})\|)-\psi_1(\|z_2-g(x^*)\|)|\cdot\|z_1-g(\tilde{x})\|\\
&\quad=|\psi_1(\|z_2-g(x^*)\|)|(1+c_1)r+|\psi_2(\|z_3\|)|\cdot\|z_3\|(1+c_1)r\|z_1-g(\tilde{x})\|.
\end{aligned}$$

This term is $O(r)$ for $r\to 0$ due to the properties of ψ, cf. 2 of Proposition 3.2.

For the second term of (4.13) we consider

$$\begin{aligned}
&|\mathfrak{n}(z_2)\mathfrak{n}(g(x^*))-\mathfrak{n}(z_1)\mathfrak{n}(g(\tilde{x}))|\\
&\quad\leqslant\mathfrak{n}(z_2)|\mathfrak{n}(g(x^*))-\mathfrak{n}(g(\tilde{x}))|+\mathfrak{n}(g(\tilde{x}))|\mathfrak{n}(z_2)-\mathfrak{n}(z_1)|\\
&\quad\leqslant c_2c_3(1+c_1)r\\
&\quad=O(r)\text{ for }r\to 0.
\end{aligned}$$

For the third term of (4.13) we have with $z_4=\tau_4(z_2-g(x^*))+(1-\tau_4)(z_1-g(\tilde{x}))$ where $\tau_4\in[0,1]$

$$\begin{aligned}
&\|\psi(\|z_2-g(x^*)\|)\nabla\mathfrak{n}(z_2)-\psi(\|z_1-g(\tilde{x})\|)\nabla\mathfrak{n}(z_1)\|\\
&\quad\leqslant|\psi(\|z_2-g(x^*)\|)|\cdot\|\nabla\mathfrak{n}(z_2)-\nabla\mathfrak{n}(z_1)\|\\
&\qquad+|\psi(\|z_2-g(x^*)\|)-\psi(\|z_1-g(\tilde{x})\|)|\cdot\|\nabla\mathfrak{n}(z_1)\|\\
&\quad\leqslant|\psi(\|z_2-g(x^*)\|)|c_4r+|\psi_1(z_4)|\cdot\|z_4\|\cdot\|z_2-z_1+g(\tilde{x})-g(x^*)\|c_3\\
&\quad\leqslant|\psi(\|z_2-g(x^*)\|)|c_4r+|\psi_1(z_4)|\cdot\|z_4\|(1+c_1)c_3r\\
&\quad=O(r)
\end{aligned}$$

for $r\to 0$ due to the properties of ψ, cf. 2 of Proposition 3.2.

For the last term of (4.13) we have

$$|\mathfrak{n}(g(x^*))-\mathfrak{n}(g(\tilde{x}))|\leqslant c_3c_1r\ =\ O(r).$$

Altogether, we obtain with (4.9), (4.10) and (4.13) $\|\lambda-\mu\|^2_{\mathcal{F}^*}=O(r^2)$. Thus, (4.7) follows by (4.8). □

5. Main results

In order to show that the approximation $v(x) = \mathfrak{n}(x)w(x)$ is a Lyapunov function, we show in Theorem 5.1 that $v'(x) < 0$ holds for all $x \in K \setminus \{\overline{x}\}$. In Theorem 5.2 we show that each connected and bounded subset of the basin of attraction can be covered by a sublevel set of the function v. Hence, we can determine each connected subset of the basin of attraction with a calculated Lyapunov function using the proposed method.

Theorem 5.1. *Let $\overline{x}$ be a fixed point of $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$ such that $|\lambda| < 1$ holds for all eigenvalues λ of $Dg(\overline{x})$.*

We consider the radial basis function $\Psi(x) = \psi_{l,k}(c\|x\|)$ with $c > 0$, where $\psi_{l,k}$ denotes the Wendland function with $k \in \mathbb{N}$ and $l := \lfloor \frac{d}{2} \rfloor + k + 1$. Let $\sigma \geqslant P \geqslant 2 + \sigma^*$, where $\sigma^* := \frac{d+1}{2} + k$ and $P \in \mathbb{N}$. Let K be a compact set such that $\overline{x} \in \overset{\circ}{K}$ and $K \subset A(\overline{x})$.

Let V be the Lyapunov function of Theorem 2.2 with $V'(x) = -\|x - \overline{x}\|^2$ and $V(\overline{x}) = 0$, and $\mathfrak{n}(x) = \sum_{2 \leqslant |\alpha| \leqslant P} c_\alpha (x - \overline{x})^\alpha + M\|x - \overline{x}\|^{2H}$ as in Definition 2.6, and let $W(x) = \frac{V(x)}{\mathfrak{n}(x)} \in C^{P-2}(A(\overline{x}), \mathbb{R})$ with $W(\overline{x}) = 1$, cf. Proposition 2.7.

Then there is a constant h', such that for all approximations $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ of W with respect to a grid $X_N \subset K \setminus \{\overline{x}\}$ with fill distance $h \leqslant h'$ in the sense of Proposition 3.5

$$v'(x) < 0 \quad \text{holds for all } x \in K \setminus \{\overline{x}\},$$

where $v(x) = w(x)\mathfrak{n}(x)$.

Proof. Let $\tilde{B}$ be an open set with $K \subset \tilde{B} \subset \overline{\tilde{B}} \subset A(\overline{x})$. Choose $\chi \in C_0^\infty(\mathbb{R}^d, [0, 1])$ to be a function satisfying $\chi(x) = 1$ for $x \in K$ and $\chi(x) = 0$ for $\mathbb{R}^d \setminus \tilde{B}$. Thus, $\chi \in C_0^\infty(\mathbb{R}^d) \subset \mathcal{F}$. Set $W_0 = W \cdot \chi$; then $W_0 \in C_0^{P-2}(\mathbb{R}^d, \mathbb{R})$ and $W_0(x) = W(x)$ holds for all $x \in K$. Since $P - 2 \geqslant \sigma^* = \frac{d+1}{2} + k$, we have $W_0 \in C_0^{P-2}(\mathbb{R}^d, \mathbb{R}) \subset H^{P-2}(\mathbb{R}^d) \subset H^{(d+1)/2+k}(\mathbb{R}^d) = \mathcal{F}$; note that W_0 has compact support.

Set $\varepsilon = \frac{1}{8}$. We choose $\delta > 0$ so small that $\overline{B_\delta(\overline{x})} \subset K$ and

$$\mathfrak{n}(g(x)) \leqslant C\|x - \overline{x}\|^2, \tag{5.1}$$

$$\mathfrak{n}'(x) \leqslant -\frac{1}{2}\|x - \overline{x}\|^2, \tag{5.2}$$

$$|W_0(x) - W_0(\overline{x})| \leqslant \varepsilon, \tag{5.3}$$

$$|W_0'(x) - W_0'(\overline{x})| \leqslant \frac{\varepsilon}{C}, \tag{5.4}$$

$$|W_0(x) - w(x)| \leqslant \varepsilon, \tag{5.5}$$

$$|W_0'(x) - w'(x)| \leqslant \frac{\varepsilon}{C} \tag{5.6}$$

hold for all $x \in \overline{B_\delta(\overline{x})}$. This is possible due to Proposition 2.7 applied to $g(\overline{B_\delta(\overline{x})})$ for (5.1); for (5.2) note that $P > 2$ and $\mathfrak{h}'(x) = -\|x - \overline{x}\|^2 + o(\|x - \overline{x}\|^P)$ implies $\mathfrak{n}'(x) = -\|x - \overline{x}\|^2 + o(\|x - \overline{x}\|^P)$. Eqs. (5.3) and (5.4) can be established by the continuity of W_0 and W_0' at $\overline{x}$, and (5.5) and (5.6) by Propositions 4.1 and 4.2. Define $h' := \min(1, \delta^2/C^*)$, where C^* is as in Theorem 4.4 for W_0, $H = 1$ and K.

