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Sums and products of ultracomplete topological spaces

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Abstract

In 1987 V.I. Ponomarev and V.V. Tkachuk characterized strongly complete topological spaces as those spaces which have countable character in their Stone–Čech compactification. On the other hand, in 1998 S. Romaguera introduced the notion of cofinally Čech complete spaces and he showed that a metrizable space admits a cofinally complete metric (otherwise, called ultracomplete metric), a term introduced independently by N.R. Howes in 1971 and A. Császár in 1975, if and only if it is cofinally Čech complete. In a recent paper the authors showed that these two notions are equivalent and in this way answered a question raised by Ponomarev and Tkachuk [Vestnik MGU 5 (1987) 16–19] about giving an internal characterization for strongly complete topological spaces (termed ultracomplete by the authors). In this paper, sums and products of ultracomplete spaces are studied.

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1. Introduction

Ultracomplete topological spaces were introduced independently by Ponomarev and Tkachuk in 1987 [12] and by Romaguera in 1998 [13]. In [12] ultracomplete spaces were called *strongly complete* and were defined by their external characterization (Theorem 2.1(1)) while in [13] ultracomplete spaces were called *cofinally Čech complete* and were defined by their internal characterization (Theorem 2.1(3)). In [2], the authors proved that the two definitions are in fact equivalent and termed such spaces *ultracomplete*.

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Ultracompleteness constitutes an interesting strong form of completeness as can be seen from results obtained in [12,13,6,2]. It is proved in [13, Theorem 1] that a metrizable space admits a cofinally complete metric if and only if it is ultracomplete (then termed cofinally Čech complete). To define the term *cofinally complete metric*, a few words about uniformities is in order. A filter $\mathcal{F}$ on a uniform space $(X, \mathfrak{U})$ is said to be *weakly Cauchy* if for each cover $\mathcal{U} \in \mathfrak{U}$ there is a filter $\mathcal{G}$ containing $\mathcal{F}$ and a $G \in \mathcal{G}$ such that $G \subset U$ for some $U \in \mathcal{U}$ [3]. In [8] Howes introduced the notion of a *cofinally complete* uniform space and proved that a uniform space is cofinally complete if and only if every weakly Cauchy filter has a cluster point in X. Cofinally complete uniform spaces were called *ultracomplete* by Császár in [4]. In his paper, Császár showed that the Euclidean metric on the real line $\mathbb{R}$ is cofinally complete and that there exists a complete metric space that is not cofinally complete, where a metric space (X, ρ) is said to be cofinally complete if the uniformity $\mathfrak{U}_\rho$ generated by ρ is cofinally complete. Howes [9] (see also [10]) later on showed that the Hilbert space ℓ_2 of square summable sequences is complete but not cofinally complete for its usual metric.

We refer the reader to [5] for undefined terms. We also use the terminology of [5] when it comes to seperation axioms, in particular, $T_{3\frac{1}{2}}$ ($\equiv$ *Tychonoff*) implies T_1. Throughout the paper, by $\mathbb{N}$ we denote the set of *natural numbers*. By $\aleph_0$ we denote the cardinality of $\mathbb{N}$. For a subset A of a space X, by $\overline{A}$ we denote the closure of A in X. Finally, all spaces are assumed to be Tychonoff.

2. Preliminaries

Let us recall that two collections of sets $\mathcal{F}$ and $\mathcal{U}$ *mesh* if every $F \in \mathcal{F}$ intersects every $U \in \mathcal{U}$. As in [5], we denote a compactification of a space X by a pair (Y, c), where Y is a compact Hausdorff space and $c : X \to Y$ is a homeomorphic embedding of X into Y such that $\overline{c(X)} = Y$. Below, by a compactification of X we shall mean not only a pair (Y, c) but also the compact space $Y \equiv cX$. Also, in many situations, we shall identify X with $c(X)$ and so $\overline{X} = cX$. The Stone–Čech compactification of a space X is denoted by βX. A collection $\mathcal{B}(A)$ of open subsets of a space X is called a *base for a set* $A \subset X$ *in* X if all the elements of $\mathcal{B}(A)$ contain A and for any open set V containing A there exists a $U \in \mathcal{B}(A)$ such that $A \subset U \subset V$. The *character of* A *in* X is defined to be the smallest cardinal number of the form $|\mathcal{B}(A)|$, where $\mathcal{B}(A)$ is a base for A in X, and is denoted by $\chi(A, X)$. Below, for a collection $\mathcal{P}$ of subsets of a set X, by $\mathcal{P}^F$ we denote the collection of all unions of finite subcollections from $\mathcal{P}$.

