JOURNAL OF COMBINATORIAL THEORY, Series B 51, 211–226 (1991)

K-Elimination Property for Circuits of Matroids

MANOEL LEMOS

Departamento de Matemática, Universidade Federal de Pernambuco, Cidade Universitária, Recife PE 50739, Brazil

Communicated by the Editors

Received March 8, 1988

For the class of matroids linearly representable over a field of characteristic 2, we prove a conjecture of Oxley relating some circuit elimination properties. Other circuit elimination properties together with some conjectures about them are defined. We settle these conjectures for special classes of matroids. © 1991 Academic Press, Inc.

0. INTRODUCTION

In this paper we shall work on a conjecture made by Oxley in [3]. We say that a matroid M has the *k-property* if for each pair of distinct circuits C_1 and C_2 of M and each subset A of $C_1 \cap C_2$ with $|A| = k$, there is a circuit of M contained in $(C_1 \cup C_2) \backslash A$. Note that the 1-property is just the "elimination axiom" for circuits of a matroid.

Observe that every binary matroid has the k-property for every k. Note also that every matroid which does not have distinct circuits with at least k elements in common must have the j-property for every $j \geqslant k$. Since a matroid has the k-property if and only if every connected component of it has the k-property, we have that a matroid whose connected components are binary or does not have distinct circuits with at least k elements in common has the k-property. Lemos [2] gave examples of connected non-binary matroids with the k-property and distinct circuits with at least k elements in common, for every $k \geqslant 5$.

Fournier in [1] showed that a matroid has the 2-property if and only if it is binary. In [3], Oxley proved that if a matroid M has the k-property for k equal to 3 or 4, then every connected component of M is binary or does not have a pair of different circuits with at least k elements in common.

Oxley [3] also conjectured that if M has the k-property for some $k \geqslant 2$ then M has the t-property for every $t \geqslant k$. In his article, Oxley proved this

0095-8956/91 $3.00

conjecture for $k \leqslant 5$. We shall prove it for all k for the class of matroids which are representable over some field of characteristic two.

THEOREM 1. *If M is linearly representable over a field of characteristic 2 and M has the k-property for some $k \geqslant 2$, then M has the t-property for every $t \geqslant k$.*

In Section 2, we shall look at a strong version of the k-property which corresponds to the "strong elimination axiom" for circuits when k is equal to 1. Some conjectures are made and proved in special cases.

We say that $L \subseteq E(M)$ is a *line* of a matroid M if $M \times L$ is a coloopless matroid with corank two. A line of M is said to be *large* if it contains at least four different circuits of M.

If L is a line of M then there is a bijection between the series classes of $M \times L$ and the circuits of $M \times L$, namely: S is a series class of $M \times L$ if and only if $L \backslash S$ is a circuit of $M \times L$. Hence, if L is a large line and contains a pair of distinct circuits with at least k elements in common ($k > 1$), then $M \times L$ does not have the k-property. A matroid M is said to be a *special m-line* if $E(M)$ is a large line of M, $|E(M)| = m$, and $E(M)$ contains a pair of distinct circuits with $m-2$ elements in common. A matroid M with the k-property ($k > 1$) does not have a minor isomorphic to a special $(k+2)$-line, since Oxley [3] proved that:

(0.1) If M has the k-property then every minor of M has the k-property.

Oxley [3] showed that these are the forbidden minors for a matroid which is to have the k-property for some integers k, namely:

THEOREM 2 (Oxley). *A matroid M has the k-property ($2 \leqslant k \leqslant 5$) if and only if M does not have a minor isomorphic to a special ($k+2$)-line.*

In Section 3, we will extend this result to the case $k = 6$. Actually, we will show that if M does not have a minor isomorphic to a special 8-line, and C_1 and C_2 are circuits of M with at least 6 elements in common, then $M \times (C_1 \cup C_2)$ is binary. Note that Oxley's conjecture for k equal to 6 is a consequence of this. Oxley [3] pointed out that this result does not hold in general; he gave examples of matroids having the k-property and no minor isomorphic to a special $(k+2)$-line, for some integers k.

1. PROOF OF THEOREM 1

Theorem 1 will be proved in this section. We shall need the following lemma of Oxley [3]:

(1.1) For any k, if M is a matroid which does not have the k-property, then M has a minor N which has two distinct spanning circuits C_1 and C_2 such that $C_1 \cup C_2 = E(N)$, $|C_1 \cap C_2| = k$, and $C_1 \triangle C_2$ is independent in N.

Let $\mathbf{M}$ be a class of matroids closed under minors and isomorphism. We say that $M \in \mathbf{M}$ is a *k-obstruction* for $\mathbf{M}$ if M has the k-property and does not have the $(k+1)$-property, but every minor of M has the $(k+1)$-property. Let obs$(k, \mathbf{M})$ be the class of all k-obstructions for $\mathbf{M}$. Oxley's conjecture can be stated as: obs$(k, \mathbf{M})$ is empty for every $k \geqslant 2$, where $\mathbf{M}$ is the class of all matroids.