We show now $v'(x) < 0$ for $v(x) = \mathfrak{n}(x)w(x)$ and $x \in K \setminus \{\overline{x}\}$, and distinguish between the two cases $x \in \overline{B_\delta(\overline{x})} \setminus \{\overline{x}\}$ and $x \in K \setminus \overline{B_\delta(\overline{x})}$.

Case 1: Let $x \in \overline{B_\delta(\overline{x})} \setminus \{\overline{x}\}$. Then, as $W_0(x) \geqslant 0$,

$$\begin{aligned} v'(x) &= \mathfrak{n}(g(x))w(g(x)) - \mathfrak{n}(x)w(x) \\ &= [w(g(x)) - w(x)]\mathfrak{n}(g(x)) + [\mathfrak{n}(g(x)) - \mathfrak{n}(x)]w(x) \\ &= \mathfrak{n}(g(x))w'(x) + \mathfrak{n}'(x)w(x) \\ &\leqslant C\|x-\overline{x}\|^2 \left(|W_0'(x)| + \frac{\varepsilon}{C}\right) - \frac{1}{2}\|x-\overline{x}\|^2 (W_0(x) - \varepsilon) \\ &\quad \text{by (5.1), (5.6), (5.2) and (5.5)} \\ &\leqslant \|x-\overline{x}\|^2 \left(2\varepsilon - \frac{1}{2}(1-2\varepsilon)\right) \\ &\quad \text{by (5.3) and (5.4)—note that } W_0'(\overline{x}) = 0 \text{ and } W_0(\overline{x}) = 1 \\ &= -\varepsilon\|x-\overline{x}\|^2 \end{aligned}$$

since $\varepsilon = \frac{1}{8}$. This shows $v'(x) < 0$ for $x \in \overline{B_\delta(\overline{x})} \setminus \{\overline{x}\}$.

Case 2: Due to Theorem 4.4 and $h \leqslant h'$ we have $|V'(x) - v'(x)| = |D_\mathfrak{n} W_0(x) - D_\mathfrak{n} w(x)| \leqslant \delta^2$ for all $x \in K$. Then

$$v'(x) \leqslant V'(x) + \delta^2 = -\|x-\overline{x}\|^2 + \delta^2 < 0$$

for all $x \in K \setminus \overline{B_\delta(\overline{x})}$. This shows $v'(x) < 0$ and proves the theorem. □

A Lyapunov function gives information about the basin of attraction: if $\overline{x}$ is an asymptotically stable fixed point and O is an open, bounded and connected neighborhood of $\overline{x}$ such that $v'(x) < 0$ holds for all $\overline{O} \setminus \{\overline{x}\}$, $v(x) < R^*$ holds for all $x \in O$ and $v(x) = R^*$ holds for all $x \in \partial O$, then $\overline{O}$ is a subset of the basin of attraction $A(\overline{x})$, cf. Theorem 2.1. Thus, the question arises whether we can cover each connected subset of the basin of attraction with a sublevel set of the calculated Lyapunov function v. The affirmative answer is given in Theorem 5.2.

Theorem 5.2. *Let $\overline{x}$ be a fixed point of $x_{n+1} = g(x_n)$, where $g \in C^\sigma(\mathbb{R}^d, \mathbb{R}^d)$ such that $|\lambda| < 1$ holds for all eigenvalues λ of $Dg(\overline{x})$.*

We consider the radial basis function $\Psi(x) = \psi_{l,k}(c\|x\|)$ with $c > 0$, where $\psi_{l,k}$ denotes the Wendland function with $k \in \mathbb{N}$ and $l := \lfloor \frac{d}{2} \rfloor + k + 1$. Let $\sigma \geqslant P \geqslant 2 + \sigma^*$, where $\sigma^* := \frac{d+1}{2} + k$ and $P \in \mathbb{N}$. Let O_0 be an open, bounded and connected set with $\overline{x} \in O_0 \subset \overline{O_0} \subset A(\overline{x})$.

Let V be the Lyapunov function of Theorem 2.2 with $V'(x) = -\|x-\overline{x}\|^2$ and $V(\overline{x}) = 0$, and $\mathfrak{n}(x) = \sum_{2 \leqslant |\alpha| \leqslant P} c_\alpha (x-\overline{x})^\alpha + M\|x-\overline{x}\|^{2H}$ as in Definition 2.6, and let $W(x) = \frac{V(x)}{\mathfrak{n}(x)} \in C^{P-2}(A(\overline{x}), \mathbb{R})$ with $W(\overline{x}) = 1$, cf. Proposition 2.7.

Then there is an open, bounded and connected set $\overline{B} \subset A(\overline{x})$ and an $h^* > 0$, such that for all approximations $w \in C^{2k}(\mathbb{R}^d, \mathbb{R})$ of W with respect to a grid $X_N \subset \overline{B} \setminus \{\overline{x}\}$ with fill distance $h < h^*$ in the sense of Proposition 3.5 there is an open, bounded and connected set O with $O_0 \subset O$ such that for $v(x) = w(x)\mathfrak{n}(x)$ we have

- *$v(x) < R^*$ holds for all $x \in O$ and $v(x) = R^*$ holds for all $x \in \partial O$ for an $R^* \in \mathbb{R}^+$,*
- *$v'(x) < 0$ holds for all $x \in \overline{O} \setminus \{\overline{x}\}$.*

Proof. Set $R := \max_{x \in \overline{O_0}} V(x) > 0$ and define the sets K_1 and K_2 as the closure of the connected components which include $\overline{x}$ of the following sets:

$$\{x \in A(\overline{x}) \mid V(x) \leqslant R\} \quad \text{for } K_1,$$
$$\{x \in A(\overline{x}) \mid V(x) \leqslant R + 4\} \quad \text{for } K_2,$$

respectively. Denote by B the connected component which includes $\overline{x}$ of $\{x \in A(\overline{x}) | V(x) < R+5\}$. Then obviously $\overline{O_0} \subset K_1 \subset K_2 \subset B \subset \overline{B} \subset A(\overline{x})$ and B is open; recall that the sets K_1, K_2 and $\overline{B}$ are compact, cf. Theorem 2.2. All these sets are positively invariant.

