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ARC COMPONENTS OF CHAINABLE HAUSDORFF CONTINUA

G.R. GORDH, Jr. and Sam B. NADLER, Jr. *
University of Kentucky, Lexington, Ky. 40506, U.S.A.
Loyola University, New Orleans, La. 70118, U.S.A.
University of North Carolina, Charlotte, N.C. 28213, U.S.A.

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Abstract: General theorems concerning one-to-one continuous functions from connected linearly ordered spaces into Hausdorff continua are first obtained. These results are applied to a study of the structure of the arc components of chainable Hausdorff continua.

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a-triodic	irreducible continuum	real curve
decomposable continuum	linear order	Stone-Čech compactification
half-ray curve	monotone function	unicoherent continuum
half-ray triod	one-point compactification	

1. Introduction

The primary purpose of this paper is to determine information about the structure of the arc components of chainable Hausdorff continua (Section 4). The two main results about arc components of such continua are the following:

(1) each arc component is a one-to-one continuous image of a connected linearly ordered space (Theorem 4.1), and

(2) assuming such a continuum to be hereditarily decomposable, each arc component is homeomorphic to a connected linearly ordered space (Theorem 4.2).

Among other results, we show that if M is a chainable Hausdorff conti-

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nuum with exactly two arc components, then one such component is an arc and the other is homeomorphic to a connected linearly ordered space with a first point and no last point (Corollary 4.12).

The results in Section 4 are proved with the aid of material developed in Section 3 concerning one-to-one continuous mappings of connected linearly ordered spaces. Aside from their usefulness in Section 4, the results and techniques in Section 3 should be of independent interest, especially in view of the fact that similar mapping techniques are currently being used in [3] to show that certain indecomposable Hausdorff continua must have uncountably many composants. Some of the results in Section 3 are generalizations of known theorems concerning one-to-one continuous mappings of a half-ray (see [7; 12]). Several of them yield new proofs of previously established theorems (see Remark 3.4).

Except where it is explicitly mentioned to the contrary, the results in this paper are new for the case of metric spaces.

We mention one possible area of application. Recently, interest has been growing in curve theory. The theory of half-ray curves and real curves was developed in [9] and [10] (a *half-ray curve*, respectively *real curve*, is a metric continuum which is a one-to-one continuous image of $[0, \infty)$, respectively the real line). Of prime importance in both papers (as well as in the proof of [8, Theorem 6]) is the structure of the arc components of chainable metric continua with exactly two arc components (i.e., [7, Theorem 1]). This structure and the resulting embedding of half-ray curves done in [9], and revisited in [10], were crucial to the development of the theory of real curves in [10]. There are obvious generalizations of the notions of half-ray and real curves to Hausdorff continua which are a one-to-one continuous image of the appropriate type of connected linearly ordered space. The results in this paper, especially Corollary 4.12, will be useful in investigating these more general "curves".

2. Definitions and preliminary remarks

A *linearly ordered space* is a linearly ordered set endowed with the natural order topology. If L is a linearly ordered space, then $+\infty$ and $-\infty$ will denote the greatest and least elements, respectively, of the order completion of L. A linear order for a set X is said to be *compatible* with a topology T for X if and only if the order topology for X is identically the same as T. A topological space for which there is a compatible linear order is called *linearly orderable*. If X is a non-degenerate connected linearly orderable topological space, then it is easy to verify that there are precisely two compatible linear orders for X.

A *continuum* is a nonempty compact connected Hausdorff space. An *arc* (not necessarily metrizable) is a continuum with precisely two noncut points. It is well known that an arc is a linearly orderable space. We shall use the standard notation $[a, b]$ to denote an arc with noncut points a and b. A *simple triod* is a continuum which is the union of two arcs such that their intersection is one and only one point, that point being a noncut of one of the arcs and a cut point of the other. By an *arc component* of a continuum we mean a maximal arcwise connected subset of the continuum (note that, for example, an arc component of an hereditarily indecomposable continuum is a single point). The term *non-degenerate* will be used to mean that a space has more than one point.

The symbol cl(S) will mean the closure of S and the symbol int(S) will mean the interior of S.