Next we prove:

(1.2) If $k \geqslant 2$ and $M \in$ obs$(k, \mathbf{M})$, then there are spanning circuits C_1 and C_2 of M such that:

(a) $C_1 \cup C_2 = E(M)$.

(b) $|C_1 \cap C_2| = k+1$.

(c) $C_1 \triangle C_2$ is a basis of M.

(d) For each $e \in E(M) \backslash C_i$, there is a partition A, B of C_i such that $A \cup e$ and $B \cup e$ are the only circuits of M in $C_i \cup e$ which contain e.

Proof. By (0.1) and (1.1) we can suppose that there are circuits C_1 and C_2 of M such that

$$E(M) = C_1 \cup C_2, \tag{1}$$

$$|C_1 \cap C_2| = k+1, \tag{2}$$

$$C_1 \text{ and } C_2 \text{ span } E(M), \tag{3}$$

$$C_1 \triangle C_2 \text{ is independent.} \tag{4}$$

As M has the k-property, it follows that for every $a \in C_1 \cap C_2$, $(C_1 \triangle C_2) \cup a$ contains a circuit of M. Hence, by (4), $C_1 \triangle C_2$ is a basis of M. By (3) we have that

$$|C_1 \triangle C_2| = |C_i| - 1$$

and by (2) we get

$$|C_i| = 2k+1. \tag{5}$$

Observe that if $e \in E(M) \backslash C_i$ then $C_i \cup e$ is a connected line of M. By (5), $C_i \cup e$ contains two different circuits which have at least k elements in common. As M has the k-property, $C_i \cup e$ cannot be a large line. Hence $C_i \cup e$ contains only three circuits of M. ∎

Let F be a field and $\mathbf{M}_F$ be the class of matroids linearly representable over F.

(1.3) If $k \geqslant 2$ and $M \in \text{obs}(k, \mathbf{M}_F)$ then there is a matrix H, all of whose entries are from $\{0, 1\}$, such that M is isomorphic to the matroid given by the linear dependence over F of the columns of the matrix $[I, H]$.

We call the attention of the reader to the fact that the proof of this lemma works for a more general result; namely, let a matroid M be linearly representable over a field of characteristic p. Suppose that M has a spanning circuit C such that $C \cup e$ contains only three circuits for every $e \in E(M) \backslash C$. Then M is linearly representable over $GF(p)$.

Proof of (1.3). Suppose that C_1 and C_2 are circuits of M as in (1.3). Let V be a vector space over F and $f\colon E(M) \to V$ be a linear representation for M. As $C_1 = \{e_1, ..., e_h, e\}$ is a circuit of M, we have

$$f(e) = \alpha_1 f(e_1) + \cdots + \alpha_h f(e_h), \tag{6}$$

where α_i are non-zero elements of F. We can replace $f(e_i)$ by $\alpha_i f(e_i)$ and still have a linear representation for M. Hence we can suppose that $\alpha_i = 1$ for every i. Let $g \in E(M) \backslash C_1$. There is a partition A, B of C_i such that $g \cup A$ and $g \cup B$ are the only circuits of M in $g \cup C_1$ which contain g. Suppose that $e \in B$. We have

$$f(g) = \sum \{\beta_i f(e_i) : e_i \in A\}, \tag{7}$$

$$f(g) = \gamma f(e) + \sum \{\gamma_i f(e_i) : e_i \in B\}. \tag{8}$$

From (6) and (8) we get

$$f(g) = \sum \{(\gamma_i + \gamma) f(e_i) : e_i \in B\} + \sum \{\gamma f(e_i) : e_i \in A\}$$

and by (7), we have that $\beta_i = \gamma$ for every $e_i \in A$, since $C_1 \backslash e$ is a basis of M. Hence

$$\gamma^{-1} f(g) = \sum \{f(e_i) : e_i \in A\},$$

and we can replace $f(g)$ by $\gamma^{-1} f(g)$ and still have a linear representation for M. The result follows. ∎

Theorem 1 follows from the next result.

(1.4) If F is a field of characteristic 2 then $\text{obs}(k, \mathbf{M}_F) = \varnothing$ for every $k \geqslant 2$.

Proof. If $N \in \text{obs}(k, \mathbf{M}_F)$ then N is binary by (1.3), and we have a contradiction. ▌

Since transversal matroids and gammoids are linearly representable over every sufficiently large field (for example, see Welsh [7]), we have as a consequence of Theorem 1 that:

(1.5) If M is a gammoid which has the k-property for some $k \geqslant 2$, then M has the t-property for every $t \geqslant k$.

