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A generic diagonalization of the 3×3 neutrino mass matrix and its implications on the μ–τ flavor symmetry and maximal CP violation

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ABSTRACT

In the flavor basis where the mass eigenstates of three charged leptons are identified with their flavor eigenstates, one may diagonalize a 3×3 Majorana neutrino mass matrix M_ν by means of the standard parametrization of the 3×3 neutrino mixing matrix V. In this treatment the unphysical phases of M_ν have to be carefully factored out, unless a special phase convention for neutrino fields is chosen so as to simplify M_ν to M'_ν without any unphysical phases. We choose this special flavor basis and establish some exact analytical relations between the matrix elements of $M'_\nu M'^{\dagger}_\nu$ and seven physical parameters – three neutrino masses (m_1, m_2, m_3), three flavor mixing angles (θ_{12}, θ_{13}, θ_{23}) and the Dirac CP-violating phase (δ). Such results allow us to derive the conditions for the μ–τ flavor symmetry with $\theta_{23} = \pi/4$ and maximal CP violation with $\delta = \pm\pi/2$, which should be useful for discussing specific neutrino mass models. In particular, we show that $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ keep unchanged when constant matter effects are taken into account for a long-baseline neutrino oscillation experiment.

1. Recent neutrino oscillation experiments [1] have provided us with very convincing evidence that neutrinos are massive and lepton flavors are mixed. Similar to the phenomenon of quark flavor mixing, which is described by the 3×3 Cabibbo–Kobayashi–Maskawa (CKM) matrix [2], the phenomenon of lepton flavor mixing can also be described by a 3×3 unitary matrix V, the so-called Maki–Nakagawa–Sakata–Pontecorvo (MNSP) matrix [3].[1] A full parametrization of the MNSP matrix V needs three rotation matrices in the complex $(1,2)$, $(1,3)$ and $(2,3)$ planes:

$$O_{12} = \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad O_{13} = \begin{pmatrix} c_{13} & 0 & \hat{s}^*_{13} \\ 0 & 1 & 0 \\ -\hat{s}_{13} & 0 & c_{13} \end{pmatrix}, \quad O_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix}, \tag{1}$$

where $c_{ij} \equiv \cos\theta_{ij}$, $s_{ij} \equiv \sin\theta_{ij}$ and $\hat{s}_{13} \equiv s_{13} e^{i\delta}$ (for $ij = 12, 13, 23$). The unitary MNSP matrix V can then be parametrized as

$$V = P_l U P_\nu, \tag{2}$$

where $P_l = \mathrm{Diag}\{e^{i\phi_e}, e^{i\phi_\mu}, e^{i\phi_\tau}\}$ and $P_\nu = \mathrm{Diag}\{e^{i\rho}, e^{i\sigma}, 1\}$ are two diagonal phase matrices, and

$$U = O_{23} O_{13} O_{12} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & \hat{s}^*_{13} \\ -s_{12}c_{23} - c_{12}s_{23}\hat{s}_{13} & c_{12}c_{23} - s_{12}s_{23}\hat{s}_{13} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}\hat{s}_{13} & -c_{12}s_{23} - s_{12}c_{23}\hat{s}_{13} & c_{23}c_{13} \end{pmatrix} \tag{3}$$

is just the standard parametrization of the CKM matrix [1]. Without loss of generality, one may arrange three mixing angles (θ_{12}, θ_{13}, θ_{23}) to lie in the first quadrant and allow three CP-violating phases (δ, ρ, σ) to vary between 0 and 2π. In the mass eigenstates of three charged leptons and three neutrinos, V appears in the charged-current interactions

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[1] Different from the CKM matrix, which must be unitary in the standard electroweak model, the MNSP matrix V can be either unitary or non-unitary in a given neutrino mass model. For example, V is unitary in the type-II seesaw mechanism, but it must be non-unitary in the type-I, type-(I+II), type-III and multiple seesaw models [4] although the effects of its unitarity violation are at most at the percent level [5]. In this Letter we simply assume that V is unitary at low energies.

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$$\mathcal{L}_{\rm cc} = -\frac{g}{\sqrt{2}} \overline{(e \quad \mu \quad \tau)_{\rm L}} \gamma^\mu V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix}_{\rm L} W^-_\mu + {\rm h.c.} \tag{4}$$

Hence the phase matrix P_l is unphysical and can be rotated away by redefining the phases of three charge-lepton fields. If neutrinos are the Dirac particles, the phase matrix P_ν can also be rotated away be redefining the phases of three neutrino fields. In this case we are left with U as the MNSP matrix. If neutrinos are the Majorana particles, however, P_ν is physical because it characterizes two irremovable relative phases of three Majorana neutrino fields [6]. In this case we are left with $V' = UP_\nu$, which contains three mixing angles and three CP-violating phases. We shall assume neutrinos to be the Majorana particles throughout this Letter (U in the Dirac case can simply be reproduced from V' in the Majorana case by setting $P_\nu = \mathbf{1}$).