Let f be a one-to-one continuous function from a connected linearly ordered space L into a continuum M. We define

$$K_+(f) = \bigcap_{t\in L} \mathrm{cl}[f((t, +\infty))],$$

$$K_-(f) = \bigcap_{t\in L} \mathrm{cl}[f((-\infty, t))].$$

If L has no last element, then $K_+(f)$ is a subcontinuum of M; if L has a last element $+\infty$, then $K_+(f) = \emptyset$ (since, for $t = +\infty$, we have $(t, +\infty) = \emptyset$. Analogous statements are true for $K_-(f)$. We point out that if L is the real line and f is a surjection, then $K_+(f)$ and $K_-(f)$ are the "singular sets" described in [10]:

A finite collection $\{C_1, C_2, ..., C_n\}$ of sets is a *chain* provided that $C_i \cap C_j \neq \emptyset$ if and only if $|i - j| \leqslant 1$. A continuum M is *chainable* (or *snake-like*) if and only if each open cover of M admits an open refinement which covers M and is a chain. It is easy to see that each subcontinuum of a chainable continuum is chainable and that every chainable continuum is a-triodic and hereditarily unicoherent. The reader is referred to [6] for a more detailed discussion of (non-metric) chainable continua and for a proof of the fact that an hereditarily decomposable continuum is chainable if and only if it is a-triodic and hereditarily unicoherent.

It follows from the above discussion and [4, Theorem 5.2] that given an hereditarily decomposable chainable continuum M, there exists an arc $[a, b]$ and a continuous monotone function f from M onto $[a, b]$

such that $\mathrm{int}(f^{-1}(t)) = \emptyset$ for each $t \in [a, b]$; also, M is irreducible between any point of $f^{-1}(a)$ and any point of $f^{-1}(b)$.

The Stone-Čech compactification of a completely regular T_1-space X is denoted by βX.

3. One-to-one continuous mappings of connected linearly ordered spaces.

Lemma 3.1. *If Λ is a collection of connected linearly orderable subspaces of a topological space X and Λ is totally ordered by set inclusion, then $L = \bigcup \Lambda$ is a one-to-one continuous image of a connected linearly ordered space.*

Proof. Assume that L is not a single point. Let L_0 be a fixed non-degenerate element of Λ with compatible linear order denoted by $\leqslant_0$. For each $L_\alpha \in \Lambda$, let $\leqslant_\alpha$ denote the compatible linear order for L_α which agrees on $L_\alpha \cap L_0$ with $\leqslant_0$. We "induce" a linear order $\leqslant$ for the set L as follows: for x and y in L, $x \leqslant y$ if and only if $x \leqslant_\alpha y$ for some L_α in Λ which contains both x and y. Since $(L_\alpha, \leqslant_\alpha)$ is connected for each $L_\alpha \in \Lambda$ and since Λ is totally ordered, $(L, \leqslant)$ is connected. Since the identity mapping of $(L_\alpha, \leqslant_\alpha)$ onto L_α with the subspace topology is a homeomorphism, the identity mapping of $(L, \leqslant)$ onto L with the subspace topology is one-to-one and continuous.

Theorem 3.2. *If X is an arcwise connected topological space which contains no simple triod, then X is a one-to-one continuous image of a connected linearly ordered space.*

Proof. Assume X contains more than one point and let x and y be points of X with $x \neq y$. Let A be an arc in X with noncut points x and y and let $p \in (A \setminus \{x, y\})$. Let $\leqslant$ be a linear order for A such that $\leqslant$ is compatible with the subspace topology for A and assume $x \leqslant y$. Let Λ' denote the collection of all connected linearly orderable subspaces of X such that $L' \cap A = [p, y]$ for each $L' \in \Lambda'$. Partially order Λ' by set inclusion. Let Λ be a maximal totally ordered subset of Λ' and let $L = \bigcup \Lambda$. Then, by Lemma 3.1, there is a connected linearly ordered space Z and a one-to-one continuous function f from Z onto L. Since X contains no simple triod, it follows that the point $f^{-1}(p)$ is a noncut point of Z. Similarly, let Λ'' denote the collection of all connected linearly orderable subspaces of X such that $L'' \cap A = [x, p]$ for each $L'' \in \Lambda''$, and partially order Λ''