A second consequence of (1.3) is

(1.6) If the characteristic of F is p then

$$\text{obs}(k, \mathbf{M}_F) = \text{obs}(k, \mathbf{M}_{GF(p)}).$$

2. A Strong Version of the k-Property

A matroid M is said to have the *strong k-property* if for each pair of circuits C_1 and C_2 of M, each subset A of $C_1 \cap C_2$ with $|A| = k$, and each $e \in C_1 \backslash C_2$, there is a circuit C_3 of M such that $e \in C_3 \subseteq (C_1 \cup C_2) \backslash A$. For k equal to 1, this is just the strong version of the "elimination axiom" for circuits of a matroid.

Observe that every binary matroid has the strong k-property. Note also that every matroid which does not have a pair of distinct circuits with at least k elements in common must have the strong j-property for every $j \geqslant k$. A natural conjecture to make is the following:

Conjecture A. A matroid has the k-property if and only if it has the strong k-property.

Note that this conjecture is true for $k \leqslant 4$. When $k = 1$, it states the equivalence between the strong and weak versions of the "elimination axiom" for circuits of a matroid. For $2 \leqslant k \leqslant 4$, this conjecture is true by the results of Oxley [3] and Fournier [1], which say that for $2 \leqslant k \leqslant 4$, a connected matroid has the k-property if and only if it is binary or it does not have a pair of distinct circuits with at least k elements in common.

We recall that in [3], Oxley conjectured that:

Conjecture B. If a matroid has the k-property for some $k \geqslant 2$ then it has the t-property for every $t \geqslant k$.

If one combines Conjecture A and Conjecture B then one obtains the following conjecture:

Conjecture C. If a matroid has the strong k-property for some $k \geqslant 2$ then it has the strong t-property for every $t \geqslant k$.

It is not difficult to see that:

(2.1) If M has the strong k-property then every minor of M has the strong k-property.

During this section $\mathbf{M}$ will be a class of matroids closed under minors and isomorphism. Suppose that Conjecture A is not true for the class of matroids $\mathbf{M}$. Choose a matroid in $\mathbf{M}$ with a minimum number of elements such that it has the k-property, but it does not have the strong k-property. Denote it by $M_k(\mathbf{M})$. Note that there are spanning circuits C_1 and C_2 of $M_k(\mathbf{M})$ and $e \in C_1 \backslash C_2$ such that:

(a) $E(M_k(\mathbf{M})) = C_1 \cup C_2$.

(b) $|C_1 \cap C_2| = k$.

(c) $C_1 \triangle C_2$ does not contain a circuit of $M_k(\mathbf{M})$ which includes e.

C_1 and C_2 are said to be a *pair of special circuits* of $M_k(\mathbf{M})$ and e an *uncovered element* of $M_k(\mathbf{M})$.

Unfortunately, we are not able to say anything "good" about $|C_1 \backslash C_2|$, as we were able to say in Section 1 for the members of obs$(k, \mathbf{M})$. This is going to be the difficulty in proving Conjecture A for the matroids linearly representable over some field of characteristic two.

By analogy with Section one, we say that $M \in \mathbf{M}$ is a *strong k-obstruction* for $\mathbf{M}$ if M has the strong k-property and does not have the strong $(k+1)$-property, but every minor of M has the strong $(k+1)$-property. Let $\text{obs}_s(k, \mathbf{M})$ be the set of all strong k-obstructions for $\mathbf{M}$.

If $M \in \text{obs}_s(k, \mathbf{M})$ then there are spanning circuits C_1 and C_2 of M and $e \in C_1 \backslash C_2$ such that:

(a′) $E(M) = C_1 \cup C_2$.

(b′) $|C_1 \cap C_2| = k+1$.

(c′) $C_1 \triangle C_2$ does not contain a circuit of M which includes e.

C_1 and C_2 are said to be a *pair of special* circuits of M and e an *uncovered* element of M.

In this case, we will be able to show that $|C_1 \backslash C_2| \geqslant k$ and hence we shall prove Conjecture C for the class of matroids linearly representable over some field of characteristic two. The next lemma will explain why $|C_1 \backslash C_2|$ is so important.

(2.2) If C_1 and C_2 are a pair of special circuits of $M_k(\mathbf{M})$ and

$$2(|C_1 \backslash C_2| + 1) \geqslant k,$$

then $C_i \cup a$ contains exactly three circuits of $M_k(\mathbf{M})$, when $a \notin C_i$.