In the flavor basis where the mass eigenstates of three charged leptons are identified with their flavor eigenstates, $V' = UP_\nu$ links the neutrino flavor eigenstates $(\nu_e, \nu_\mu, \nu_\tau)$ to the neutrino mass eigenstates (ν_1, ν_2, ν_3). The effective Majorana neutrino mass term can be written as

$$\mathcal{L}_\nu = \frac{1}{2} \overline{(\nu_e \quad \nu_\mu \quad \nu_\tau)_{\rm L}} M_\nu \begin{pmatrix} (\nu_e)^{\rm c} \\ (\nu_\mu)^{\rm c} \\ (\nu_\tau)^{\rm c} \end{pmatrix}_{\rm R} + {\rm h.c.}, \tag{5}$$

where $(\nu_\alpha)^{\rm c} \equiv \mathcal{C}\overline{\nu_\alpha}^T$ denotes the charge-conjugate counterpart of ν_α (for $\alpha = e, \mu, \tau$), and M_ν is a symmetric 3×3 matrix which totally has six complex entries or twelve real parameters:

$$M_\nu = \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix}. \tag{6}$$

In the chosen basis one may obtain three neutrino masses (m_1, m_2, m_3), three neutrino mixing angles $(\theta_{12}, \theta_{13}, \theta_{23})$ and three CP-violating phases (δ, ρ, σ) by diagonalizing M_ν. We stress that it is not a trivial job to diagonalize M_ν and derive the generic expressions of nine physical parameters in terms of the elements of M_ν. Such an attempt has been made by Aizawa and Yasuè [7], but their treatment is subject to the transformation[2]

$$V'^\dagger M_\nu V'^* = \hat{M}_\nu \equiv \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}. \tag{7}$$

In view of the fact that $V' = UP_\nu$ and $\hat{M}_\nu$ totally consist of nine physical parameters but M_ν generally contains twelve free parameters, one immediately encounters a parameter mismatching problem. This ambiguity will be clarified in this Letter.

The purpose of our work is two-fold. First, we point out that one should use the transformation $V^\dagger M_\nu V^* = \hat{M}_\nu$ instead of the one in Eq. (7) to diagonalize the 3×3 Majorana neutrino mass matrix M_ν. In this case the aforementioned parameter mismatching problem does not occur, because $V = P_l V'$ contains three unphysical phases which can exactly eliminate the unphysical phases of M_ν. Second, we choose a special flavor basis of three neutrino fields to factor out the unphysical phases of M_ν such that M_ν is reduced to M'_ν which only contains nine free parameters. Then we diagonalize the Hermitian matrix $H'_\nu \equiv M'_\nu M'^\dagger_\nu$ via the transformation $U^\dagger H'_\nu U = \hat{M}^2_\nu$. Different from Ref. [7], here a much simpler and more transparent way is found to establish some exact analytical relations between the physical parameters of U and $\hat{M}_\nu$ and the matrix elements of H'_ν. Our results can be used to work out the conditions for the μ–τ flavor symmetry with $\theta_{23} = \pi/4$ and maximal CP violation with $\delta = \pm\pi/2$. Their usefulness is illustrated by taking a simple example of M_ν. In particular, we show that $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ keep unchanged when constant matter effects are taken into account for a long-baseline neutrino oscillation experiment.

2. Let us diagonalize M_ν by means of the transformation $V^\dagger M_\nu V^* = \hat{M}_\nu$. Namely, M_ν can be parametrized as follows:

$$M_\nu = V\hat{M}_\nu V^T = P_l V' \hat{M}_\nu V'^T P_l^T = P_l M'_\nu P_l^T, \tag{8}$$

where $V = P_l V'$ with $V' = UP_\nu$, and $M'_\nu \equiv V'\hat{M}_\nu V'^T$. Since U, P_ν and $\hat{M}_\nu$ contain four, two and three real parameters respectively, M'_ν totally consists of nine parameters which are all physical or experimentally observable. Given three unphysical phases in P_l, the total number of real parameters of M_ν is therefore twelve. In other words, M_ν can be reduced to M'_ν after its three unphysical phases are factored out. This parameter counting is certainly consistent with Eq. (6), in which six independent elements of M_ν are totally composed of twelve real parameters. After substituting Eq. (8) into Eq. (5), we obtain