by set inclusion. Then the union K of a maximal totally ordered subset of Λ'' is a one-to-one continuous image under a function g of a connected linearly ordered space W, and $g^{-1}(p)$ is a noncut point of W. By identifying $f^{-1}(p)$ with $g^{-1}(p)$, we can obtain a connected linearly ordered space Y and a one-to-one continuous function from Y onto $L \cup K$ (such a function is the function which is f for points from Z and g for points from W). In view of this, it suffices to show that $L \cup K = X$. Suppose $L \cup K \neq X$ and choose $b \in X \setminus (L \cup K)$. Since X is arcwise connected, there is an arc B in X with noncut points b and p. Since x contains no simple triod and since $b \notin (L \cup K)$, it follows that $B \cap A = [p, y]$ or $B \cap A = [x, p]$. Whichever is the case, B can be used to contradict maximality of the appropriate supposed maximal totally ordered subset of Λ' or of Λ'' (for example, if $B \cap A = [p, y]$ then $\Lambda \cup \{B\}$ is a totally ordered subset of Λ' properly containing the maximal totally ordered subset Λ of Λ'. The proof that $\Lambda \cup \{B\}$ is totally ordered uses the fact that $b \notin (L \cup K)$ and that X contains no simple triod). This completes the proof of the theorem.

As a consequence of Theorem 3.2 and results in [9] and [10], we have the following corollary. For the definitions of half-ray curve, real curve and singular set, the reader is referred to [10] (also see Section 1 above). Note that a metrizable simple triod is homeomorphic to a figure "T".

Corollary 3.3. *A metric continuum X is arcwise connected and contains no simple triod if and only if X is one of the following:*

(1) *a point,*
(2) *a metric arc,*
(3) *a circle,*
(4) *a non-locally connected half-ray curve, or*
(5) *a real curve neither of whose singular sets is a single point.*

Proof. If X is an arcwise connected metric continuum which contains no simple triod then, by Theorem 3.2, X is a one-to-one continuous image of a point, the unit interval $[0, 1]$ of real numbers, the interval $[0, +\infty)$ of real numbers, or the entire real line. The corollary now follows by analyzing some of the structure results in [9] and [10].

Remark 3.4. Using results in [9] and [10], Corollary 3.3 can be applied to give a somewhat different proof of [11, Theorem 2] which determines those metric continua which are arcwise connected and contain no half-ray triod.

The following important lemma is essentially the observation that the Stone-Čech compactification of the ordinals less than the first uncountable ordinal, with the order topology, is the one-point compactification (see [2, p. 75]). To clarify terminology we point out that we use the term *sequence* to mean a function whose domain is the set of natural numbers.

Lemma 3.5.
(i) *If* $[a, b)$ *contains no cofinal sequence, then* $\beta[a, b)$ *is homeomorphic to* $[a, b]$.
(ii) *If* (a, b) *contains no coinitial sequence and no cofinal sequence, then* $\beta(a, b)$ *is homeomorphic to* $[a, b]$.

Proof. (i). Let i denote the identity mapping of $[a, b)$ into $[a, b]$, and let $i^*: \beta[a, b) \to [a, b]$ denote the continuous extension of i to $\beta[a, b)$ (see [2, Theorem 6, p. 86]). Suppose i^* is not one-to-one Then, by a simple continuity argument using nets, there exist distinct points r and s in $\beta[a, b)$ such that $i^*(r) = b = i^*(s)$. Let U and V be open sets in $\beta[a, b)$ such that $r \in U$, $s \in V$, and $\text{cl}(U) \cap \text{cl}(V) = \emptyset$. Let $A = i^*(\text{cl}(U)) \cap [a, b)$ and let $B = i^*(\text{cl}(V)) \cap [a, b)$. Then A and B are disjoint closed cofinal subsets of $[a, b)$ (cofinal because neither r nor s is in $[a, b)$). Choose an increasing sequence $\{t_n\}_{n=1}^{\infty}$ in $[a, b)$ such that $t_n \in A$ if n is even and $t_n \in B$ if n is odd. Since $[a, b)$ contains no cofinal sequence and since $\{t_n\}_{n=1}^{\infty}$ is increasing, $\{t_n\}_{n=1}^{\infty}$ converges to a point t in $[a, b)$. Since A and B are each closed, $t \in (A \cap B)$, which is a contradiction. Hence, i^* is one-to-one and, therefore, is a homeomorphism. This proves (i).
(ii) This can be dome in a similar way. Part (ii) can also be considered as a consequence of (i).

Lemma 3.6. *If* f *is a one-to-one continuous function from a connected linearly ordered space* L *with a first point into an hereditarily unicoherent continuum* M *such that* $\text{cl}(f(L)) = M$ *and* $\text{int}(f(L)) = \emptyset$, *then* M *is indecomposable.*

Proof. The proof is analogous to that given in [12] for the case when L is a half-open interval of real numbers.