The following theorem was proved in [2].

Theorem 2.1. *For every space X the following conditions are equivalent*:

(1) *X has countable character in one* (*equivalently, in all*) *of its Hausdorff compactifications cX, i.e., $\chi(X, cX) \leqslant \aleph_0$.*

(2) *There exists a locally compact space Z and a homeomorphic embedding $e : X \to Z$ of X into Z satisfying $\chi(e(X), Z) \leqslant \aleph_0$.*

(3) *There exists a sequence* $\{\mathcal{U}_n\}$ *of open covers of* X *such that, if* $\mathcal{F}$ *is a filter base on* X *which meshes with some sequence* $\{U_n\colon U_n \in \mathcal{U}_n\}$, *then* $\mathcal{F}$ *clusters in* X (*the sequence* $\{U_n\colon U_n \in \mathcal{U}_n\}$ *is called an* ultracomplete sequence of open covers).
(4) *There exists a sequence* $\{\mathcal{U}_n\colon n \in \mathbb{N}\}$ *of open covers of* X, *such that for every open cover* $\mathcal{V}$ *of* X *there exists an* $n \in \mathbb{N}$ *satisfying* $\mathcal{U}_n < \mathcal{V}^F$.

The following implications are thus evident:

locally compact → ultracomplete → Čech complete

Examples show that none of the above implications are reversible, even in the realm of metrizable spaces (see [13,2]).

In [12] and [6] it is proved that ultracompleteness is invariant and inverse invariant under perfect maps (in the realm of Tychonoff spaces). It is also proved in [6] that unlike Čech completeness, ultracompleteness is preserved by open maps (in the realm of Tychonoff spaces). On the other hand, an example is given in [2] to show that ultracompleteness is not an invariant of closed maps.

The following result, which is evident from the definition, is known.

Proposition 2.2. *Ultracompleteness is hereditary with respect to closed subsets.*

It is known that Čech completeness is hereditary with respect to G_δ-subsets. Therefore, one would expect that if X is ultracomplete and a subset $A \subset X$ has countable character in the X, then A is also ultracomplete. The following example shows that this is not true.

Example 2.3. Let $X \subset \beta\mathbb{N}$ be defined by $X = \beta\mathbb{N} \setminus \{x_i\colon i \in \mathbb{N}\}$, where $x_i \in \beta\mathbb{N} \setminus \mathbb{N}$ for every $i \in \mathbb{N}$. Then X is a countably compact, ultracomplete, non-locally compact space. From Corollary 4.14 we have that $X \times \beta X$ is also ultracomplete. Now $X \times \mathbb{N}$ is open and dense in $X \times \beta X$ and Theorem 4.1 shows that $X \times \mathbb{N}$ is not ultracomplete.

3. Sums of ultracomplete spaces

Let X be a space, if we denote the topology on X by $\tau(X)$ then $\tau^*(X) = \tau(X) \setminus \{\emptyset\}$. The subset $A \subset X$ is said to be *bounded in* X if every continuous real function on X is bounded on A. This is equivalent to saying that the collection $\mathcal{B}_A = \{U \in \mathcal{B}\colon U \cap A \neq \emptyset\}$ is finite for every discrete subcollection $\mathcal{B} \subset \tau^*(X)$. The following result was given independently in [12,6].

Proposition 3.1. *Let* X *be an ultracomplete space and let* $X_C = \{x \in X\colon X$ *is not locally compact at the point* $x\}$. *Then* X_C *is bounded in* X.