Let $\tilde{B}$ be an open set with $\overline{B} \subset \tilde{B} \subset \overline{\tilde{B}} \subset A(\overline{x})$, e.g. $\tilde{B} = \{x \in A(\overline{x}) | V(x) < R+6\}$. Choose $\chi \in C_0^{\infty}(\mathbb{R}^d, [0,1])$ to be a function satisfying $\chi(x) = 1$ for $x \in \overline{B}$ and $\chi(x) = 0$ for $\mathbb{R}^d \setminus \tilde{B}$. Thus, $\chi \in C_0^{\infty}(\mathbb{R}^d) \subset \mathcal{F}$. Set $W_0 = W \cdot \chi$; then $W_0 \in C_0^{P-2}(\mathbb{R}^d, \mathbb{R})$ and $W_0(x) = W(x)$ holds for all $x \in \overline{B}$. Since $P - 2 \geqslant \sigma^*$, $W_0 \in H^{(d+1)/2+k}(\mathbb{R}^d) = \mathcal{F}$.

For $\varepsilon = \frac{1}{2}$ choose $0 < \delta \leqslant 1$ with Proposition 4.2 for W_0 such that both $\overline{B_\delta(\overline{x})} \subset O_0$ holds and we have for all $x \in \overline{B_\delta(\overline{x})}$

$$|W_0(x) - w(x)| \leqslant \frac{\varepsilon}{C}, \tag{5.7}$$

where C was defined in 2 of Proposition 2.7 for $\overline{B_1(\overline{x})}$.

Choose $1 \geqslant r_0 > 0$ so small that

$$U := \{x \in A(\overline{x}) | V(x) < r_0\} \subset \overline{B_\delta(\overline{x})} \quad \text{holds.}$$

We define a function $T: A(\overline{x}) \to \mathbb{N}_0$ denoting the minimal number $T(x) \in \mathbb{N}_0$ such that $S_{T(x)}x \in U$. The function fulfills $T(x) = 0$ if and only if $x \in U$. Moreover, by definition $T'(x) \leqslant 0$ holds. Since $\overline{B} \subset A(\overline{x})$ is a compact set, there is a $\theta_0 > 0$ such that $S_{\theta_0}\overline{B} \subset U$. Thus, $0 \leqslant T(x) \leqslant \theta_0$ holds for all $x \in \overline{B}$.

Let $h' > 0$ be the constant from Theorem 5.1 for the set $\overline{B}$ and define $h^* := \min(h', (2\theta_0 C^*)^{-1})$ where C^* was defined in Theorem 4.4 for the set $\overline{B}$ and $H = h'$. Thus,

$$v'(x) < 0 \quad \text{holds for all } x \in \overline{B} \setminus \{\overline{x}\}, \tag{5.8}$$

$$\text{and} \quad |V'(x) - v'(x)| \leqslant \frac{1}{2\theta_0} \quad \text{holds for all } x \in \overline{B} \tag{5.9}$$

by Theorem 5.1 and Theorem 4.4.

Now let $x \in U \subset \overline{B_\delta(\overline{x})}$. Hence, $V(x) \in [0, r_0)$. For $v(x) = \mathrm{n}(x)w(x)$ we have for all $x \in \overline{B_\delta(\overline{x})}$

$$|V(x) - v(x)| = \mathrm{n}(x)[W_0(x) - w(x)] \leqslant \underbrace{\max_{x \in \overline{B_\delta(\overline{x})}} \mathrm{n}(x)}_{\leqslant C \cdot \delta^2} |W_0(x) - w(x)| \leqslant \tfrac{1}{2}$$

by (5.7) since $\varepsilon = \frac{1}{2}$ and $\delta \leqslant 1$. Thus, $v(x) \leqslant V(x) + \frac{1}{2} < r_0 + \frac{1}{2}$ and $v(x) \geqslant V(x) - \frac{1}{2} \geqslant -\frac{1}{2}$, i.e.

$$v(x) \in \left[-\frac{1}{2}, r_0 + \frac{1}{2}\right) \quad \text{for all } x \in U. \tag{5.10}$$

Now define O as the connected component including $\overline{x}$ of

$$\{x \in B | v(x) < R + 2 =: R^*\}.$$

We will show that $K_1 \subset O \subset K_2$ holds. Then O is bounded and $\overline{O} \subset K_2 \subset \overline{B}$, i.e. $v'(x) < 0$ holds for all $x \in \overline{O} \setminus \{\overline{x}\}$. Furthermore, $v(x) < R^*$ holds for all $x \in O$ and $v(x) = R^*$ holds for all $x \in \partial O \subset K_2$. On the other hand, this shows $O_0 \subset K_1 \subset O$, i.e. $O_0 \subset O$.

To prove $K_1 \subset O$ we let $x \in K_1$. Then we have in particular $\theta_0 \geqslant T(x) \geqslant 0$ and $V(y) \in [0, r_0)$ for all $y \in U$. Hence, we obtain

$$\begin{aligned} v(x) &= v(S_{T(x)}x) - \sum_{k=0}^{T(x)-1} v'(S_k x) \\ &< r_0 + \frac{1}{2} - \sum_{k=0}^{T(x)-1} \left(V'(S_k x) - \frac{1}{2\theta_0} \right) \text{ by (5.10) and (5.9)} \\ &\leqslant \underbrace{V(S_{T(x)}x) - \sum_{k=0}^{T(x)-1} V'(S_k x)}_{=V(x)} + r_0 + \frac{1}{2} + \frac{T(x)}{2\theta_0} \\ &\leqslant V(x) + r_0 + 1 \leqslant R + 2 = R^* \end{aligned}$$

since $r_0 \leqslant 1$. This shows $x \in O$.

For the inclusion $O \subset K_2$ we show that for $x \in \partial K_2$ we have $v(x) > R^*$. For $x \in \partial K_2 \subset \overline{B}$ we have $V(x) = R + 4$. With $T(x) \leqslant \theta_0$ and $V(y) \in [0, r_0)$ for all $y \in U$ we have

$$\begin{aligned} v(x) &= v(S_{T(x)}x) - \sum_{k=0}^{T(x)-1} v'(S_k x) \\ &\geqslant -\frac{1}{2} - \sum_{k=0}^{T(x)-1} \left(V'(S_k x) + \frac{1}{2\theta_0} \right) \text{ by (5.10) and (5.9)} \\ &> \underbrace{V(S_{T(x)}x) - \sum_{k=0}^{T(x)-1} V'(S_k x)}_{=V(x)} - r_0 - \frac{1}{2} - \frac{T(x)}{2\theta_0} \\ &\geqslant V(x) - 1 - \frac{1}{2} - \frac{1}{2} = R + 4 - 2 = R^*, \end{aligned}$$

i.e. $x \notin O$. This proves the theorem. □

6. Examples

6.1. A toy example

Consider the difference equation

$$\begin{cases} x_{n+1} = \frac{1}{2}x_n + x_n^2 - y_n^2, \\ y_{n+1} = -\frac{1}{2}y_n + x_n^2 \end{cases} \tag{6.1}$$

with fixed point $\overline{x} = (0, 0)$. This is Example 1 of [4]. We have $d = 2$ and fix $k = 1$, thus $\sigma^* = \frac{5}{2}$. A function n for $P = 5$ has been calculated in (2.6) and (2.7).