$$\mathcal{L}_\nu = \frac{1}{2} \overline{(\nu'_e \quad \nu'_\mu \quad \nu'_{\tau\,{\rm L}})} M'_\nu \begin{pmatrix} (\nu'_e)^{\rm c} \\ (\nu'_\mu)^{\rm c} \\ (\nu'_\tau)^{\rm c} \end{pmatrix}_{\rm R} + {\rm h.c.}, \tag{9}$$

where $\nu'_\alpha = \nu_\alpha e^{-i\phi_\alpha}$ (for $\alpha = e, \mu, \tau$). In this new basis of three neutrino fields, the corresponding Majorana neutrino mass matrix can be parametrized as

$$M'_\nu = V'\hat{M}_\nu V'^T = UP_\nu \hat{M}_\nu P_\nu^T U^T. \tag{10}$$

[2] Note that the notations used in Ref. [7] are different from ours.

Now it becomes obvious that the treatment in Ref. [7] is equivalent to a choice of the special flavor basis given in Eq. (9). This observation does not change even if one considers the following Hermitian matrices:

$$H'_\nu \equiv M'_\nu M'^\dagger_\nu = V' \hat{M}^2_\nu V'^\dagger = U P_\nu \hat{M}^2_\nu P^\dagger_\nu U^\dagger = U \hat{M}^2_\nu U^\dagger,$$
$$H_\nu \equiv M_\nu M^\dagger_\nu = V \hat{M}^2_\nu V^\dagger = P_l V' \hat{M}^2_\nu V'^\dagger P^\dagger_l = P_l H'_\nu P^\dagger_l. \tag{11}$$

Two comments are in order.

- H'_ν only contains seven real parameters and has nothing to do with the Majorana phase matrix P_ν. Hence one may establish the direct relations between the elements of H'_ν and the physical parameters of U and $\hat{M}_\nu$ (i.e., m_1, m_2, m_3; θ_{12}, θ_{13}, θ_{23} and δ), no matter whether massive neutrinos are the Dirac or Majorana particles.
- To diagonalize H_ν or M_ν itself, one has to take into account the unphysical phase matrix P_l. In the literature many authors have chosen the flavor basis defined in Eq. (9) to reconstruct the effective Majorana neutrino mass matrix M'_ν. This special phase convention or basis choice is useful in the study of neutrino phenomenology, but one should keep in mind that a neutrino mass model generally predicts M_ν in the flavor basis defined in Eq. (5).

After clarifying the difference between the flavor bases associated with M_ν (or H_ν) and M'_ν (or H'_ν), we shall follow a phenomenological way to derive the exact analytical expressions of three neutrino masses, three flavor mixing angles and the Dirac CP-violating phase in terms of the matrix elements of H'_ν. Our present treatment is much simpler and more transparent than the one given Ref. [7], and it leads us to two constraint equations for the matrix elements of H'_ν which were not presented in Ref. [7].

To be more specific, we denote the matrix elements of the Hermitian matrix H'_ν as

$$H'_\nu = \begin{pmatrix} A & B & C \\ B^* & D & E \\ C^* & E^* & F \end{pmatrix}, \tag{12}$$

where A, D and F are real, and B, C and E are in general complex. Note that $H'_\nu = P^\dagger_l H_\nu P_l = P^\dagger_l M_\nu M^\dagger_\nu P_l$ holds, and the matrix elements of M_ν have been expressed in Eq. (6). Therefore,

$$A = |a|^2 + |b|^2 + |c|^2, \qquad B = \left(ab^* + bd^* + ce^*\right) e^{i(\phi_\mu - \phi_e)}, \qquad C = \left(ac^* + be^* + cf^*\right) e^{i(\phi_\tau - \phi_e)},$$
$$D = |b|^2 + |d|^2 + |e|^2, \qquad E = \left(bc^* + de^* + ef^*\right) e^{i(\phi_\tau - \phi_\mu)}, \qquad F = |c|^2 + |e|^2 + |f|^2. \tag{13}$$