It is not difficult to see that if X and Y are two ultracomplete spaces, then their sum $X \oplus Y$ is also ultracomplete. We therefore have the following result.

Theorem 3.2. *If X_k is ultracomplete for every $k = 1, \dots, n$, then their sum $\bigoplus_{k=1}^{n} X_k$ is also ultracomplete.*

For infinite sums we have

Theorem 3.3. *Let $\mathcal{A}$ be some infinite indexing set. The sum $X = \bigoplus_{\alpha \in \mathcal{A}} X_\alpha$ is ultracomplete if, and only if, there exists a finite subset $\mathcal{A}_0 \subset \mathcal{A}$ such that X_α is locally compact for every $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$ and X_α is ultracomplete for every $\alpha \in \mathcal{A}_0$.*

Proof. We only need to prove the *only if* part. Since X_α is clopen in X for every $\alpha \in \mathcal{A}$ we have that X_α is ultracomplete for every $\alpha \in \mathcal{A}$.

Suppose that there exists a sequence $\mathcal{A}' = \{\alpha(n)\colon\ n \in \mathbb{N}\} \subset \mathcal{A}$ such that $X_{\alpha(n)}$ is not locally compact for all $n \in \mathbb{N}$. Therefore, for every $n \in \mathbb{N}$, there exists $a_n \in X_{\alpha(n)}$ such that a_n admits no compact neighbourhood in $X_{\alpha(n)}$. Let $A = \{a_n\colon\ n \in \mathbb{N}\}$, then it is not difficult to see that A is not bounded in X and therefore, neither is the set X_C. By Proposition 3.1 one concludes that X is not ultracomplete. □

4. Products of ultracomplete spaces

We begin this section by showing that the product of two ultracomplete spaces does not have to be ultracomplete.

Theorem 4.1. *Let X be a non-locally compact space and let Y be non-countably compact, then the product $X \times Y$ is not ultracomplete.*

Proof. Since ultracompleteness is an invariant of open maps, if either X or Y is not ultracomplete then neither is their product $X \times Y$. Therefore, assume that X and Y are ultracomplete. Since X is not locally compact, neither is the space $X_n = X \times \{n\}$ for every $n \in \mathbb{N}$. Thus, by Theorem 3.3, we have that $X \times \mathbb{N} \cong \bigoplus_{n \in \mathbb{N}} X_n$ is not ultracomplete. But the space Y is not countably compact, and therefore there exists a closed subset $A \subset Y$ such that $A \cong \mathbb{N}$. Consequently, $X \times \mathbb{N} \cong \bigoplus_{n \in \mathbb{N}} X_n$ is a closed subset of $X \times Y$, and by Proposition 2.2 one concludes that $X \times Y$ cannot be ultracomplete. □

Let κ be an infinite cardinal number and let $\mathcal{D}(\kappa)$ denote the discrete topological space of cardinality κ. We will consider the Stone–Čech compactification $\beta\mathcal{D}(\kappa)$ of $\mathcal{D}(\kappa)$. Let $\mathcal{D}(\kappa)^* = \beta\mathcal{D}(\kappa) \setminus \mathcal{D}(\kappa)$. Our objective is to prove the following two theorems.

Theorem 4.2. *Let X and Y be two ultracomplete, countably compact spaces. Then, their product $X \times Y$ is also ultracomplete, countably compact.*

Theorem 4.3. *Let X and Y be two ultracomplete spaces such that X is countably compact, locally compact while Y is non-countably compact, non-locally compact. Then, their product $X \times Y$ is also ultracomplete.*

The proofs of the above theorems will be preceded by lemmas, the proofs of which (unless given) are similar to the proofs of the corresponding results for the particular case of $\kappa = \aleph_0$ where one can consult [14].

