GEORG CANTOR. HIS MATHEMATICS AND PHILOSOPHY OF THE INFINITE.

By Joseph Warren Dauben. Cambridge, Mass./London (Harvard University Press). 1979. ix + 404 pp. $27.50.

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Without question Georg Cantor is one of the most fascinating figures in the history of mathematics. Declaring that the essence of mathematics lies in its freedom, Cantor dared to make the actual infinite the subject of mathematical investigation, thereby creating the Paradise from which Hilbert vowed mathematicians would never be driven. Cantor's personal life was as sensational as his mathematics. Tales of his bitter struggle against the opposition of leading mathematicians, such as Kronecker, and of his insanity have been told often, although frequently without much concern for veracity or significance. Even before his death in 1918, Cantor and his transfinite numbers had become the subject of historical research, and a sizable literature has accrued over the years. In recent times one thinks especially of the contributions of Ivor Grattan-Guinness, Herbert Meschkowski, and Joseph Dauben, the author of the book under review. With its publication Dauben has provided what was still lacking and needed after such extensive scholarly activity: a comprehensive, critical, and well-documented synthesis of the fruits of that activity into a literate intellectual biography of Cantor.

The skeleton of the book is formed by a series of straightforward expositions of the contents of Cantor's most important publications, beginning with the seminal investigations on trigonometric series (1870-1872) and ending with the monographic "Beitrage zur Begrundung der transfiniten Mengenlehre" (1895-1897), where Cantor attempted his clearest and most defensible presentation of the theory of transfinite numbers. This expository material, roughly half of the text, serves to delineate the gradual progression of Cantor's mathematics from relatively orthodox analysis to his radically different and controversial theory of transfinite numbers. It should prove especially beneficial to anyone unable or disinclined to read Cantor's actual publications and the Noether-Cavaillès edition of Cantor's correspondence with Dedekind. In particular, the material is a valuable source of readings for courses in history of mathematics. Completing the skeleton are an introductory chapter tracing the analytical background to Cantor's work with trigonometric series and a penultimate one on post-Cantorian set theory.

The introductory chapter treats the researches of Dirichlet, Riemann, Lipschitz, and Hankel that were motivated by Fourier's ground-breaking contributions to the methods of analysis. Although Riemann's *Habilitationsschrift* is the most directly relevant to Cantor's work on the uniqueness of trigonometric series representations, all of the above-mentioned mathematicians set an important precedent for Cantor by introducing the consideration of infinite sets of points in their analysis. As Dauben indicates, however, none of them developed the set-theoretic aspects of their work. In this respect, it would have been enlightening if more attention had been paid to the misconceptions regarding the possible structure of infinite point sets that underlay--and to a certain extent undermined--the deliberations of Hankel and Lipschitz. In particular, although over three pages are devoted to Lipschitz, the reader is left with the impression that Lipschitz, largely out of lack of interest, simply failed to pursue the analysis of infinite sets when, in fact, underlying misconceptions (not mentioned by Dauben) led Lipschitz to a rather naive analysis of all the structural possibilities for infinite sets of points, thereby obviating the need for further investigation [1]. Had greater emphasis been accorded to these matters, a further dimension would have been added to the discussion of the interesting question, raised at the end of Chapter 1, as to why Cantor's predecessors failed to develop "a more general and autonomous theory."

Such a tantalizing question certainly admits no simple, nor absolute, answer. Dauben suggests that, prior to Cantor, there was a natural tendency to focus more attention upon the function than upon its domain since, after all, the theory of functions was under investigation. He also suggests that the development of the theory of sets was linked to a rigorous theory of real numbers: "Cantor ... was led to focus his attention upon the ways in which point sets with various specified properties might be defined. His approach, moreover, required the development of a rigorous theory of real numbers. This was a necessary step before point sets of complicated structure could be satisfactorily identified, described, and analyzed" (p. 28). In what sense was it a necessary step? Surely a rigorous theory of real numbers is not needed to grasp distinct possibilities for the structure of infinite sets. This point is clearly illustrated by a paper published by H. J. S. Smith in 1875 to which Dauben himself refers prior to the above statement. Without knowledge of Cantor's work and without employing a rigorous theory of real numbers, Smith constructed lucid, geometrically conceivable examples of diverse types of nowhere dense sets. Using Cantor's terminology, Smith constructed sets of the first species, perfect nowhere dense sets of zero content, and perfect nowhere dense sets of positive content. Smith did not proceed to develop systematically the theory of point sets, whereas Cantor did. But can this difference be explained by the presence of a rigorous theory of real numbers in Cantor's work? Yes and no.