Proof. Suppose that there is $a \notin C_i$ such that $C_i \cup a$ contains more than three circuits of $M_k(\mathbf{M})$. If $C'_1, ..., C'_n$ $(n \geqslant 3)$ are the circuits contained in $C_i \cup a$ which are different from C_i, then $C_i \backslash C'_1, ..., C_i \backslash C'_n$ is a partition of C_i. Observe that

$$C_i \cap C'_j = \bigcup \{C_i \backslash C'_h : h \neq j\}.$$

As $M_k(\mathbf{M})$ has the k-property and $n \geqslant 3$, we have for $1 \leqslant j \leqslant n$ that

$$|C_i \cap C'_j| = \sum_{h \neq j} |C_i \backslash C'_h| < k. \tag{1}$$

If $m = |C_1 \backslash C_2|$ then

$$k + m = |C_i| = \sum_{i=1}^{n} |C_i \backslash C'_h| = |C_i \cap C'_j| + |C_i \backslash C'_j|$$

and by (1)

$$k + m < k + |C_i \backslash C'_j|.$$

For $1 \leqslant j \leqslant n$, we have

$$|C_i \backslash C'_j| \geqslant m + 1. \tag{2}$$

By (1) and (2) we have

$$k > |C_i \cap C'_j| \geqslant (n-1)(m+1).$$

Since $n \geqslant 3$ we get

$$2(m+1) < k,$$

and we have a contradiction. ∎

Suppose now that $M \in \operatorname{obs}_s(k, \mathbf{M})$, C_1 and C_2 are a pair of special circuits of M, and e is an uncovered element of M. If $a \in C_1 \cap C_2$ then there is a circuit C of M such that $e \in C \subseteq (C_1 \triangle C_2) \cup a$, since M has the strong k-property. Note that $a \in C$ because by (c′) there is not a circuit of M contained in $C_1 \triangle C_2$ which includes e. Hence $C_1 \triangle C_2$ is a spanning set for M. As a consequence of this we have

$$|C_1 \triangle C_2| = 2\,|C_1 \backslash C_2| \geqslant |C_1| - 1,$$

and as

$$|C_1| = |C_1 \backslash C_2| + k + 1,$$

it follows that

$$|C_1 \backslash C_2| \geqslant k.$$

As in (2.2) we can prove that:

(2.3) If $M \in \mathrm{obs}_s(k, \mathbf{M})$ and C_1, C_2 are a pair of special circuits of M, then $C_i \cup a$ contains exactly three circuits of M, when $a \notin C_i$ and $k \geqslant 2$.

Until the end of this section, F will be a field of characteristic two. We remind the reader that $\mathbf{M}_F$ is the class of all matroids linearly representable over F. Conjecture C will be proved for the class $\mathbf{M}_F$, but first we need the following lemma:

(2.4) If M is linearly representable over F, and has a spanning circuit C such that $C \cup a$ contains exactly three circuits of M for every $a \in E(M) \backslash C$, then M is binary.

Proof. As we pointed out before, the proof of (1.4) works in this case. ∎

Suppose that $M \in \mathrm{obs}_s(k, \mathbf{M}_F)$ and $k \geqslant 2$. By (2.3) and (2.4) we have that M is binary. Since every binary matroid has the strong t-property, we have a contradiction. Hence:

(2.5) $\mathrm{obs}_s(k, \mathbf{M}_F) = \varnothing$ for $k \geqslant 2$.

As a consequence of this we have that Conjecture C holds for $\mathbf{M}_F$:

(2.6) If $M \in \mathbf{M}_F$ has the strong k-property for some $k \geqslant 2$ then it has the strong t-property for every $t \geqslant k$.

To prove Conjecture A for the class $\mathbf{M}_F$ for some k, we need to give a lower bound for $|C_1 \backslash C_2|$ and use (2.2) and (2.4) to show that $M_k(\mathbf{M}_F)$ is binary, and get a contradiction.

(2.7) If C_1 and C_2 are a pair of special circuits of $M_k(\mathbf{M})$ then $|C_1 \backslash C_2| \geqslant 3$.

Proof. If $|C_1 \backslash C_2| = 1$, then $C_1 \triangle C_2$ is a circuit of $M_k(\mathbf{M})$, since $M_k(\mathbf{M})$ has the k-property. Hence $M_k(\mathbf{M})$ has the strong k-property, so we have a contradiction.

If $|C_1 \backslash C_2| = 2$, and e is an uncovered element of $M_k(\mathbf{M})$, then $C_1 \triangle C_2$ contains a circuit C such that $e \notin C$ and hence $|C| \leqslant 3$. If $|C| = 2$, then C_1 and $C_2' = C_2 \triangle C$ are circuits of $M_k(\mathbf{M})$ such that $|C_1 \backslash C_2'| = 1$ and $|C_1 \cap C_2'| = k + 1$. As $M_k(\mathbf{M}) \times (C_1 \cup C_2')$ has the k-property, we have that $M_k(\mathbf{M}) \times (C_1 \cup C_2')$ is binary and hence $M_k(\mathbf{M})$ is binary, a contradiction.

If $|C|=3$ and $C_1 \cap C=\{a\}$ then $C_2 \cup a$ contains a circuit C_2' different from C_2 such that $C_2' \supseteq C_1 \cap C_2$. As $|C_2' \cap C_2| \geqslant k$, $C_2 \cup a$ contains only three circuits of $M_k(\mathbf{M})$ and hence $(C_1 \cap C_2) \cup a$ is a circuit of $M_k(\mathbf{M})$, a contradiction since $(C_1 \cap C_2) \cup a \subset C_1$. ∎

Now we can prove that Conjecture A is true for $\mathbf{M}_F$ when $k \leqslant 8$.