Lemma 4.4. *Every clopen subset of $\mathcal{D}(\kappa)^*$ is of the form $W(M) = \overline{M} \cap \mathcal{D}(\kappa)^*$, where $M \subset \mathcal{D}(\kappa)$.*

Lemma 4.5. *If $M_1, M_2 \subset \mathcal{D}(\kappa)$, then $W(M_1) \subset W(M_2)$ if and only if $M_1 \setminus M_2$ is finite.*

Lemma 4.6. *Every G_δ set $A \subset \mathcal{D}(\kappa)^*$ has a non-empty interior.*

Lemma 4.7. *Let $\{x_n\}$ be a sequence of distinct points in $\beta\mathcal{D}(\kappa)$. Then, there exists a subsequence $\{x_{n(k)}\}$ of $\{x_n\}$ such that the subspace $\{x_{n(k)}\}$ is discrete and therefore, there exist clopen sets V_k such that $x_{n(k)} \in V_k$ and $V_k \cap V_h = \emptyset$ whenever $k \neq h$.*

Lemma 4.8. *Let X be an ultracomplete space satisfying $\mathcal{D}(\kappa) \subset X \subset \beta\mathcal{D}(\kappa)$. Let $\{U_n\colon n \in \mathbb{N}\}$ be a countable base for X in $\beta X \cong \beta\mathcal{D}(\kappa)$. If $x_n \in U_n \setminus X$ for $n \in \mathbb{N}$, then $\{x_n\}$ has cluster points in X.*

Proof. Let $F = \overline{\{x_n\}}$. If $\{x_n\}$ does not have cluster points in X, then $X \subset \beta X \setminus F$ and there does not exist $n \in \mathbb{N}$ such that $X \subset U_n \subset \beta X \setminus F$. $\square$

Lemma 4.9. *Let X be an ultracomplete space satisfying $\mathcal{D}(\kappa) \subset X \subset \beta\mathcal{D}(\kappa)$. Let $\{x_n\}$ be a sequence in βX. There exists a subsequence $A = \{x_{n(k)}\}$ of $\{x_n\}$ such that $\overline{A} = \beta A \cong \beta\mathbb{N}$.*

Proof. Using Lemma 4.7 one can go on to prove that every continuous function $g : A \to I$ can be extended to a function $G : \overline{A} \to I$, which proves our assertion (see, for example, [5, Theorem 3.6.14]). $\square$

Lemma 4.10. *Let κ_1 and κ_2 be two infinite cardinal numbers, and let X and Y be two ultracomplete, countably compact spaces satisfying $\mathcal{D}(\kappa_1) \subset X \subset \beta\mathcal{D}(\kappa_1)$ and $\mathcal{D}(\kappa_2) \subset Y \subset \beta\mathcal{D}(\kappa_2)$. Then, their product $X \times Y$ is also ultracomplete, countably compact.*

Proof. That $X \times Y$ is countably compact follows from the fact that every ultracomplete space is a k-space.

Assume that X and Y are not locally compact. There exist countable bases $\{U_n\colon n \in \mathbb{N}\}$ of X in $\beta X \cong \beta\mathcal{D}(\kappa_1)$ and $\{V_n\colon n \in \mathbb{N}\}$ of Y in $\beta Y \cong \beta\mathcal{D}(\kappa_2)$. One can take both bases to be monotonically decreasing. We will show that $\{U_n \times V_n\colon n \in \mathbb{N}\}$ is a base for $X \times Y$ in $\beta X \times \beta Y$.

Assume the contrary, then there exists an open set $W \subset \beta X \times \beta Y$ such that $X \times Y \subset W$ and $(U_n \times V_n) \setminus W \neq \emptyset$ for every $n \in \mathbb{N}$. Take arbitrary points $(x_n, y_n) \in (U_n \times V_n) \setminus W$ for every $n \in \mathbb{N}$. Without loss of generality, assume that there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that $y_{n(k)} \in V_{n(k)} \setminus Y$ for $k \in \mathbb{N}$ (otherwise consider the sequence $\{x_n\}$). By Lemma 4.7 one can assume that both $\{y_{n(k)}\}$ and $\{x_{n(k)}\}$ are discrete subspaces (if $\{x_{n(k)}\}$ is a stationary sequence then the proof would also follow easily).