In making the above-quoted statement, Dauben had in mind the fact that in Cantor's paper of 1872, where set-theoretical notions are first introduced in his research, Cantor began by presenting a construction of the real number system, which is carefully distinguished from the geometrical continuum by means of Cauchy sequences. I have always been struck by the unexpected nature of this part of Cantor's paper. To compose a paper that would have been praised by the likes of Dirichlet, Riemann, and Lipschitz, all Cantor had to do was to introduce the concept of a derived set and then proceed to show how his uniqueness theorem for a finite set of exceptional points can be extended by induction to the case in which the set has an nth derived set which is finite. But Cantor felt compelled to do more. Starting from the system A of rational numbers, he constructed a system B identifiable with the real numbers and then from B a system C, and so on. The hierarchy of systems A, B, C, ... was used to describe a limit point of the nth order and, thereby, a set with finite nth derived set. There are, however, much simpler ways to conceive such sets, such as Smith's, as Cantor undoubtedly realized. Nonetheless, Cantor did not follow the most expeditious course, and therein lie further reasons why it was Cantor who was destined to become the creator of the theory of transfinite sets and numbers.

The need that Cantor felt to establish rigorously the existence of limit points on the basis of a construction of the real number system was certainly reinforced, if not originated, by his training at the University of Berlin, where, in fact, Weierstrass had already attempted such a construction in his lectures on the foundations of analysis. Furthermore, the emphasis at Berlin upon foundational matters and upon a rigorous and systematic investigation of mathematical problems helps explain why Cantor, rather than Smith, proceeded to investigate systematically the theory of transfinite sets. Cantor's construction of the unending progression of number systems, A, B, C, ..., also reflects his characteristic penchant for constructing conceptual hierarchies. His belief in the legitimacy and importance of such a "dialectical generation of concepts" (as he later called it) was a driving force behind his creation of transfinite numbers. In these respects, the presence in Cantor's work of a rigorous theory of real numbers helps to explain why he became founder of the theory of transfinite sets and numbers. Is this what Dauben had in mind by his assertion? In the exposition of Cantor's paper of 1872 (in Chapter 2) he never tells us, although such conclusions could perhaps be inferred from reading the exposition. The exposition of Cantor's mathematics throughout the book is distinguished by its cautious adherence to the original. The highest priority has been given to a faithful, occasionally prolix, representation of Cantor's work.

The chapter on post-Cantorian set theory is a fitting capstone to the presentation of Cantor's mathematics since Dauben concentrates upon developments directly linked to Cantor's work: the paradoxes; Zermelo's proof of the Well-Ordering Theorem and his concomitant axiomatization of the theory of sets; the celebrated letters of Baire, Borel, Hadamard, and Lebesgue; the programs of Frege, Russell, and Hilbert; and the discoveries of Cohen. Naturally these topics are too extensive and complex to receive much more than a cursory treatment aimed at rounding off the presentation of Cantor's mathematics. For a more detailed and penetrating historical analysis of post-Cantorian set theory the recent work of Moore should be consulted [2]. It is an encouraging sign of the increasing vitality of the discipline of the history of mathematics that no sooner is such a substantial work as Dauben's off the press than it can be supplemented by more recent, equally significant work. In this respect, Dauben's discussion of early attempts to establish the invariance of dimension--prompted by Cantor's startling discovery that R and R^n can be placed in one-to-one correspondence, but otherwise tangential to Dauben's story--should be supplemented by Johnson's recent essay [3]. The choice of material on post-Cantorian set theory also reflects Dauben's decision to concentrate upon those aspects of Cantor's mathematics that pertain to his theory of transfinite numbers. Consequently, scant attention is given to tracing the considerable impact Cantor's results on point sets and his method of transfinite induction had upon mathematicians working in more traditional areas of mathematics.

So much for the bare bones of the book. The flesh and blood are provided by the material dealing with nonmathematical matters such as Cantor's relations with Kronecker; Cantor's mental breakdowns; his concern for the physical, philosophical, and theological ramifications of the theory of transfinite sets and numbers, particularly his dialogue with Catholic theologians in the aftermath of the Papal encyclical *Aeterni Patris*; his role in the formation of the *Deutscher Mathematiker-Vereinigung* and in bringing about the first international congress of mathematicians; his harsh, dogmatic rejection of Veronese's infinite and infinitesimal numbers--an ironic counterpart to Kronecker's criticism of his transfinite numbers; his deep-seated religious sentiments, and the overall nature of his personality. It is in the treatment of such extramathematical matters that Dauben's book excels. By thus distinguishing the mathematical and extramathematical elements of the book I do not mean to suggest that Dauben disengages them. Quite the contrary. He fully appreciates that the two are inextricable when it comes to a historical understanding of the mathematics Cantor created, a type of mathematics so unorthodox and inherently controversial that it undoubtedly required a personality as extraordinary as Cantor's to pursue it doggedly as he did. Cantor came to see himself as an oracle, the means by which God chose to communicate knowledge of the infinite. Although Cantor's remarkable religious sentiments are expressed most fully in extant documents postdating his first mental breakdown in 1884, Dauben adduces evidence that the views expressed after his breakdown are rooted in earlier convictions [4].