Let us reiterate that the above six matrix elements of H'_ν are not fully independent. Because H'_ν totally consists of seven real parameters, there must exist two constraint equations among A, B, C, D, E and F. This point can also be understood in another way. The Majorana phases of M_ν (i.e., ρ and σ) are completely canceled in the elements of H'_ν, and the unphysical phases of M_ν (i.e., ϕ_e, ϕ_μ and ϕ_τ) are also canceled in those elements. So the number of real parameters of H'_ν is not nine but seven, leading to two correlative equations of its six matrix elements. Now we substitute $U = O_{23} O_{13} O_{12}$ into the expression of H'_ν in Eq. (11). Then we arrive at

$$O^\dagger_{23} H'_\nu O_{23} = O_{13} O_{12} \hat{M}^2_\nu O^\dagger_{12} O^\dagger_{13} = O_{13} N_\nu O^\dagger_{13}, \tag{14}$$

where

$$N_\nu \equiv O_{12} \hat{M}^2_\nu O^\dagger_{12} = \begin{pmatrix} m_1^2 + s_{12}^2 \Delta m_{21}^2 & c_{12} s_{12} \Delta m_{21}^2 & 0 \\ c_{12} s_{12} \Delta m_{21}^2 & m_1^2 + c_{12}^2 \Delta m_{21}^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} \equiv \begin{pmatrix} N_{11} & N_{12} & 0 \\ N_{12} & N_{22} & 0 \\ 0 & 0 & N_{33} \end{pmatrix} \tag{15}$$

with $\Delta m^2_{21} \equiv m_2^2 - m_1^2$. The left- and right-hand sides of Eq. (14) explicitly read

$$O^\dagger_{23} H'_\nu O_{23} = \begin{pmatrix} A & c_{23} B - s_{23} C & s_{23} B + c_{23} C \\ c_{23} B^* - s_{23} C^* & c_{23}^2 D + s_{23}^2 F - 2 c_{23} s_{23} \mathrm{Re}(E) & c_{23} s_{23} (D - F) + c_{23}^2 E - s_{23}^2 E^* \\ s_{23} B^* + c_{23} C^* & c_{23} s_{23} (D - F) + c_{23}^2 E^* - s_{23}^2 E & s_{23}^2 D + c_{23}^2 F + 2 c_{23} s_{23} \mathrm{Re}(E) \end{pmatrix},$$
$$O_{13} N_\nu O^\dagger_{13} = \begin{pmatrix} c_{13}^2 N_{11} + s_{13}^2 N_{33} & c_{13} N_{12} & c_{13} \hat{s}^*_{13} (N_{33} - N_{11}) \\ c_{13} N_{12} & N_{22} & -\hat{s}^*_{13} N_{12} \\ c_{13} \hat{s}_{13} (N_{33} - N_{11}) & -\hat{s}_{13} N_{12} & c_{13}^2 N_{33} + s_{13}^2 N_{11} \end{pmatrix}. \tag{16}$$

The equality $(O^\dagger_{23} H'_\nu O_{23})_{12} = (O_{13} N_\nu O^\dagger_{13})_{12}$ yields

$$c_{13} N_{12} = c_{23} B - s_{23} C. \tag{17}$$

Since the left-hand side of Eq. (17) is real and positive, we immediately obtain $\mathrm{Im}(c_{23} B - s_{23} C) = 0$. As a result, the neutrino mixing angle θ_{23} is simply given by

$$\tan\theta_{23} = \frac{\mathrm{Im}(B)}{\mathrm{Im}(C)}. \tag{18}$$

On the other hand, the equality $(O_{23}^{\dagger} H'_{\nu} O_{23})_{13} = (O_{13} N_{\nu} O_{13}^{\dagger})_{13}$ yields

$$c_{13}\hat{s}_{13}^{*}(N_{33} - N_{11}) = s_{23}B + c_{23}C. \tag{19}$$

This equation allows us to obtain the Dirac CP-violating phase:

$$\tan\delta = -\frac{s_{23}\,\mathrm{Im}(B) + c_{23}\,\mathrm{Im}(C)}{s_{23}\,\mathrm{Re}(B) + c_{23}\,\mathrm{Re}(C)} = -\frac{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2}{\mathrm{Re}(B)\,\mathrm{Im}(B) + \mathrm{Re}(C)\,\mathrm{Im}(C)}. \tag{20}$$

With the help of the equality $(O_{23}^{\dagger} H'_{\nu} O_{23})_{23} = (O_{13} N_{\nu} O_{13}^{\dagger})_{23}$, we have

$$-\hat{s}_{13}^{*} N_{12} = c_{23}s_{23}(D - F) + c_{23}^2 E - s_{23}^2 E^{*}, \tag{21}$$

which can also lead us to an expression of δ:

$$\begin{aligned}\tan\delta &= -\frac{\mathrm{Im}(E)}{c_{23}s_{23}(D - F) + (c_{23}^2 - s_{23}^2)\,\mathrm{Re}(E)} \\ &= -\frac{\mathrm{Im}(E)\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}}{\mathrm{Im}(B)\,\mathrm{Im}(C)(D - F) - \{[\mathrm{Im}(B)]^2 - [\mathrm{Im}(C)]^2\}\,\mathrm{Re}(E)}.\end{aligned} \tag{22}$$