Consider the sequence of points $\{(x_{n(k)}, y_{n(k)})\}$. Since $(x_{n(k)}, y_{n(k)}) \in (\beta X \times \beta Y) \setminus W$ we should have that $\overline{\{(x_{n(k)}, y_{n(k)})\}} \cap X \times Y = \emptyset$. We show that this does not hold and thus arrive at a contradiction. By Lemma 4.8 we have $A = \overline{\{y_{n(k)}\}} \cap Y \neq \emptyset$ and by Lemma 4.9, $K = \overline{\{y_{n(k)}\}} \cong \beta\mathbb{N}$. Since A is a G_δ-set in $K^* = K \setminus \{y_{n(k)}\}$, it has a non-empty interior (see Lemma 4.6). In other words, there exists a subsequence $\{y_{n(k,j)}\}$ such that $\overline{\{y_{n(k,j)}\}} \setminus \{y_{n(k,j)}\} \subset Y$ (see Lemma 4.4). Now, let $H = \overline{\{x_{n(k,j)}\}}$. Since X is countably compact, one can take a point $x \in (H \setminus \{x_{n(k,j)}\}) \cap X$. There exists a homeomorphism $h : H \to K'$, where $K' = \overline{\{y_{n(k,j)}\}}$, satisfying $h(x_{n(k,j)}) = y_{n(k,j)}$. Let $y = h(x)$. Then $(x, y) \in X \times Y$ and (x, y) is a cluster point of $\{(x_{n(k,j)}, y_{n(k,j)})\}$. This gives a contradiction.

Finally, if both X and Y are locally compact then $X \times Y$ is locally compact (and so is ultracomplete), while if only one of X and Y is locally compact, the proof is analogous (but simpler) to the above proof. □

Lemma 4.11. *Let κ_1 and κ_2 be two infinite cardinal numbers, and let X and Y be two ultracomplete spaces satisfying $\mathcal{D}(\kappa_1) \subset X \subset \beta\mathcal{D}(\kappa_1)$ and $\mathcal{D}(\kappa_2) \subset Y \subset \beta\mathcal{D}(\kappa_2)$ such that X is countably compact, locally compact while Y is non-countably compact, non-locally compact. Then, their product $X \times Y$ is also ultracomplete.*

Proof. Since Y is ultracomplete, there exists a monotonically decreasing countable base $\{V_n : n \in \mathbb{N}\}$ of Y in $\beta Y \cong \beta\mathcal{D}(\kappa_2)$. We will show that $\{X \times V_n : n \in \mathbb{N}\}$ is a base for $X \times Y$ in $\beta X \times \beta Y$.

Assume the contrary, then there exists an open set $W \subset \beta X \times \beta Y$ such that $X \times Y \subset W$ and $(X \times V_n) \setminus W \neq \emptyset$ for every $n \in \mathbb{N}$. Take arbitrary points $(x_n, y_n) \in (X \times V_n) \setminus W$ for every $n \in \mathbb{N}$. By Lemma 4.7, there exists a subsequence $\{y_{n(k)}\}$ of $\{y_n\}$ such that the subspace $\{y_{n(k)}\}$ is discrete.

Consider the sequence of points $\{(x_{n(k)}, y_{n(k)})\}$. Since $(x_{n(k)}, y_{n(k)}) \in (\beta X \times \beta Y) \setminus W$ we should have that $\overline{\{(x_{n(k)}, y_{n(k)})\}} \cap X \times Y = \emptyset$. As in Lemma 4.10, one can show that this does not hold and thus arrive at a contradiction. □

Proof of Theorem 4.2. Let X and Y be two ultracomplete, countably compact spaces. Then $X \times Y$ is also countably compact.