The final chapter, an epilogue entitled "Cantor's Personality," should be read at least twice: once before reading the book and then again afterward. Although the entire book is written with a sensitivity to the bearing of Cantor's idiosyncratic personality upon his mathematics, these matters are first considered systematically in the Epilogue because of the more conjectural nature of the analysis. But the conjectures are plausible and, in any case, based upon evidence which is helpful to have considered before proceeding through the book. I am not suggesting that the Epilogue should have been an introduction; it is placed appropriately, but the reader may find, as I did, that foreknowledge of its contents adds considerably to the understanding of many episodes in Cantor's professional and mathematical life.

The Epilogue contains as detailed a depiction of Cantor's breakdowns as the evidence permits. Dauben concurs with Grattan-Guinness' assessment of Cantor's mental illness as essentially endogenous. Although he rejects E. T. Bell's crass portrayal of Cantor's father, Georg Woldemar Cantor, as the source of all his son's later mental troubles, Dauben does suggest that Georg Woldemar did indeed leave a lasting imprint upon his son and, in particular, was instrumental in instilling in him an overriding sense of the importance of success and of a faith in God, especially as a source of strength in times of distress. The basis for this view is the remarkable letter written by Georg Woldemar to Georg on the occasion of his confirmation at age 15. The letter, published by Fraenkel in 1930, is indeed uncanny in the way it foreshadows Cantor's subsequent perception of his mathematics and the fate of his career. Even the delusions of persecution which are manifest (to this reviewer at least) in the documents quoted throughout the book are prophesied in Georg Woldemar's warning of "enemies" capable of forcing Georg "to stand in the second or third rank" (p. 275). On the more positive side, the religious sentiments which Georg Woldemar encouraged in his son seem reflected in Cantor's conviction that his theory of transfinite numbers was divinely inspired and infallible. According to Dauben:

> *Later generations might forget the philosophy, smile at the abundant references to St. Thomas and the Church fathers, overlook his metaphysical pronouncements and miss entirely the deep religious roots of Cantor's later faith in the veracity of his work. But these all*

> *contributed to Cantor's resolve not to abandon his transfinite numbers for less controversial and more acceptable interests... His forbearance, as much as anything else he might have contributed, ensured that set theory would survive the early years of doubt and denunciation to flourish eventually as a vigorous, revolutionary force in scientific thought in the twentieth century.* (p. 299)

By considering the relevance of Cantor's personality to the development of his mathematical ideas, Dauben has kept the promise made in the introduction "to go beyond names, dates and theorems" and to present "a study of the pulse, metabolism, even in part the psychodynamics of an intellectual process: the creation of a new mathematical theory" (p. 4). In this respect I think that Dauben could have gone even further than he does by considering more carefully and fully the relevance of Cantor's experiences at the University of Berlin. I have attempted something along these lines in the case of Wilhelm Killing [5], and Cantor seems another, especially suitable candidate for such an approach. In addition to the disciplinary ideals fostered by Weierstrass, I suspect that the type of arithmetic cultivated at Berlin, notably Kummer's theory of ideal complex numbers, was highly supportive of the spirit and direction of Cantor's mathematical research. Cantor himself, in defending "free mathematics," invoked the precedent of Kummer's theory of ideal numbers. Although, as Dauben notes (p. 133), by making reference to Kummer, Cantor could point out that, without Kummer's theory of ideal numbers, the world would be unable to appreciate the work of Kronecker and Dedekind, the reference reflects more than Cantor's cunning; it reflects the pervasive influence of the implications of Kummer's theory. For example, in a letter to Dedekind dated 15 April 1882, Cantor referred to "the ideal nature in Kummer's sense" of the concept of an irrational number in his own and Dedekind's theories of the real numbers [6]. Although Cantor's mathematics of course reflects his extraordinary personality, the view of mathematics he encountered at Berlin was certainly congenial to its development.