Now we approximate $W(x) = \frac{V(x)}{\mathrm{n}(x)}$, where $V'(x) = -\|x\|^2$, by w. We use a hexagonal grid of the form $0.2\left(i + \frac{j}{2}, j\frac{\sqrt{3}}{2}\right)$ with $N = 118$ points without the point $\overline{x}$ ($i = j = 0$). Note that the grid used in Section 4.1 of [4] is of the form $0.2\left(i - \frac{j}{2}, j\frac{\sqrt{3}}{2}\right)$ and consists of

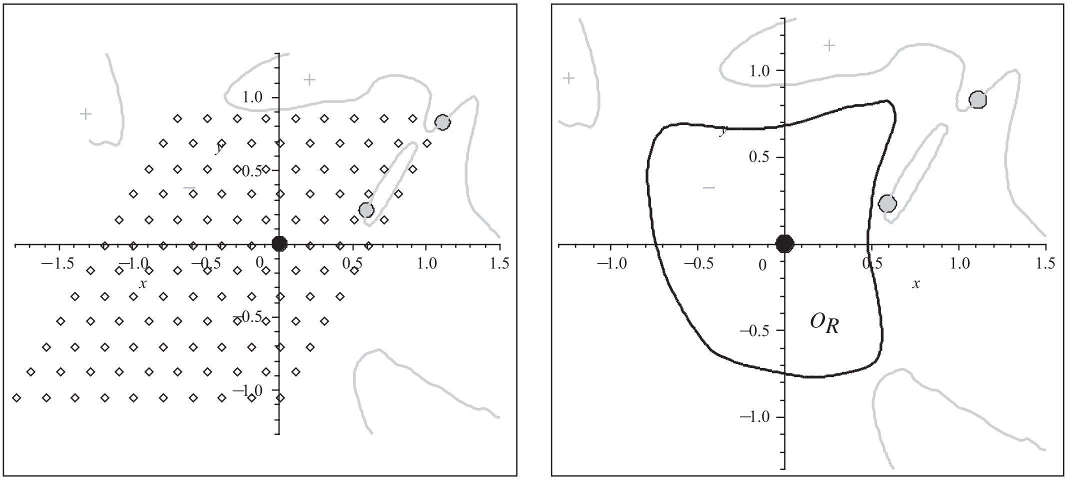

Fig. 1. All figures correspond to Example (6.1). The black point $\overline{x} = (0, 0)$ denotes the stable fixed point, whereas the gray points mark two unstable fixed points. Left: the grid with $N = 118$ points (black diamonds) and the sign of $v'(x)$ (gray). Right: the sign of $v'(x)$ (gray) and a sublevel set O_R (black) of v for $R = 1.05$, which is a subset of the basin of attraction $A(0, 0)$.

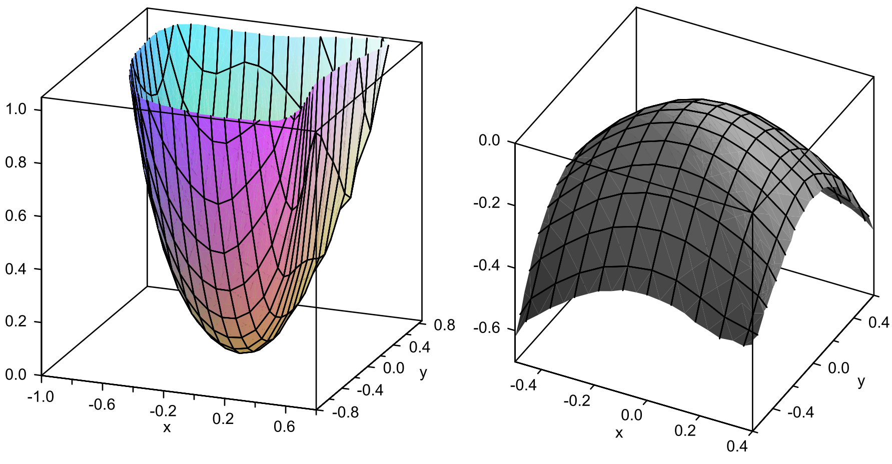

Fig. 2. All figures correspond to Example (6.1). Left: graph of the function $v(x) = \mathrm{n}(x)w(x)$ near $\overline{x} = (0, 0)$. Right: graph of the orbital derivative $v'(x) = \mathrm{n}(g(x))w(g(x)) - \mathrm{n}(x)w(x)$ near $\overline{x} = (0, 0)$.

$N = 119$ points. The radial basis function is given by $\Psi(x) = \psi_{3,1}(0.6\|x\|)$. For the function $v(x) = w(x)\,\mathrm{n}(x)$ we show the sign of $v'(x)$ (gray) in Fig. 1 as well as the sublevel set $O_R = \{x \in \mathbb{R}^2 | v(x) < R\}$ (black) with $R = 1.05$; note that $\overline{O_R}$ is a subset of the basin of attraction $A(0, 0)$. This subset of the basin of attraction is of similar size as the one obtained in [4, Section 4.1]. However, the discrete orbital derivative is now negative also in a neighborhood of $\overline{x}$ in contrast to [4]. Fig. 2 shows the local structure of $v(x)$ and $v'(x)$ near the equilibrium $\overline{x} = (0, 0)$.

6.2. A model for the demand for education

As a second example we consider a model proposed in [15] to model the human decision for or against higher education and the development of wages. We summarize the model here, for more details cf. [15,14]. For the model it is assumed that in each time period n a continuum of agents of mass one is born that lives for two periods. They decide in the first time unit of their life, whether they invest into education. If they do so, they can work in the high-level sector, otherwise they have to work in the low-level sector for all their life.

If the agent i invests into education, the wages will be $1 - e_i$ in the first year and w_{n+1} in the following; note that the wages are set 1 in the low-level sector and w_n in the high-level sector at time n. e_i reflects the costs of education for agent i and the costs are equally distributed over the continuum of agents. If the agent does not invest into education, his wages will be 1 in each year. The result is calculated by a utility function $U(1 - e_i, w_{n+1})$, $U(1, 1)$, respectively. The utility function is given by the standard Cobb–Douglas utility function

$$U(a, b) = a^\gamma b^{1-\gamma}$$

with $0 < \gamma < 1$, and we set $\delta := \frac{1-\gamma}{\gamma}$.

The agent i thus will invest into education, if $U(1 - e_i, w^e_{n+1}) > U(1, 1)$, where w^e_{n+1} is the expected wage in the following year; hence, all agents with $e_i < e(w^e_{n+1})$ will invest into education, where

$$e(w^e_{n+1}) = 1 - (w^e_{n+1})^{-\delta}. \tag{6.2}$$

We assume that there are two possibilities for the forecast of the expected wage of the following year w^e_{n+1}: the **adaptive agents** have the wage forecast

$$w^e_{n+1} = \rho w_n + (1 - \rho) w_{n-1}$$

with $0 \leqslant \rho \leqslant 1$, using a weighted average of the past two year's wages. The **fixed-point forecasters** (or steady state forecasters) use the fixed point w^* of the dynamical system as their forecast; we assume that it costs the information cost $C_s > 0$ to find out this fixed point.