Denote by $\widetilde{X}$ (respectively $\widetilde{Y}$) the set X (respectively Y) with the discrete topology. The continuous maps $\mathrm{id}_X : \widetilde{X} \to X$ and $\mathrm{id}_Y : \widetilde{Y} \to Y$ allow perfect extensions $F_X : \beta\widetilde{X} \to \beta X$ and $F_Y : \beta\widetilde{Y} \to \beta Y$. Consider the perfect maps $F'_X : F_X^{-1}(X) \to X$ and $F'_Y : F_Y^{-1}(Y) \to Y$. The space $F_X^{-1}(X) \times F_Y^{-1}(Y)$ is ultracomplete by Lemma 4.10 and therefore, $X \times Y = (F'_X \times F'_Y)(F_X^{-1}(X) \times F_Y^{-1}(Y))$ is also ultracomplete. □

Proof of Theorem 4.3. This is analogous to the proof of Theorem 4.2 but using Lemma 4.11 instead of Lemma 4.10. □

Theorems 4.1, 4.2 and 4.3 can be combined to give the following result.

Theorem 4.12. *Let X and Y be two ultracomplete spaces. Then $X \times Y$ is ultracomplete if, and only if, one of the following conditions holds*:
(i) *both X and Y are locally compact, or*
(ii) *either X or Y is countably compact, locally compact, or*
(iii) *both X and Y are countably compact.*

As a corollary to Theorem 4.12 one can cite the following result obtained in [6].

Corollary 4.13. *Let X and Y be two paracompact ultracomplete spaces. Then $X \times Y$ is paracompact and ultracomplete if, and only if, one of the following conditions holds*:
(i) *both X and Y are locally compact, or*
(ii) *either X or Y is compact.*

The next result can also be given as a corollary.

Corollary 4.14. *The Tychonoff product $X \times Y$ of an ultracomplete space X and a compact space Y is ultracomplete.*

By induction we have that:

Corollary 4.15. *Let X_k be an ultracomplete, countably compact space for every $k = 1, \ldots, n$, then $\prod_{k=1}^{n} X_k$ is also ultracomplete, countably compact.*

We now show that the above result can be extended to countable products.

Theorem 4.16. *Let X_n be an ultracomplete, countably compact space for every $n \in \mathbb{N}$, then $X = \prod_{n\in\mathbb{N}} X_n$ is also ultracomplete, countably compact.*

Proof. We only need to show that X is ultracomplete. Let $\{U_k^n\colon k \in \mathbb{N}\}$ be a countable base for X_n in βX_n. Consider the following countable collection of open sets in $\prod_{n\in\mathbb{N}} \beta X_n$:

$$\left\{U_n^1 \times \cdots \times U_n^k \times \prod_{i=k+1}^{\infty} \beta X_i\colon k, n \in \mathbb{N}\right\}.$$

We show that this collection is a base for X in $cX = \prod_{n\in\mathbb{N}} \beta X_n$. Let U be an open set in cX such that $X \subset U \subset cX$. For every $x = \{x_n\} \in X$ there exists an elementary neighbourhood $V(x) = \prod_{n\in\mathbb{N}} V_n(x_n) \subset U$. Let $m(x) = \min\{i\colon V_n(x_n) = \beta X_n$ for every $n \geqslant i\}$. Let $W_k = \bigcup\{V(x)\colon m(x) = k\}$ and let $W = \bigcup_{k\in\mathbb{N}} W_k$. Then the set W is open in cX and $X \subset W \subset$

U. Since X is countably compact, there exists some $n \in \mathbb{N}$ such that $X \subset \bigcup_{k=1}^{n} W_k \subset U$. Denote by pr the projection $\mathrm{pr} : \prod_{k\in\mathbb{N}} \beta X_k \to \prod_{k=1}^{n} \beta X_k$. We therefore have

$$\prod_{k=1}^{n} X_k \subset \bigcup_{k=1}^{n} \mathrm{pr}(W_k) \subset \prod_{k=1}^{n} \beta X_k$$

and consequently, by Corollary 4.15, there exists some $i \in \mathbb{N}$ such that

$$\prod_{k=1}^{n} X_k \subset \prod_{k=1}^{n} U_i^k \subset \bigcup_{k=1}^{n} \mathrm{pr}(W_k).$$