The next theorem and its corollary are generalizations of [7, Lemmas 2, 3].

Theorem 3.7. *If f is a one-to-one continuous function from a connected linearly ordered space L into an hereditarily unicoherent, hereditarily decomposable continuum M, then*

(i) $K_+(f) \cap f(L) = \emptyset$,

(ii) $K_-(f) \cap f(L) = \emptyset$,

(iii) $K_+(f) \cap K_-(f) = \emptyset$.

Proof. (i) Let $S = K_+(f) \cap f(L)$. Suppose that $S \neq \emptyset$. It follows from the hereditary unicoherence of M and the arcwise connectivity of $f(L)$ that

(*) if $a < b$ in L, and $f(a)$ and $f(b)$ are in S, then $f([a, b]) \subset S$.

Consequently, $f^{-1}(S)$ is a connected subset of L.

If $f^{-1}(S)$ is bounded above, then choose $r \in L$ such that each point of $f^{-1}(S)$ is strictly less than r (such a choice is possible for, if not, L would have a last element and $K_+(f)$, hence S, would be empty). It follows that $\mathrm{cl}[f(r, +\infty))] \cap f((-\infty, r]) = S \cup \{f(r)\}$. Hence, $\mathrm{cl}[f([r, +\infty))] \cap f((-\infty, r])$ is not connected. But then, from the arcwise connectivity of $f((-\infty, r])$, we can easily obtain a contradiction to the hereditary unicoherence of M. Thus, $f^{-1}(S)$ is not bounded above. Hence, using (*), it follows that there exists $s_0 \in L$ such that $S = f([s_0, +\infty))$.

Next we show that L contains an increasing cofinal sequence. Suppose not. Let f^*: $\beta L \to M$ denote the continuous extension of f to βL (see [2, Theorem 6.5, p. 86]). Since we are supposing that L contains no cofinal sequence, we can now apply Lemma 3.5 to conclude that $K_+(f)$ consists of at most one point. This contradicts the fact that $f^{-1}(S)$ is not bounded above and completes the proof that L contains an increasing cofinal sequence $\{t_n\}_{n=1}^{\infty}$. We assume, without loss of generality, that $t_1 > s_0$. Let $A = \mathrm{cl}(f([s_0, +\infty)))$.

The rest of the proof is similar to the latter part of the proof of [7, Lemma 2] but we include it for completeness. For each $n = 1, 2, \ldots$, $f([s_0, t_n])$ is nowhere dense in A. Hence $f([s_0, +\infty))$ is of the first category in A. Therefore, by a theorem of Baire (see [5, Theorem 2-77, p. 87]), the interior of $f([s_0, +\infty))$ relative to A is empty. Thus by Lemma 3.6, A is indecomposable. This is a contradiction. Thus $S = \emptyset$ and (i) is proved.

(ii) This is similar.

(iii) Suppose $K_+(f) \cap K_-(f) \neq \emptyset$. Let $a < b$ be two points of L. Then using (i) and (ii),

$$f([a, b]) \cap [\mathrm{cl}(f([b, +\infty))) \cup \mathrm{cl}(f((-\infty, a]))]$$

is not connected, which contradicts the hereditary unicoherence of M. This completes the proof of Theorem 3.7.

The following corollary is an extension of [7, Lemma 3].

Corollary 3.8. *If f is a one-to-one continuous function from a connected linearly ordered space L into an hereditarily unicoherent, hereditarily decomposable continuum M, then f is a homeomorphism onto $f(L)$.*

Proof. Let (x, y) be an open subinterval of L. It follows from Theorem 3.7 that

$$f((x, y)) = f(L) \setminus \mathrm{cl}[f(L \setminus (x, y))].$$

Thus $f((x, y))$ is open relative to $f(L)$. If L has a first point a, the same argument shows that $f([a, y))$ is open relative to $f(L)$; an analogous comment holds if L has a last point. The result follows.

4. Arc components of chainable continua

Theorem 4.1. *If M is a chainable continuum, then each arc component of M is a one-to-one continuous image of some connected linearly ordered space.*

Proof. Each arc component of M is an arcwise connected topological space which contains no (simple) triod. Theorem 3.2 now applies to give the desired conclusion.