We denote the amount of agents following the fixed-point forecast by z_n and the adaptive agents by $1 - z_n$. The supply of high-skilled labor depending on the wage w_{n+1} at time $n + 1$ is thus

$$l^s = (1 - z_n) e(\rho w_n + (1 - \rho) w_{n-1}) + z_n e(w^*). \tag{6.3}$$

The demand for high-skilled labor depending on the wage is given by

$$l^d(w) = \left(\frac{\alpha}{w}\right)^{1/(1-\mu)} \tag{6.4}$$

with productivity parameter $\alpha > 0$ and $\mu = \frac{\delta-1}{\delta}$. The market clearing condition $l^d(w_{n+1}) = l^s$ determines w_{n+1} as a function of w_n, w_{n-1} and z_n, i.e. cf. (6.3), (6.4) and (6.2)

$$w_{n+1} = \alpha \left[(1 - z_n)[1 - (\rho w_n + (1 - \rho) w_{n-1})^{-\delta}] + z_n (1 - (w^*)^{-\delta}) \right]^{-1/\delta}.$$

The amount of agents z_n, $1 - z_n$, respectively, following the different strategies will depend on how successful the respective strategies have been in the last year; this is called the generation overspill. Agents who choose the fixed-point strategy will regret their decision of education if they had earned more without education, i.e. if $e(w_{n+1}) \leqslant e \leqslant e(w^*)$. The regret is the difference between the possible utility $U(1,1)$ and the actual one $U(1-e, w_{n+1})$. Hence, considering the accumulated regret R_f for the fixed-point forecasters we obtain

$$R_f(w^*, w_{n+1}) = \int_{e(w_{n+1})}^{e(w^*)} R(e, w_{n+1})\, de,$$
$$\text{where} \quad R(e, w_{n+1}) = U(1,1) - U(1-e, w_{n+1}).$$

Similarly we obtain the accumulated regret for the choice of the adaptive forecasters as

$$R_a(w_{n-1}, w_n, w_{n+1}) = \int_{e(w_{n+1})}^{e(\rho w_n + (1-\rho) w_{n-1})} R(e, w_{n+1})\, de.$$

We define the difference $R^d = R_a - R_f = \int_{e(w^*)}^{e(\rho w_n+(1-\rho)w_{n-1})} (1-(1-e)^\gamma w_{n+1}^{1-\gamma})\, de$. If R^d is large, then the next generation prefers the fixed-point strategy *f*, if it is small or negative, the next generation rather prefers the adaptive strategy *a*. The costs C_s of the fixed-point strategy have a negative influence on the choice of the fixed-point strategy. Altogether $z_{n+1} = H(\beta(R^d - C_s))$, where $H(x) = \frac{1}{1+\exp(-x)}$. Note that the parameter $\beta > 0$ measures how sensible the next generation is to the advice of the former one.

Altogether, we obtain the following model, where we set $x_n = w_n^{-\delta}$ and $y_n = x_{n-1}$, cf. [14, Eq. (10)]

$$\begin{cases} x_{n+1} = G_s(x_n, y_n, z_n), \\ y_{n+1} = x_n, \\ z_{n+1} = H(\beta(R^d(x_n, y_n, G_s(x_n, y_n, z_n)) - C_s)) \end{cases} \tag{6.5}$$

with the functions, cf. [14, Eq. (11)]

$$G_s(x,y,z) = \frac{1}{\alpha^\delta}(1-z)\left(1 - (\rho x^{-1/\delta} + (1-\rho) y^{-1/\delta})^{-\delta}\right) + \frac{1}{1+\alpha^\delta} z,$$
$$R^d(x,y,\phi) = x^* - (\rho x^{-1/\delta} + (1-\rho) y^{-1/\delta})^{-\delta} - \frac{1+\delta}{2+\delta}\phi^{-1/(1+\delta)} \times \left((x^*)^{(2+\delta)/(1+\delta)} - (\rho x^{-1/\delta} + (1-\rho) y^{-1/\delta})^{-\delta((2+\delta)/(1+\delta))}\right),$$
$$H(x) = \frac{1}{1+\exp(-x)}$$

and $x^* = (w^*)^{-\delta} = \frac{1}{1+\alpha^\delta}$. By Proposition 7 of [14], $\overline{x} = \overline{y} = x^*$ and $\overline{z} = \frac{1}{1+\exp(\beta C_s)}$ is a fixed point.

We choose the constants $\alpha = \frac{3}{4}$, $\delta = 2$, $C_s = \frac{1}{4}$, $\rho = 0.4$, and $\beta = 10$. Since $\alpha < 1$ and $\rho < 1 - \alpha^\delta = \frac{7}{16}$, 3 of Proposition 7 implies, since $\beta C_s < \ln\left(\frac{\alpha^\delta}{1-\rho-\alpha^\delta}\right)$, that the fixed point is locally stable. However, there exists also the following stable period-three cycle, which lies outside the area of the model: $\mathbf{x}_1 = (0.160, 1.236, 0.473)$, $\mathbf{x}_2 = (0.845, 0.160, 0.101)$, $\mathbf{x}_3 = (1.236, 0.845, 0.194)$.

We are interested in the basin of attraction $A(\overline{x}, \overline{y}, \overline{z})$. We have $d = 3$ and fix $k = 1$, thus $\sigma^* = 3$. A function ň for $P = 5$ is given in Appendix A. We use a hexagonal grid of the form $(\overline{x}, \overline{y}, \overline{z}) + 0.08\left(i + \frac{j}{2} + \frac{k}{2}, j\frac{\sqrt{3}}{2} + \frac{k}{2\sqrt{3}}, k\sqrt{\frac{2}{3}}\right)$ with 99 points without the point $i = j = k = 0$ and with one additional point to obtain the Lyapunov function v, so that $N = 100$.

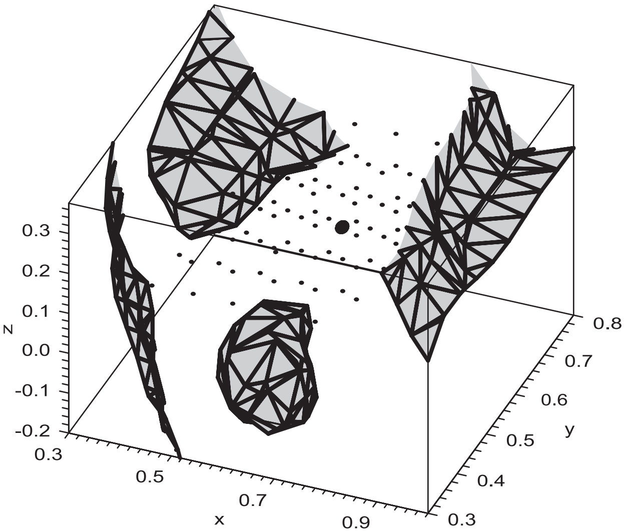

Fig. 3. We consider Example (6.5). The figure shows the stable fixed point $(\overline{x}, \overline{y}, \overline{z})$ (black point), the grid with $N = 100$ points (black diamonds) and the level set $v'(x, y, z) = 0$ (gray); note that the sign of v' is negative near the fixed point.