But, for every $k \leqslant n$ we have $\mathrm{pr}^{-1}\mathrm{pr}(W_k) = W_k$ and therefore,

$$\begin{aligned} X &\subset \mathrm{pr}^{-1}\left(\prod_{k=1}^{n} X_k\right) \subset \mathrm{pr}^{-1}\left(\prod_{k=1}^{n} U_i^k\right) = \left(\prod_{k=1}^{n} U_i^k\right) \times \left(\prod_{k=n+1}^{\infty} \beta X_k\right) \\ &\subset \mathrm{pr}^{-1}\left(\bigcup_{k=1}^{n} \mathrm{pr}(W_k)\right) = \bigcup_{k=1}^{n} W_k \subset U. \qquad \square \end{aligned}$$

The following corollary follows from Proposition 2.2 and Theorem 4.16.

Corollary 4.17. *If X is the limit of an inverse sequence of ultracomplete, countably compact spaces, then it is also ultracomplete, countably compact.*

From the above results one can give the following result on countable products of ultracomplete spaces.

Theorem 4.18. *Let X_n be an ultracomplete space for every $n \in \mathbb{N}$, then $X = \prod_{n\in\mathbb{N}} X_n$ is ultracomplete if, and only if, either*

(i) *X_n is countably compact for every $n \in \mathbb{N}$, or*

(ii) *there exists $n_0 \in \mathbb{N}$ and a finite set $\mathbb{N}_0 \subset \mathbb{N} \setminus \{n_0\}$ such that X_{n_0} is not locally compact (or not countably compact), X_n is locally compact, countably compact for all $n \in \mathbb{N}_0$ and X_n is compact for all $n \in \mathbb{N} \setminus \{\{n_0\} \cup \mathbb{N}_0\}$, or*

(iii) *there exists a finite set $\mathbb{N}_0 \subset \mathbb{N}$ such that X_n is locally compact for all $n \in \mathbb{N}_0$ and X_n is compact for all $n \in \mathbb{N} \setminus \mathbb{N}_0$.*

Remark 4.19. One can note that due to Archangel'skiĭ's result (see [1]) that the product $\prod_{a\in\mathcal{A}} X_\alpha$, where $X_\alpha \neq \emptyset$ for $\alpha \in \mathcal{A}$, is of pointwise countable type if and only if all spaces X_α are of pointwise countable type and there exists a countable set $\mathcal{A}_0 \subset \mathcal{A}$ such that X_α is compact for $\alpha \in \mathcal{A} \setminus \mathcal{A}_0$, it is enough to study countable products of ultracomplete spaces. One can also note that the countable product of locally compact spaces need not be ultracomplete as the space $\mathbb{R}^{\aleph_0} \cong \ell_2$ shows, where $\mathbb{R}$ is the set of real numbers with the standard topology and ℓ_2 is the Hilbert space of square summable sequences.

5. Ultracompleteness and countable compactness

The proof of Theorem 4.2 would be evident if one can prove that every Čech complete, countably compact space is ultracomplete. Unfortunately, we do not know the answer to this question and we therefore have the following problem.

Problem 5.1. Is every Čech complete, countably compact space, ultracomplete?

We do have some partial results to Problem 5.1.

Theorem 5.2. *Every Čech complete, countably compact GO-space is ultracomplete.*

Proof. Let X be a Čech complete, countably compact GO-space. Consider the Dedekind compactification X^* of X (see, for example, [11]) and let $\{U_n\colon n \in \mathbb{N}\}$ be a collection of open subsets of X^* satisfying (i) $U_{n+1} \subset U_n$ for every $n \in \mathbb{N}$ and (ii) $X = \bigcap_{n\in\mathbb{N}} U_n$. We will prove that $\{U_n\colon n \in \mathbb{N}\}$ is in fact a base for X in X^*. If not, there exists an open set U in X^* such that $U_n \setminus U \neq \emptyset$ for every $n \in \mathbb{N}$. For each n take a point $x_n \in U_n \setminus U$ and consider the set $\{x_n\colon n \in \mathbb{N}\}$. One can assume that the points x_n are distinct. Since countable compactness and sequential compactness are equivalent in GO-spaces, there exists a convergent subsequence $\{x_{n(k)}\colon k \in \mathbb{N}\}$. Let $x = \lim x_{n(k)}$, then $x \in X^* \setminus X$. Without loss of generality, one can assume that the points $x_{n(k)}$ are monotonically increasing. Let $a_{n(k)} \in X$ be such that $x_{n(k-1)} < a_{n(k)} < x_{n(k)}$ for $k = 2, 3, \ldots$. Then, since X is sequentially compact, there exists $a \in X$ such that $a = \lim a_{n(k,j)}$ for some subsequence $\{a_{n(k,j)}\}$ of $\{a_{n(k)}\}$. Consequently, we get that