Theorem 4.2. *If M is an hereditarily decomposable chainable continuum, then each arc component of M is a connected linearly orderable space.*

Proof. Let A denote an arc component of M. By Theorem 4.1, there is a one-to-one continuous function f from some connected linearly ordered space L onto A. Since a chainable continuum is hereditarily unicoherent and since we are assuming M is hereditarily decomposable, Corollary 3.8 implies that f is a homeomorphism. Hence A is a connected linearly orderable space. This proves the theorem.

The following corollary shows that there are only four possible topological types of arc components of an hereditarily decomposable *metric* continuum.

Corollary 4.3. *If M is an hereditarily decomposable chainable metric continuum, then an arc component of M must be either a point, a (metric) arc, a half-ray of reals (i.e. homeomorphic to the interval $[0, \infty)$ of real numbers), or the real line.*

Lemma 4.4. *Let M be an hereditarily decomposable chainable continuum which is not an arc or a point. Then there exists a proper subcontinuum N of M such that N contains more than one point and any arc in M is contained in N or in $M \setminus N$.*

Proof. By [4, Theorem 5.2] (see the discussion near the end of Section 2) there exists an arc $[a, b]$ and a continuous monotone function f from M onto $[a, b]$ such that $\mathrm{int}(f^{-1}(t)) = \emptyset$ for each $t \in [a, b]$; also, M is irreducible between any point of $f^{-1}(a)$ and any point of $f^{-1}(b)$. Since M is not an arc, $f^{-1}(d)$ is not a single point for some $d \in [a, b]$. We consider two cases.

Case 1: $a < d < b$. Since f maps M onto $[a, b]$ and is monotone, $f^{-1}([a, d])$ and $f^{-1}([d, b])$ are each non-degenerate proper subcontinua of M. We show that at least one of $f^{-1}([a, d])$ and $f^{-1}([d, b])$ has the desired property for N. Suppose that neither $f^{-1}([a, d])$ nor $f^{-1}([d, b])$ has the desired property. Then there are arcs α_1 and α_2 such that $\alpha_1 \subset f^{-1}([a, d])$, $\alpha_2 \subset f^{-1}([d, b])$, and $\alpha_i \cap f^{-1}(d)$ is precisely one of the noncut points of α_i, $i = 1, 2$. Let e_i denote the noncut point of α_i not in $\alpha_i \cap f^{-1}(d)$, $i = 1, 2$. Since M is irreducible between any point of $f^{-1}(a)$ and any point of $f^{-1}(b)$,

$$M = f^{-1}([a, f(e_1)]) \cup \alpha_1 \cup f^{-1}(d) \cup \alpha_2 \cup f^{-1}([f(e_2), b]).$$

Therefore, it follows that

$$f^{-1}(d) \setminus [f^{-1}([a, f(e_1)]) \cup \alpha_1 \cup \alpha_2 \cup f^{-1}([f(e_2), b])]$$

is a nonempty open subset of M (the nonemptiness is a consequence of the fact that $f^{-1}(d)$ consists of more than two points and $\alpha_i \cap f^{-1}(d)$ is a single point for each $i = 1, 2$). However, this contradicts the fact that $\mathrm{int}(f^{-1}(t)) = \emptyset$ for each $t \in [a, b]$ and completes the proof of Case 1.

Case 2: $d = a$ or $d = b$. We show that $f^{-1}(d)$ has the desired property for N. Assume $d = a$. Suppose $f^{-1}(d)$ does not have the desired property. Then there is an arc γ in M such that $\gamma \cap f^{-1}(d)$ is precisely one of the noncut points of γ. Let e denote the noncut point of γ not in $\gamma \cap f^{-1}(d)$. Since M is irreducible between any point of $f^{-1}(a)$ and any point of $f^{-1}(b)$, $M = \gamma \cup f^{-1}([f(e), b])$. Hence $f^{-1}(d)$ must be the single point in

$\gamma \cap f^{-1}(d)$, a contradiction to the fact that $f^{-1}(d)$ is not a single point. An analogous argument holds for $d = b$. This completes the proof of Case 2 and of the lemma.

The major result in [7] is that if a chainable metric continuum has exactly two arc components, then one of them is an arc and the other is a half-ray. A large portion of the proof of that result is devoted to showing that, for such a chainable metric continuum, one of the arc components must be compact. Most of the results below represent extensions of these facts in two directions – the condition of being metric is relaxed to that of being Hausdorff and, where appropriate, the condition of having exactly two arc components is replaced by that of being hereditarily decomposable (Lemma 4.8 shows that the latter condition is weaker than having exactly two arc components even for the Hausdorff setting).