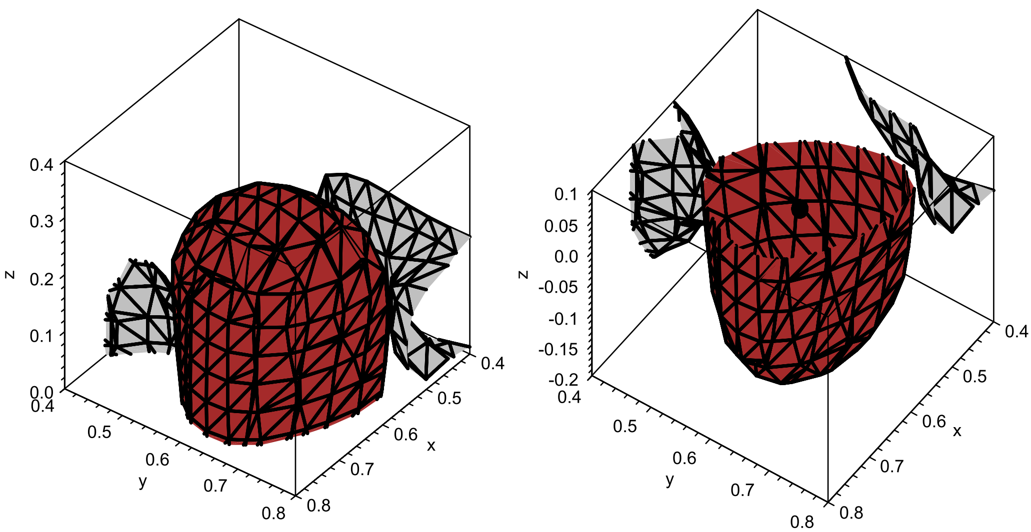

Fig. 4. All figures correspond to Example (6.5). The black point is the stable fixed point $(\overline{x}, \overline{y}, \overline{z})$. We show the level set $v'(x, y, z) = 0$ (gray) together with the sublevel set O_R (brown) of v for $R = 0.8$, which is a subset of the basin of attraction $A(\overline{x}, \overline{y}, \overline{z})$. Left: the part with $z \in [0, 0.4]$; note that this is the relevant part since the model is valid only for $z \geqslant 0$. This part is also positively invariant, since $z_{n+1} > 0$ by (6.5). Right: the part with $z \in [-0.2, 0.1]$.

We choose the radial basis function $\Psi(x) = \psi_{3,1}(\frac{3}{2}\|x\|)$. The sign of $v'(x)$ together with the grid is shown in Fig. 3. In Fig. 4 we show the level set $v'(x) = 0$ (gray) as well as the sublevel set $O_R = \{(x, y, z) | v(x, y, z) < R\}$ with $R = 0.8$ (brown) which is a subset of the basin of attraction $A(\overline{x}, \overline{y}, \overline{z})$. Note that the set O_R includes also points with negative z-values, whereas the model only makes sense for $z \geqslant 0$. The set $O_R \cap \{(x, y, z) | z \geqslant 0\}$ is shown in Fig. 4, left, and is also a subset of $A(\overline{x}, \overline{y}, \overline{z})$, since it is positively invariant; this can be seen from (6.5) since $z_{n+1} = H(\zeta) = \frac{1}{1+\exp(-\zeta)} > 0$ for all $\zeta \in \mathbb{R}$.

Appendix A. A function ռ for Section 6.2

A function ռ for the example of Section 6.2, cf. (6.5) is given by

$$
\begin{aligned}
&ռ(x, y, z)\\
&= 810 \frac{(e^5 + 2e^{5/2} + 1)(31e^{5/2} + 15)}{-1525e^{15/2} + 21\,201e^5 + 24\,705e^{5/2} + 6075}(x - \overline{x})^2\\
&\quad + \frac{18\,432(1 + e^{5/2})e^5}{-1525e^{15/2} + 21\,201e^5 + 24\,705e^{5/2} + 6075}(x - \overline{x})(y - \overline{y})\\
&\quad + \frac{1}{5}\frac{135\,223e^{15/2} + 175\,125e^5 + 123\,525e^{5/2} + 30\,375}{-1525e^{15/2} + 21\,201e^5 + 24\,705e^{5/2} + 6075}(y - \overline{y})^2 + (z - \overline{z})^2\\
&\quad -111.8301485(x - \overline{x})^3 - 318.2321415(x - \overline{x})^2(y - \overline{y})\\
&\quad -37.36154017(x - \overline{x})^2(z - \overline{z}) + 25.35481402(x - \overline{x})(y - \overline{y})^2\\
&\quad -167.3283992(x - \overline{x})(y - \overline{y})(z - \overline{z}) - 166.9291335(y - \overline{y})^2(z - \overline{z})\\
&\quad +34.80534914(y - \overline{y})^3\\
&\quad -467.5966683(x - \overline{x})^4 - 687.0022561(x - \overline{x})^3(y - \overline{y})\\
&\quad +290.8657722(x - \overline{x})^3(z - \overline{z}) - 774.2211474(x - \overline{x})^2(y - \overline{y})^2\\
&\quad +481.594964(x - \overline{x})^2(y - \overline{y})(z - \overline{z}) + 40.14033576(x - \overline{x})^2(z - \overline{z})^2\\
&\quad -194.8335032(x - \overline{x})(y - \overline{y})^3 - 182.5493215(x - \overline{x})(y - \overline{y})^2(z - \overline{z})\\
&\quad +120.4210073(x - \overline{x})(y - \overline{y})(z - \overline{z})^2 - 536.3814964(y - \overline{y})^4\\
&\quad -191.2350810(y - \overline{y})^3(z - \overline{z}) + 90.3157554(y - \overline{y})^2(z - \overline{z})^2\\
&\quad +1138.088078(x - \overline{x})^5 + 1115.953760(x - \overline{x})^4(y - \overline{y})\\
&\quad +6.907579760(x - \overline{x})^4(z - \overline{z}) - 1569.068708(x - \overline{x})^3(y - \overline{y})^2\\
&\quad +1697.730028(x - \overline{x})^3(y - \overline{y})(z - \overline{z}) - 105.8818797(x - \overline{x})^3(z - \overline{z})^2\\
&\quad -4208.607644(x - \overline{x})^2(y - \overline{y})^3 + 2603.716790(x - \overline{x})^2(y - \overline{y})^2(z - \overline{z})\\
&\quad -9.294101910(x - \overline{x})^2(y - \overline{y})(z - \overline{z})^2 - 3963.856339(x - \overline{x})(y - \overline{y})^4\\
&\quad +2286.60626(x - \overline{x})(y - \overline{y})^3(z - \overline{z}) + 418.695483(x - \overline{x})(y - \overline{y})^2(z - \overline{z})^2\\
&\quad -190.3537367(y - \overline{y})^5 + 2287.846586(y - \overline{y})^4(z - \overline{z})\\
&\quad +291.603610(y - \overline{y})^3(z - \overline{z})^2\\
&\quad +5000((x - \overline{x})^2 + (y - \overline{y})^2 + (z - \overline{z})^2)^3.
\end{aligned}
$$

Appendix B.