$$a = \lim a_{n(k,j)} = \lim x_{n(k,j)} = x,$$

while $a \in X$ and $x \in X^* \setminus X$, leading to a contradiction. □

On the other hand, we have an example of a Čech complete, pseudocompact space which is not ultracomplete.

Example 5.3. Let $\{U_i\colon i \in \mathbb{N}\}$ be a collection of clopen infinite disjoint subsets of $\beta\mathbb{N}$. For each $i \in \mathbb{N}$ choose a countably infinite discrete $A_i \subset U_i \setminus \mathbb{N}$ and let $A = \bigcup\{A_i\colon i \in \mathbb{N}\}$. It is clear that A is a discrete subspace of $\beta\mathbb{N} \setminus \mathbb{N}$ and hence $\overline{A}$ is homeomorphic to $\beta\mathbb{N}$. Let $A^* = \overline{A} \setminus A$ and $A_i^* = \overline{A_i} \setminus A_i$. It is immediate that A_i^* is open in A^* and hence the set $F = A^* \setminus (\bigcup\{A_i^*\colon i \in \mathbb{N}\})$ is compact. As a consequence, the set $H = F \cup A$ is σ-compact and hence $X = \beta\mathbb{N} \setminus H$ is Čech complete.

To prove that X is pseudocompact, use a result of Hewitt [7] which says that a space X is pseudocompact if and only if the remainder $\beta X \setminus X$ does not contain non-empty G_δ-subsets of βX. In our case $\beta X \setminus X = H$ and if H contained a non-empty G_δ-subset of $\beta\mathbb{N}$, then its interior in $\beta\mathbb{N} \setminus \mathbb{N}$ would be non-empty, while it is clear that H is nowhere dense in $\beta\mathbb{N} \setminus \mathbb{N}$.

To prove that X is not ultracomplete, observe first that for any open (in $\beta\mathbb{N}$) set $G \supset X$ we have $A_i \setminus G$ is finite for any $i \in \mathbb{N}$ and therefore $A_i \cap G$ is infinite. Now if $\{W_i\colon i \in \mathbb{N}\}$

is an external base of X in $\beta X = \beta\mathbb{N}$, then $A_i \cap W_i$ is infinite (and hence non-empty) for all $i \in \mathbb{N}$, which makes it possible to choose an $s_i \in A_i \cap W_i$. Let $S = \{s_i\colon i \in \mathbb{N}\}$. Remember that $\overline{A}$ is homeomorphic to $\beta\mathbb{N}$, apply a simple fact about $\beta\mathbb{N}$: if $P, Q \subset \mathbb{N}$ and $P \cap Q$ is finite, then $\overline{P} \cap \overline{Q} \cap (\beta\mathbb{N} \setminus \mathbb{N}) = \emptyset$. Since $S \cap A_i$ consists of one point, we have $S^* \cap A_i^* = \emptyset$, where $S^* = \overline{S} \setminus S$. Thus, $\overline{S} = S \cup S^*$ is a closed subset of βX which lies in $\beta X \setminus X$ and intersects every W_i, which is a contradiction.

Finally, X is not countably compact because if we take a point $x_i \in A_i^*$ for all $i \in \mathbb{N}$ then the set $\{x_i\colon i \in \mathbb{N}\}$ is closed and discrete in X.

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