Theorem 4.5. *If M is an hereditarily decomposable chainable continuum, then some arc component of M is compact. In particular, some arc component of M is an arc or a point.*

Proof. Let $\kappa' = \{H: H$ is a subcontinuum of M and no arc in M intersects both H and $M \setminus H\}$. Partially order κ' by set inclusion. Let κ be a maximal totally ordered subset of κ' and let $S = \bigcap \kappa$. Then S is a subcontinuum of M and $S \in \kappa'$ (if $S \notin \kappa'$, then there is an arc γ in M such that γ intersects both S and $M \setminus S$; consideration of an open subset of M which contains S but not all of γ leads, by standard results, to a contradiction). If S is not an arc or a point then, applying Lemma 4.4 to S, we obtain a contradiction to the maximality of κ. Hence S is an arc or a point, which, since $S \in \kappa'$, implies that S is an arc component of M. The theorem now follows.

Remark 4.6. Theorem 4.5 is false if we delete the condition "hereditarily decomposable" from its statement. There are well-known examples of chainable metric continua such that no arc component is compact (see, for example, [5, Fig. 8–6]). We also point out that each arc component of an hereditarily decomposable chainable continuum *can* be a single point. Finally, we refer the reader to [7, Fig. 7] and to [6] to see how important a-triodicity is in Theorem 4.5.

Corollary 4.7. *If M is an hereditarily decomposable chainable continuum such that each arc component of M is non-degenerate, then some arc component of M is an arc.*

Lemma 4.8. *If M is a chainable continuum with at most countably many arc components, then M is hereditarily decomposable.*

Proof. Let C be a subcontinuum of M. Since M is hereditarily unicoherent, C has at most countably many arc components (use the pigeon-hole principle). Let f be a one-to-one continuous function from a connected linearly ordered space L onto an arc component A of C. There are several cases to consider.

First assume that L is of the form $[a, b)$. If $[a, b)$ contains no cofinal sequence, then, by Lemma 3.5, f can be extended to a continuous function f^* from $[a, b]$ into C. Clearly, $A = f^*([a, b])$; hence, A is a locally connected subcontinuum of C. If $[a, b)$ contains a cofinal sequence $\{t_n\}_{n=1}^{\infty}$, then $A = \bigcup_{n=1}^{\infty} f([a, t_n])$. Thus A is the union of at most countably many locally connected subcontinua of C.

A similar argument is valid in case L is of the form (a, b). Consequently, each arc component of C is the union of at most countably many locally connected subcontinua of C. Since C has at most countably many arc components, C is the union of at most countably many locally connected subcontinua. By a theorem of Baire (see [5, Theorem 2-77, p. 87]), at least one of these locally connected subcontinua has nonempty interior relative to C. Consequently, C is decomposable by [5, Theorem 3-41, p. 139].

The next theorem extends [8, Theorem 5] to the Hausdorff setting. The theorem has added significance in view of the fact that it does not seem to be known (in the Hausdorff setting) whether or not an arcwise connected continuum must be decomposable. Compare the proof for the metric case in [1] with that in [8]; also see the "Added in proof" statement at the end of [8]).

Theorem 4.9. *A non-degenerate arcwise connected chainable continuum is an arc.*

Proof. By Lemma 4.8, M is hereditarily decomposable. Hence by Theorem 4.2, M must be an arc.

Theorem 4.10. *If M is a non-degenerate chainable continuum with only finitely many arc components, then no arc component of M is a single point and some arc component of M is an arc.*

Proof. From Lemma 4.8 and Corollary 4.7, it suffices to show that no arc component of M is a single point. We prove this by induction on the number n of arc components.

If we have a chainable continuum for which $n = 1$, then Theorem 4.9 completes the proof.

Assume inductively that any non-degenerate chainable continuum with no more than $n = k$ arc components has no arc component which is a single point. Now let M be a chainable continuum which is not a single point, such that M has exactly $n = k + 1$ arc components. By Lemma 4.8, M is hereditarily decomposable. Let $f : M \to [a, b]$ be a monotone function of the type used in the proof of Lemma 4.4 above. Note that the proper subcontinuum of M guaranteed by Lemma 4.4 has the following properties:

(1) it contains more than one point,

(2) each arc component of it is an arc component of M, and

(3) it has at most $n - 1$ arc components.