Accompanying the text are an extensive bibliography, very complete through the mid-1970s, and an admirably thorough index. Several brief appendixes contain hitherto unpublished material of interest. For example, one appendix contains a portion of a letter from Sophie Kowalevsky to Mittag-Leffler, humorously portraying Cantor's unsuccessful attempt to lecture on philosophy at Halle University. Two appendixes add a bit more to the sparse documentary evidence relating to Kronecker's opposition to Weierstrassian analysis. The format of the book is attractive and convenient. Even though the numerous footnotes are placed at the end of the book, the pages containing them have headings specifying the corresponding pages of the text, so that it is relatively easy to pass back and forth from text to footnotes. There are many phrases and sentences quoted in Latin without translation which may be a source of annoyance to some readers. Another source of annoyance stems from Dauben's occasionally unorthodox use of mathematical terms and his use of unfamiliar terms without sufficient explication, both of which may cause mathematicians to misconstrue his statements or, at least, to fail to understand them [7].

Historians of mathematics can only be grateful for the effort Professor Dauben has expended to create the synthesis of Cantor scholarship found in his book. But the book can, and I hope will, be read with profit by a far more extensive audience. Any student, mathematician, philosopher, theologian, or general historian with an interest in Georg Cantor and the wondrous revolution in mathematical and philosophical thought that his work did so much to precipitate will find this book of considerable interest.

NOTES

1. Paul Montel called attention to the flaws in Lipschitz' reasoning in his French translation of the paper (*Acta Mathematica* 36, No. 3 (1912), 284, n. 2; 286, n. 1). For a discussion of Lipschitz' reasoning along the lines of Montel's observations, see pp. 14-15 of my book, *Lebesgue's Theory of Integration. Its Origins and Development*, 2nd ed., New York (Chelsea), 1975. Dauben's claim (p. 10) that Dirichlet, in a letter to Gauss of 1853 discussing the extension of his covergence theorem to functions with an infinity of maxima and minima, introduced "what today would be termed ... sets of zero measure" also seems unlikely. A more reasonable interpretation of Dirichlet's remarks, when taken within the context of his proof for the case of finitely many extrema, would be that Dirichlet presumed, as did Lipschitz later, that the infinite set of points giving rise to extreme values possesses a finite number of limit points.

2. Gregory H. Moore, *Zermelo's axiom of choice: Its origins and role in the development of mathematics (1821-1940)*, Doctoral thesis, University of Toronto, 1979 (publication planned by Springer-Verlag, New York). See also Moore's papers: The origins of Zermelo's axiomatization of set theory, *Journal of Philosophical Logic* 7 (1978), 307-329; Beyond first-order logic: The historical interplay between mathematical logic and axiomatic set theory, *History and Philosophy of Logic* 1, (1980), 95-137. On Russell's logic see also the recent book by I. Grattan-Guinness, *Dear Russell-Dear Jourdain*, London (Duckworth), 1977; New York (Columbia Univ. Press), 1979.

3. Dale M. Johnson, The problem of invariance of dimension in the growth of modern topology, Part I, *Archive for History of Exact Sciences* 20, No. 2 (1979), 97-188.

4. See pp. 146; 146, n. 107; 229; 229, n. 33; 290. An additional piece of evidence is contained in Cantor's letter to Dedekind of 5 November 1882 (E. Noether and J. Cavaillès, eds., *Briefwechsel Cantor-Dedekind* (Paris, 1937), p. 55).

5. Non-Euclidean geometry and Weierstrassian mathematics: The background to Killing's work on Lie algebras, *Historia Mathematica* 7 (1980) 289-342.

6. *Briefwechsel Cantor-Dedekind*, p. 51.

7. Examples are Dauben's use of "functional analysis" (p. 6 and throughout), "finite" (p. 13), "negligible" (p. 22), "continuous" and "linear" (p. 92), "connected " and "everywhere dense" (p. 110), "category" (p. 151).

SADI CARNOT ET L'ESSOR DE LA THERMODYNAMIQUE. Table Ronde du Centre National de la Recherche Scientifique. Paris, École Polytechnique, 11-13 juin 1974. Paris (Éditions du Centre National de la Recherche Scientifique). 1976. 435 pp.

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On the occasion of the 150th anniversary of Sadi Carnot's *Réflexions sur la puissance motrice du feu,* the Centre National de la Recherche Scientifique (CNRS), in collaboration with the École Polytechnique, in June of 1974 sponsored a round-table symposium to discuss the significance of Sadi Carnot's work, to identify his precursors and the scientific and technological traditions of his day, and to explore the advances in thermodynamics in modern and recent times. The symposium, which was organized by Professors René Taton and Pierre Costabel, drew together an international constellation of specialists.

Among the 37 papers delivered, about one-half were historically oriented. Together they provide a valuable persepective and assessment of the current state of our knowledge concerning the genesis, reception, and later reformulation and extension of Carnot's ideas. The themes that were explored in this connection encompass a broad spectrum of biographical, scientific, and contextual aspects of Carnot's contributions to thermodynamics, and are placed within the scientific and technological climate of