Proof of Lemma 2.4. By Remark 2.5, (2.2) can be solved by considering (2.3), which is a system of linear equations for c_α when considering each order of $(x - \overline{x})$. We will show that this system has a unique solution.

We will first transform the problem so that $Dg(\overline{x})$ becomes an upper diagonal matrix. Indeed, there is an invertible matrix S such that $J = SDg(\overline{x})S^{-1}$ is the (complex) Jordan normal form of $Dg(\overline{x})$. In particular, J is an upper diagonal matrix. Note that S and J are complex-valued matrices. However, if we can show that there is a unique solution $\mathfrak{h}$, it is obvious that all coefficients c_α are in fact real. Define $y := S(x - \overline{x})$, then $x = S^{-1}y + \overline{x}$. The iteration $x_{n+1} = g(x_n)$ then is equivalent to $y_{n+1} = S(x_{n+1} - \overline{x}) = S(g(x_n) - \overline{x}) = S(g(S^{-1}y_n + \overline{x}) - \overline{x}) =: f(y_n)$, and we have $Df(y) = SDg(S^{-1}y + \overline{x})S^{-1}$ and $Df(0) = SDg(\overline{x})S^{-1} = J$, which is an upper diagonal matrix. We will show that the following eq. (B.1) has a unique solution $\mathfrak{h}(y) = \sum_{2\leqslant|\alpha|\leqslant P} c_\alpha y^\alpha$:

$$\mathfrak{h}(f(y)) - \mathfrak{h}(y) = -y^T C y + o(\|y\|^P), \tag{B.1}$$

where $C = (S^{-1})^T S^{-1}$.

Note that $\mathfrak{h}(y) = \sum_{2\leqslant|\alpha|\leqslant P} c_\alpha y^\alpha$ is a solution of (B.1) if and only if $\tilde{\mathfrak{h}}(x) := \mathfrak{h}(S(x-\overline{x})) = \sum_{2\leqslant|\alpha|\leqslant P} c_\alpha (S(x-\overline{x}))^\alpha = \sum_{2\leqslant|\alpha|\leqslant P} c'_\alpha (x-\overline{x})^\alpha$ is a solution of (2.2), since we have with $y = S(x-\overline{x})$

$$\begin{aligned}
\mathfrak{h}(f(y_n)) - \mathfrak{h}(y_n) &= \mathfrak{h}(y_{n+1}) - \mathfrak{h}(y_n) \\
&= \mathfrak{h}(S(x_{n+1}-\overline{x})) - \mathfrak{h}(S(x_n - \overline{x})) \\
&= \tilde{\mathfrak{h}}(x_{n+1}) - \tilde{\mathfrak{h}}(x_n) \\
\text{and} \quad -y^T C y + o(\|y\|^P) &= -\|S^{-1}y\|^2 + o(\|S^{-1}y\|^P) \\
&= -\|x-\overline{x}\|^2 + o(\|x-\overline{x}\|^P).
\end{aligned}$$

Hence, we have to show that there exists a unique solution $\mathfrak{h}(y) = \sum_{2\leqslant|\alpha|\leqslant P} c_\alpha y^\alpha$ of (B.1), i.e. cf. Remark 2.5

$$\sum_{2\leqslant|\alpha|\leqslant P} c_\alpha \left[\left(Jy + \sum_{2\leqslant|\beta|\leqslant P-1} \frac{\partial^\beta f(0)}{\beta!} y^\beta \right)^\alpha - y^\alpha \right] = -y^T C y + o(\|y\|^P). \tag{B.2}$$

Note that $\partial^\beta f(0) = (\partial^\beta f_1(0), \ldots, \partial^\beta f_d(0))^T$ is a vector, C is a symmetric matrix and $P \geqslant 2$. We consider the terms order by order in y. The lowest appearing order is two and the terms of this order in y of both sides of (B.2) are

$$\sum_{|\alpha|=2} c_\alpha \left[(Jy)^\alpha - y^\alpha \right] = -y^T C y. \tag{B.3}$$

Writing the terms of order two as $\sum_{|\alpha|=2} c_\alpha y^\alpha = y^T B y$, (B.3) becomes $J^T B J - B = -C$. Since the eigenvalues λ of J are the eigenvalues of $Dg(\overline{x})$ and thus satisfy $|\lambda| < 1$, this equation has a unique solution B; the proof for the complex-valued matrix J is the same as for the real case, cf. [16,4, Lemma 2.5].

Now we show by induction with respect to $k := |\alpha| \leqslant P$ that the constants c_α are uniquely determined by (B.2): For $|\alpha| = 2$ this has just been done. Now let $P \geqslant |\alpha| = k \geqslant 3$. Consider (B.2): the terms of order $\leqslant k-1$ satisfy the equation by induction. Now consider the terms of order $|\alpha| = k$: all terms of order k are contained in the following expression:

$$\sum_{2\leqslant|\alpha|\leqslant k-1} c_\alpha \left(Jy + \sum_{2\leqslant|\beta|\leqslant P-1} \frac{\partial^\beta f(0)}{\beta!} y^\beta \right)^\alpha + \sum_{|\alpha|=k} c_\alpha \left[(Jy)^\alpha - y^\alpha \right]. \tag{B.4}$$

The constants c_α with $2 \leqslant |\alpha| \leqslant k-1$ are fixed. We will show that there is a unique solution for the constants c_α with $|\alpha| = k$ such that $\sum_{2\leqslant|\alpha|\leqslant k-1} c_\alpha \left(Jy + \sum_{2\leqslant|\beta|\leqslant P-1} \frac{\partial^\beta f(0)}{\beta!} y^\beta\right)^\alpha +$ $\sum_{|\alpha|=k} c_\alpha \left[(Jy)^\alpha - y^\alpha\right]$ has no terms of order k. Since all c_α in the left-hand term of (B.4) are known and all c_α in the right-hand term are unknown, this is equivalent to an inhomogeneous system of linear equations. It has a unique solution, if and only if the corresponding homogeneous system has only the zero solution. Therefore, we study the corresponding homogeneous problem

$$\sum_{|\alpha|=k} c_\alpha \left[(Jy)^\alpha - y^\alpha\right] = 0, \tag{B.5}$$

and show that $c_\alpha = 0$ is its only solution.

J is an upper diagonal matrix such that all eigenvalues λ satisfy $|\lambda| < 1$ since J is the Jordan Normal Form of $Dg(\overline{x})$. The eigenvalues are on the diagonal and thus $|J_{ii}| < 1$ holds for all $1 \leqslant i \leqslant d$. Hence,

$$Jy = (J_{11}y_1 + J_{12}y_2 + \cdots + J_{1d}y_d, J_{22}y_2 + \cdots + J_{2d}y_d, \ldots, J_{dd}y_d)^T. \tag{B.6}$$

We prove by induction that all coefficients c_α in (B.5) vanish. Note that $|\alpha| = k$. We introduce an order on $\tilde{A} := \left\{\alpha \in \mathbb{N}_0^d | \alpha_i \in \{0, \ldots, k\} \text{ for all } i \in \{1, \ldots, d\}\right\}$. Note that $\{\alpha \in \mathbb{N}_0^d \,|\, |\alpha| = k\} \subset \tilde{A}$. The order $\|\alpha\|$ on $\tilde{A}$ is such that $\|\alpha\|$ is the $(k+1)$-adic expansion of α, i.e.

$$\mathbb{N}_0 \ni \|\alpha\| = \sum_{l=1}^{d} \alpha_l (k+1)^{l-1}.$$

Now we start the induction with respect to $\|\alpha\|$. The minimal α with $|\alpha| = k$ is $\alpha = (k, 0, \ldots, 0)$. The coefficient of y_1^k in (B.5) is, due to (B.6), $c_{(k,0,\ldots,0)}(J_{11}^k - 1)$. Since $|J_{11}^k| = |J_{11}|^k < 1$, we have $c_{(k,0,\ldots,0)} = 0$.