(2.8) Suppose that $k \leqslant 8$ and let M be a linearly representable matroid over F. Then M has the k-property if and only if M has the strong k-property.

Proof. By (2.7) we have that

$$2(|C_1 \backslash C_2|+1) \geqslant k,$$

if C_1 and C_2 are a pair of special circuits of $M_k(\mathbf{M}_F)$. By (2.2) and (2.4) $M_k(\mathbf{M}_F)$ is binary. Hence $M_k(\mathbf{M}_F)$ has the strong k-property and we have a contradiction. ∎

3. Matroids with the 6-Property

When k is equal to 5 or 6, we will prove the equivalence between the k-property and the strong k-property and that the k-property implies the t-property for every $t \geqslant k$. These results follow easily from the fact that if M has the k-property and C_1 and C_2 are circuits of M with at least k elements in common then $M \times (C_1 \cup C_2)$ is binary. From this we will also conclude that a matroid has the k-property if and only if it does not have a minor isomorphic to a special $(k+2)$-line.

We will leave the proofs of the next two lemmas to the next section.

(3.1) Suppose that a circuit C spans elements a and b of M. If $M \times (C \cup a)$ and $M \times (C \cup b)$ are binary then $M \times (C \cup \{a, b\})$ is also binary.

We say that a matroid M has the *k-property for lines* if $M \times L$ has the k-property for every line L of M. Note that every matroid which has the k-property also has the k-property for lines, but the converse is not true, as Oxley pointed out in [3]. Now we can state the other lemma whose proof can be found in the next section.

(3.2) Suppose that k is equal to 5 or 6 and let M be a series extension of a 3-connected matroid. Suppose also that M has spanning circuits C_1 and C_2 such that

(a) $|C_1 \cap C_2| \geqslant k$.
(b) $C_1 \cup C_2 = E(M)$.
(c) $M \times (C_i \cup a)$ is binary when $a \in E(M) \backslash C_i$.

If M has the k-property for lines then M is binary.

In [5], Seymour showed that:

(3.3) If M is a non-binary 3-connected matroid and $\{x, y\} \subseteq E(M)$ then M has a minor isomorphic to U_4^2 whose ground set contains $\{x, y\}$.

As a consequence of this result we have:

(3.4) If M is a series extension of a non-binary 3-connected matroid and $\{x, y\} \subseteq E(M)$ then M has a large line L such that $\{x, y\} \subseteq L$.

Now we can prove:

(3.5) If M is a matroid having the k-property for lines (k equal to 5 or 6) and a pair of distinct circuits C_1 and C_2 such that $|C_1 \cap C_2| \geqslant k$ and $C_1 \cup C_2 = E(M)$, then M is binary.

Proof. Suppose that this is not true. Choose a non-binary matroid M with a minimum number of elements such that M has the k-property for lines and a pair of circuits C_1 and C_2 such that $|C_1 \cap C_2| = k$ and $C_1 \cup C_2 = E(M)$.

Let $\{A_1, A_2\}$ be a 2-separation of M. We have one of the cases:

Case 1. $A_i \subseteq C_1 \backslash C_2$ or $A_i \subseteq C_2 \backslash C_1$ for some i.

Note that the elements of A_i are in series in M. If $e \in A_i$ then $M/(A_i \backslash e)$ is contrary to the choice of M.

Case 2. $A_i \subseteq C_1 \cap C_2$ for some i.

As before, we have that the elements of A_i are in series in M. Observe that in this case we do not arrive at a contradiction.

Case 3. $A_i \cap (C_1 \backslash C_2) \neq \varnothing$ and $A_i \cap (C_2 \backslash C_1) \neq \varnothing$ for both i.

By the choice of M we have that $M_i = M \backslash (A_i \cap (C_2 \backslash C_1))$ is binary for both i. As M is isomorphic to a minor of the 2-sum of M_1 with M_2, M is binary and we have a contradiction.

We will show now that those are the only cases. Suppose that Case 1 and Case 2 do not happen. If $A_i \cap (C_2 \backslash C_1) = \varnothing$ then

$$\varnothing \neq A_i \cap C_2 = A_i \cap C_1 \cap C_2 \subset A_i \cap C_1,$$

and we have a contradiction by (2.3) of Seymour [4]. Hence

$A_i \cap (C_2 \backslash C_1) \neq \varnothing$. In the same way we can show that $A_i \cap (C_1 \backslash C_2) \neq \varnothing$ and Case 3 happens.

So M is a series extension of a 3-connected matroid, since if $\{A_1, A_2\}$ is a 2-separation of M, then the elements of A_i are in series for some i. By (3.4) there is a large line L of M such that $|L \cap C_1 \cap C_2| \geqslant 2$.