Suppose there is a point $p \in M$ such that $\{p\}$ is an arc component of M. From the proof of Lemma 4.4 and the induction hypothesis, we can conclude that $f^{-1}(f(p)) = \{p\}$ (otherwise, $f(p)$ would be a choice for "d" in the proof of Lemma 4.4). If $a < f(p) < b$, then (under the assumption that $\{p\}$ is an arc component of M) $f^{-1}([f(p), b])$ would be a proper subcontinuum of M satisfying (1), (2) and (3); by the induction hypothesis, this would complete the proof. Therefore we may assume, without loss of generality, that $f(p) = a$.

Next we prove:

(*) there are at most $n - 1$ points $t \in [a, b]$ such that $f^{-1}(t)$ is not a single point.

Suppose there were n points $t_1 < t_2 < ... < t_n$ such that $f^{-1}(t_i)$, $1 \leq i \leq n$, is not a single point. Note that $a < t_1$ because $f^{-1}(a) = \{p\}$. By the induction hypothesis and the proof of Lemma 4.4, we may assume that each of the continua $f^{-1}([t_i, b])$ satisfies (1), (2) and (3). Thus, since $f^{-1}([t_i, b])$ is properly contained in $f^{-1}([t_{i-1}, b])$ for each $i = 2, 3, ..., n$, it follows (since $a < t_1$) that M has at least $n + 1$ arc components. This contradicts the definition of n and completes the proof of (*).

It now follows that there is a point $c \in [a, b]$ such that $c > a$ and $f^{-1}(t)$ is a single point for all $t \in [a, c]$. Hence $f^{-1}([a, c])$ is an arc in M containing p, a contradiction to the supposition that $\{p\}$ is an arc component of M. This completes the proof of the theorem.

Remark 4.11. We remark, in relation to Theorem 4.10, that there are non-degenerate chainable continua with only countably many arc components such that some arc component is a single point and no arc component is an arc. The object in Fig. 1 below is an example of such a continuum. Its arc components are the sets C_n, $n = 0, 1, 2, \ldots$, where $C_0 = \{0\}$ and, for $n = 1, 2, \ldots$, C_n is "above" and homeomorphic to the half-open interval $(1/2^n, 1/2^{n-1}]$ of reals.

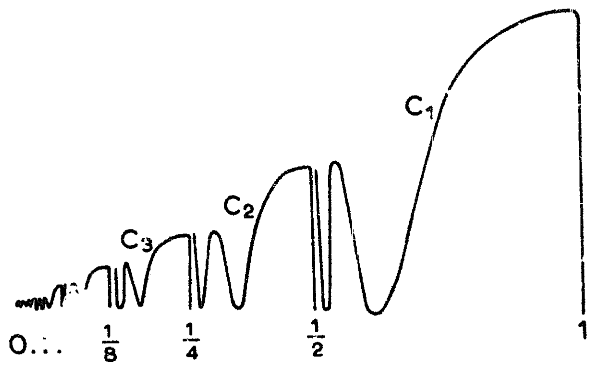

Fig. 1.

The following corollary is an extension to the Hausdorff setting of [7, Theorem 1].

Corollary 4.12. *If M is a chainable continuum with precisely two arc components, then one of them is an arc and the other is homeomorphic to a connected linearly ordered space with a first point and no last point.*

Proof. Let A_1 and A_2 be the arc components of M. By Theorem 4.9, we can assume that A_1 is an arc. Suppose A_2 is not homeomorphic to a connected linearly ordered space with a first point and no last point. Then, by Lemma 4.8 and Theorem 4.2, A_2 is homeomorphic to a connected linearly ordered space L with no first point and no last point. Let f denote a homeomorphism of L onto A_2. According to Theorem 3.7, $K_+(f)$ and $K_-(f)$ are disjoint subcontinua of A_1. Consequently, $A_1 \cap \mathrm{cl}(A_2)$ is not connected, which contradicts the hereditary unicoherence of M.

Remark 4.13. Note that not every connected linearly ordered space with a first point and no last point can be an arc component of a chainable continuum with exactly two arc components. In fact, by Lemma 3.5 (i) and the fact that the Stone-Čech compactification "follows" every other

Hausdorff compactification, such a connected linearly ordered space must have a cofinal sequence.

Question 4.14. Is every chainable continuum irreducible between some pair of points?

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