Now we assume that all coefficients c_α with $|\alpha| = k$ and $\|\alpha\| \leqslant A^*$ for some $A^* \in \mathbb{N}$ are zero. Let $\beta \in \mathbb{N}_0^d$ be minimal with $\|\beta\| > A^*$ and $|\beta| = k$. We will show that $c_\beta = 0$. Consider the coefficient of y^β in (B.5). Due to (B.6), only the terms $\sum_{|\alpha|=k} (Jy)^\alpha$ with

$$\begin{aligned}
\beta_1 &\leqslant \alpha_1 \quad \text{(to obtain } y_1^{\beta_1}\text{)}\\
\beta_1 + \beta_2 &\leqslant \alpha_1 + \alpha_2 \quad \text{(to obtain } y_2^{\beta_2}\text{)}\\
&\vdots\\
\sum_{l=1}^{d-1} \beta_l &\leqslant \sum_{l=1}^{d-1} \alpha_l \quad \text{(to obtain } y_{d-1}^{\beta_{d-1}}\text{)}\\
\sum_{l=1}^{d} \beta_l &\leqslant \sum_{l=1}^{d} \alpha_l \quad \text{(to obtain } y_d^{\beta_d}\text{)}
\end{aligned}$$

contribute to terms with y^β.

We will show that $\|\alpha\| \leqslant \|\beta\|$. The last inequality is $k = k$, since $|\alpha| = |\beta| = k$. By subtracting this last equation from the inequality before, we obtain $\alpha_d \leqslant \beta_d$. If $\alpha_d < \beta_d$, then $\|\alpha\| < \|\beta\|$. If $\alpha_d = \beta_d$, then $\sum_{l=1}^{d-1} \beta_l = \sum_{l=1}^{d-1} \alpha_l$ and we can proceed by subtracting this second-last equation from the third-last inequality, and obtain $\alpha_{d-1} \leqslant \beta_{d-1}$, etc.

Since $c_\alpha = 0$ holds for all $\|\alpha\| < \|\beta\|$ by induction, the only term left with y^β is the one with $\|\alpha\| = \|\beta\|$, i.e.

$$c_\beta[z^\beta y^\beta - y^\beta] = 0,$$

where $z = (J_{11}, J_{22}, \ldots, J_{dd})^T$. Since $|J_{ii}| < 1$ for all $i = 1, \ldots, d$, we have $|z^\beta| < \max_{i=1,\ldots,d} |J_{ii}|^{|\beta|} < 1$ and thus $z^\beta - 1 \neq 0$. Hence, we can conclude $c_\beta = 0$.

We show that $\mathfrak{h}$ is the Taylor polynomial of V of order P: Let $V(x) = T(x) + \tilde{\varphi}(x)$ with $\tilde{\varphi}(x) = o(\|x - \overline{x}\|^P)$, where $T(x) = \sum_{0 \leqslant |\alpha| \leqslant P} d_\alpha (x - \overline{x})^\alpha$ is the Taylor polynomial of V of order P. Then

$$\begin{aligned} -\|x - \overline{x}\|^2 &= V'(x) \\ &= V(g(x)) - V(x) \\ &= T(g(x)) - T(x) + \tilde{\varphi}(g(x)) - \tilde{\varphi}(x) \\ &= T'(x) + o(\|x - \overline{x}\|^P) \end{aligned} \tag{B.7}$$

since $\tilde{\varphi}(x) = o(\|x - \overline{x}\|^P)$ by definition and $\tilde{\varphi}(g(x)) = o(\|x - \overline{x}\|^P)$ as we show now:

Set $m := \max_{y \in \overline{B_1(\overline{x})}} \|Dg(y)\|$. Let $\varepsilon > 0$. Then there is a $0 < \delta < 1$ such that

$$\left| \frac{\tilde{\varphi}(x)}{\|x - \overline{x}\|^P} \right| < \frac{\varepsilon}{m^P} \tag{B.8}$$

for all $x \in B_\delta(\overline{x}) \setminus \{\overline{x}\}$, since $\tilde{\varphi}(x) = o(\|x - \overline{x}\|^P)$. Since $g(x) - g(\overline{x}) = g(x) - \overline{x} = o(1)$ by Taylor's Theorem, for the above $\delta > 0$, there is a $0 < \delta' < 1$ such that $\|g(x) - \overline{x}\| < \delta$, i.e. $g(x) \in B_\delta(\overline{x})$ for all $x \in B_{\delta'}(\overline{x})$. On the other hand, the mean value theorem yields $\|g(x) - g(\overline{x})\| \leqslant m\|x - \overline{x}\|$, i.e.

$$\frac{1}{\|x - \overline{x}\|} \leqslant \frac{m}{\|g(x) - \overline{x}\|} \tag{B.9}$$

for all $x \in B_{\delta'}(\overline{x}) \setminus \{\overline{x}\}$. With (B.8) and (B.9) we conclude that for all $\varepsilon > 0$ there is a $0 < \delta' < 1$ such that

$$\left| \frac{\tilde{\varphi}(g(x))}{\|x - \overline{x}\|^P} \right| \leqslant m^P \left| \frac{\tilde{\varphi}(g(x))}{\|g(x) - \overline{x}\|^P} \right| < \varepsilon$$

for all $x \in B_{\delta'}(\overline{x}) \setminus \{\overline{x}\}$. This shows $\tilde{\varphi}(g(x)) = o(\|x - \overline{x}\|^P)$.

Note that $V(\overline{x}) = 0$, and hence $d_0 = 0$. The terms of order one are also zero. Indeed, denote $d_i := c_{e^i}$ for $i = 1, \ldots, d$ such that $\sum_{|\alpha|=1} d_\alpha (x - \overline{x}) = d^T (x - \overline{x})$. Then (B.7) implies $d^T Dg(\overline{x})(x - \overline{x}) - d^T (x - \overline{x}) = 0$, i.e. $d^T (Dg(\overline{x}) - I) = 0$. Assume in contradiction that $d \neq 0$. Then $\det(Dg(\overline{x}) - I) = 0$, i.e. that $\lambda = 1$ is an eigenvalue of $Dg(\overline{x})$, which is a contradiction. By the uniqueness of $\mathfrak{h}$, we have $\sum_{0 \leqslant |\alpha| \leqslant P} d_\alpha (x - \overline{x})^\alpha = \mathfrak{h}(x)$, since it satisfies (2.2). □

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