If C is a circuit of M/C_i then $C \cup C_i$ is a line of M. Hence $C \cup C_i$ contains only 3 circuits of M, since M has the k-property for lines. As $L \neq C_i \cup C$ then $|L \backslash C_i| \geqslant 2$. If $|L \backslash C_i| = 2$ then C_i spans $L \backslash C_i$, otherwise $L \backslash C_i$ is a circuit of $M \backslash C_i$. By (3.1), $M \times (L \cup C_i)$ is binary. Hence $|L \backslash C_i| \geqslant 3$ and $|L| \geqslant 8$.

If $h \in L \backslash C_i$ then C_i spans h, otherwise M/h contradicts the choice of M (note that L/h is a large line of M/h since every series class of $M \times L$ has at least two elements). If $h \notin C_i$ and C_i does not span h then L does not span h and M/h contradicts the choice of M. Hence C_1 and C_2 are spanning circuits of M. The result follows from (3.2). ∎

As a consequence of this result we have:

(3.6) If M has the k-property (k equal to 5 or 6) and C_1 and C_2 are circuits of M such that $|C_1 \cap C_2| \geqslant k$, then $M \times (C_1 \cup C_2)$ is binary.

(3.7) Suppose that k is equal to 5 or 6. M has the k-property if and only if M has the strong k-property.

As a consequence of (3.6) we have also proved Oxley's conjecture for k equal to 6.

(3.8) If M has the 6-property then M has the t-property for every $t \geqslant 6$.

Now we give a forbidden-minor characterization for matroids with the 6-property.

(3.9) M has the 6-property if and only if M has no minor isomorphic to a special 8-line.

Proof. If M has a minor isomorphic to a special 8-line then M does not have the 6-property, since a special 8-element line does not have the 6-property. If M has no minor isomorphic to a special 8-line then M has the 6-property for lines, because $M \times L$ has a minor isomorphic to a special 8-line when L is a large line of M with distinct circuits with at least 6 elements in common. The result follows from (3.5). ∎

Oxley in [3] showed that there are matroids with the k-property with no minor isomorphic to a k-element line, for some k. Hence a result like (3.5) would not hold for every k, but we conjecture that (3.6) holds, namely:

Conjecture D. For any integer $k \geqslant 2$, if M has the k-property and C_1 and C_2 are circuits of M such that $|C_1 \cap C_2| \geqslant k$, then $M \times (C_1 \cup C_2)$ is binary.

Observe that this conjecture implies both Oxley's conjecture and the equivalence between the k-property and the strong k-property.

4. Proof of the Lemmas

In this section we shall prove the lemmas that we stated in the last section without a proof.

A *flat* of a matroid M is a closed subset of $E(M)$. One flat is said to be *on* another if it is contained in or contains the other flat. An *n-flat* of M is a flat with rank equal to n. The terms *copoint*, *coline*, and *coplane* will refer to flats with rank equal to $\rho(M)-1$, $\rho(M)-2$ and $\rho(M)-3$ respectively. Clearly, with this terminology, copoint is just another word for hyperplane.

A coline is said to be *large* if it is on at least four distinct copoints. In [6], Tutte proved the following result:

(4.1) A matroid is binary if and only if every coline of it is not large.

Observe that (3.1) is equivalent to:

(4.2) Suppose that H is a hyperplane of a matroid M. Suppose also that a and b are coloops of $M \times H$. If $M/(H \backslash a)$ and $M/(H \backslash b)$ are binary then $M/(H \backslash \{a, b\})$ is also binary.

Proof. We need to prove this only when a and b are distinct elements of M which belong to H. In this case $H \backslash \{a, b\}$ is a coplane P of M. There are exactly two colines $L_a = H \backslash a$ and $L_b = H \backslash b$ on both P and H. If L is another coline on P then its joins with L_a and L_b are copoints A_1 and B_1, respectively, both different from H. Let A_2 and B_2 be the third copoints on L_a and L_b, respectively. If H' is a copoint on L distinct from A_1 and B_1 then there is a coline L' on both P and H' such that $L \neq L'$. Note that L' is on the copoints A_2 and B_2. Hence L is on at most 3 copoints and by (4.1) is binary. ∎

We say that a matroid M has the *k-property for colines*, when for each coline L of M, each pair of hyperplanes H_1 and H_2 of M such that $H_1 \cap H_2 = L$, and each subset A of $E(M) \backslash (H_1 \cup H_2)$ with k elements, there is a hyperplane H_3 of M such that $H_3 \supseteq L \cup A$. Note that (3.2) is equivalent to:

(4.3) Suppose that k is equal to 5 or 6 and let M be a parallel extension of a 3-connected matroid. Suppose also that M has hyperplanes H_1 and H_2 such that:

(a) $\rho(M \times H_i) = |H_i|$.

(b) $|E(M) \backslash (H_1 \cup H_2)| \geqslant k$.

(c) $H_1 \cap H_2 = \varnothing$.

(d) $M/(H_i \backslash a)$ is binary when $a \in H_i$.

If M has the k-property for colines then M is binary.

Proof. Suppose that the result is not true. Let M be a non-binary matroid which is a parallel extension of a 3-connected matroid. Suppose also that M^* has the k-property for colines ($k=5$ or 6) and M has a pair of hyperplanes C_1 and C_2 such that

(a) $\rho(M \times H_i) = |H_i|$.

(b) $|E(M) \backslash (H_1 \cup H_2)| \geqslant k$.

(c) $|H_1 \cap H_2| = \varnothing$.

(d) $M/(H_i \backslash a)$ is binary when $a \in H_i$.

Step 1. M has a large coline L such that $|H_i \backslash L| = 3$ for some i.

Choose $A \subseteq E(M) \backslash (H_1 \cup H_2)$ with $|A| = 2$. By (3.4) there is a large coline L of M such that $L \cap A = \varnothing$. As $|E(M) \backslash L| \leqslant 9$, it follows that $|H_i \backslash L| \leqslant 3$ for some i. By (4.1) and (4.2), if $|H_i \backslash L| \leqslant 3$ then the equality occurs.

In the same way, we can show that

Step 2. If L is a large coline then $|E(M) \backslash (L \cup H_1 \cup H_2)| \leqslant 2$ when $k=5$, and $|E(M) \backslash (L \cup H_1 \cup H_2)| \leqslant 3$ when $k=6$.

By Step 1 there is a large coline L such that $|H_i \backslash L| = 3$ for some i. Observe that $J = H_i \cap L$ is a $(\rho(M)-4)$-flat of M. Suppose that $H_i \backslash L = \{a, b, c\}$. There are exactly three colines on both J and H_i: $L_a = H_i \backslash a$, $L_b = H_i \backslash b$, and $L_c = H_i \backslash c$. There are exactly three coplanes on both J and H_i: $\Pi_{ab} = L_a \cap L_b$, $\Pi_{ac} = L_a \cap L_c$, and $\Pi_{bc} = L_b \cap L_c$—by (4.1) and (4.2) these coplanes are not on large colines. Let α_1 and α_2 be the copoints on L_a distinct from H_i. We define β_1, β_2, γ_1, and γ_2 similarly.

Step 3. If Π is a coplane on J and not on H_i then $\Pi = \alpha_i \cap \beta_j \cap \gamma_k$, where α_i, β_j, and γ_k are the joins of Π with L_a, L_b, and L_c, respectively.

If α_i, β_j, and γ_k are on the same coline then γ_k is on Π_{ab} since α_i and β_j are distinct and on Π_{ab}. Hence $\Pi_{ab} = J$, and we have a contradiction.

So, we have at most 11 coplanes on J. We denote by Π_{ijk} the coplane

$\alpha_i \cap \beta_j \cap \gamma_k$ if it exists. If the colines $\alpha_i \cap \beta_j$, $\beta_j \cap \gamma_k$, and $\alpha_i \cap \gamma_k$ are on a copoint different from α_i, β_j, and γ_k then we denote them by δ_{ij}, ε_{jk}, and ζ_{ik} respectively. Note that $\delta_{ij} = \delta_{rs}$ when $r \neq i$ and $s \neq j$, and the same holds for ε_{ij} and ζ_{ij}.

Now we shall study the joins of the coplanes on J with a coplane Π_{ijk}—these joins are all the colines on Π_{ijk}. Suppose that $r \neq i$, $s \neq j$, and $t \neq k$. We have:

Possible coplane on J	Its join with Π_{ijk}
Π_{ab}	$\alpha_i \cap \beta_j$
Π_{ac}	$\alpha_i \cap \gamma_k$
Π_{bc}	$\beta_j \cap \gamma_k$
Π_{rjk}	$\beta_j \cap \gamma_k$
Π_{isk}	$\alpha_i \cap \gamma_k$
Π_{ijt}	$\alpha_i \cap \beta_j$
Π_{rsk}	$\delta_{ij} \cap \gamma_k$
Π_{rjt}	$\zeta_{ik} \cap \beta_j$
Π_{ist}	$\varepsilon_{jk} \cap \alpha_i$
Π_{rst}	$\varepsilon_{jk} \cap \delta_{ij}$

For example, the join L or Π_{ijk} and Π_{ist} is on the copoint α_i since α_i is on both coplanes. As L and $\beta_j \cap \gamma_k$ are colines on Π_{ijk}, their join is a copoint H. If β_j (or γ_k) is on L, then β_j (or γ_k) is on Π_{ist}, and hence $L_b = \beta_1 \cap \beta_2$ (or $L_c = \gamma_1 \cap \gamma_2$) is on Π_{ist}, so we have a contradiction since H_i is not on Π_{ist}. Hence β_i and γ_k are not on L, and $H = \varepsilon_{jk}$, as $\beta_j \cap \gamma_k$ is on at most 3 copoints. Hence $L = \varepsilon_{jk} \cap \alpha_i$.

Note that if $\delta_{ij} \cap \varepsilon_{jk}$ is a coline on J then it is on exactly 3 copoints. So the possible large lines on Π_{ijk} are $\delta_{ij} \cap \gamma_k$, $\zeta_{ik} \cap \beta_j$, and $\varepsilon_{jk} \cap \alpha_i$.

Step 4. Suppose that Π_{ijk} is on a large coline and $i \neq r$, $j \neq s$, $k \neq t$. The coplanes Π_{rsk}, Π_{rjt}, and Π_{ist} exist and the join of any two of these coplanes is a coline on exactly 4 copoints.

If Π_{ijk} is on a large coline then the colines $\delta_{ij} \cap \gamma_k$, $\zeta_{ik} \cap \beta_j$, and $\varepsilon_{jk} \cap \alpha_i$ exist, their join is $\varnothing$ and each of these colines is on exactly 4 copoints. Hence the coplanes Π_{rst}, Π_{rjt}, and Π_{ist} exist.

As J is on a large coline, there is a plane Π_{ijk} on a large coline. To simplify the notation we suppose that Π_{111} is on a large coline. By Step 4 the same happens with the coplanes Π_{221}, Π_{212}, and Π_{122}. Note that

$$\begin{aligned} J &= (\alpha_1 \cap \varepsilon_{11}) \cap (\alpha_2 \cap \varepsilon_{12}) \\ &= (\beta_1 \cap \zeta_{11}) \cap (\beta_2 a \zeta_{12}) = (\gamma_1 \cap \delta_{11}) \cap (\gamma_2 \cap \delta_{12}). \end{aligned}$$

(Suppose that $P=(\alpha_1\cap\varepsilon_{11})\cap(\alpha_2\cap\varepsilon_{12})\neq J$. In this case P is a coplane. As α_1 and α_2 are on P, $\alpha_1\cap\alpha_2$ is on P. So H_i is on P and P is Π_{ab}, Π_{ac}, or Π_{bc}, and we have a contradiction.) By Step 2 we have that

$$|E(M)\backslash(\alpha_i\cap\varepsilon_{1i})\cup H_1\cup H_2|\leqslant 2 \qquad \text{if} \quad k=5$$

and

$$|E(M)\backslash(\alpha_i\cap\varepsilon_{1i})\cup H_1\cup H_2|\leqslant 3 \qquad \text{if} \quad k=6.$$

As $J=(\alpha_1\cap\varepsilon_{11})\cap(\alpha_2\cap\varepsilon_{12})$ and $J\subseteq H_1\cup H_2$, we have

$$\sum_{i=1}^{2} |E(M)\backslash((\alpha_i\cap\varepsilon_{1i})\cup H_1\cup H_2)|\geqslant|E(M)\backslash(H_1\cup H_2)|\geqslant k.$$

Hence $k=6$ and

$$E(M)\backslash((\alpha_i\cap\varepsilon_{1i})\cup H_1\cup H_2) \qquad \text{for} \quad i=1,2$$

are disjoint sets with 3 elements each.

As the coline $\alpha_1\cap\varepsilon_{11}$ and J are both only on the coplanes Π_{111} and Π_{122}, we have

$$(\alpha_1\cap\varepsilon_{11})\backslash J=(\Pi_{111}\backslash J)\cup(\Pi_{122}\backslash J).$$

If $a_{ijk}=|E(M)\backslash((\Pi_{ijk}\backslash J)\cup H_1\cup H_2)|-3$ then

$$a_{111}+a_{122}=3.$$

Similarly

$$a_{212}+a_{221}=3$$
$$a_{111}+a_{212}=3$$
$$a_{122}+a_{221}=3$$
$$a_{111}+a_{221}=3$$
$$a_{122}+a_{212}=3,$$

and we have a contradiction since we cannot find integers a_{ijk} which satisfy these equations. ▮

Acknowledgments

The author thanks CNPq (Brazil) for financial support and Prof. J. G. Oxley for suggesting the problem to him.

References

1. J.-C. Fournier, A characterization of binary geometries by a double elimination axiom, *J. Combin. Theory Ser. B* **31** (1981), 249–250.
2. M. Lemos, "Some Problems in Matroid Theory," D. Phil. Thesis, Oxford, 1987.
3. J. G. Oxley, On circuit exchange properties for matroids, *European J. Combin.* **9** (1988), 331–336.
4. P. D. Seymour, Decomposition of regular matroids, *J. Combin. Theory Ser. B* **28** (1980), 305–359.
5. P. D. Seymour, On minors of non-binary matroids, *Combinatorica* **1** (1981), 387–394.
6. W. T. Tutte, Lectures on matroids, *J. Res. Nat. Bur. Standards* **69B** (1965), 1–47.
7. D. J. A. Welsh, "Matroid Theory," Academic Press, New York/London, 1976.