A straightforward comparison between Eqs. (20) and (22) yields a constraint equation for the matrix elements of H'_{ν}:

$$D - F = \frac{\{\mathrm{Re}(B)\,\mathrm{Im}(B) + \mathrm{Re}(C)\,\mathrm{Im}(C)\}\,\mathrm{Im}(E) + \{[\mathrm{Im}(B)]^2 - [\mathrm{Im}(C)]^2\}\,\mathrm{Re}(E)}{\mathrm{Im}(B)\,\mathrm{Im}(C)}. \tag{23}$$

Combining Eqs. (17) and (21), we obtain the smallest neutrino mixing angle θ_{13} as follows:

$$\begin{aligned}\tan\theta_{13} &= \left|\frac{c_{23}s_{23}(D - F) + c_{23}^2 E - s_{23}^2 E^{*}}{c_{23}B - s_{23}C}\right| \\ &= \left|\mathrm{Im}(E)\right| \frac{\sqrt{\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}^2 + \{\mathrm{Re}(B)\,\mathrm{Im}(B) + \mathrm{Re}(C)\,\mathrm{Im}(C)\}^2}}{\sqrt{\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}\{\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)\}^2}}.\end{aligned} \tag{24}$$

It should be noted that θ_{13} can also be derived in another way. The difference between the equalities $(O_{23}^{\dagger} H'_{\nu} O_{23})_{11} = (O_{13} N_{\nu} O_{13}^{\dagger})_{11}$ and $(O_{23}^{\dagger} H'_{\nu} O_{23})_{33} = (O_{13} N_{\nu} O_{13}^{\dagger})_{33}$ reads

$$\left(c_{13}^2 - s_{13}^2\right)(N_{33} - N_{11}) = s_{23}^2 D + c_{23}^2 F + 2c_{23}s_{23}\,\mathrm{Re}(E) - A. \tag{25}$$

Combining Eqs. (19) and (25), we obtain

$$\begin{aligned}\tan 2\theta_{13} &= \left|\frac{2(s_{23}B + c_{23}C)}{s_{23}^2 D + c_{23}^2 F + 2c_{23}s_{23}\,\mathrm{Re}(E) - A}\right| \\ &= \frac{2|B\,\mathrm{Im}(B) + C\,\mathrm{Im}(C)|\sqrt{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2}}{|(D - A)[\mathrm{Im}(B)]^2 + (F - A)[\mathrm{Im}(C)]^2 + 2\,\mathrm{Im}(B)\,\mathrm{Im}(C)\,\mathrm{Re}(E)|}.\end{aligned} \tag{26}$$

In view of $\tan 2\theta_{13} = 2\tan\theta_{13}/(1 - \tan^2\theta_{13})$, one may do a straightforward but lengthy calculation to work out another constraint equation for the elements of H'_{ν} from Eqs. (24) and (26). The result is

$$\begin{aligned}A = {} & \frac{\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)}{\mathrm{Im}(E)} + \frac{[\mathrm{Im}(B)]^2 D + [\mathrm{Im}(C)]^2 F + 2\,\mathrm{Im}(B)\,\mathrm{Im}(C)\,\mathrm{Re}(E)}{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2} \\ & - \frac{\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}^2 + \{\mathrm{Re}(B)\,\mathrm{Im}(B) + \mathrm{Re}(C)\,\mathrm{Im}(C)\}^2}{\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}\{\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)\}}\,\mathrm{Im}(E).\end{aligned} \tag{27}$$

Eqs. (23) and (27) clearly reflect the fact that H'_{ν} only contains seven independent real parameters.

We proceed to derive the expressions of θ_{12} and m_i^2 (for $i = 1, 2, 3$) by using Eq. (15). To do so, we have to first express N_{11}, N_{12}, N_{22} and N_{33} in terms of the matrix elements of H'_{ν}. These four quantities can be derived from Eq. (16) with the help of two constraint equations and the results of θ_{13}, θ_{23} and δ obtained above. After an algebraic exercise, we find

$$\begin{aligned}N_{11} &= A - \frac{\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)}{\mathrm{Im}(E)}, \\ N_{12} &= \left[\frac{[\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)]^2}{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2} + \left[\frac{\{\mathrm{Re}(B)\,\mathrm{Im}(B) + \mathrm{Re}(C)\,\mathrm{Im}(C)\}^2}{\{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2\}^2} + 1\right]\left[\mathrm{Im}(E)\right]^2\right]^{1/2}, \\ N_{22} &= \frac{[\mathrm{Im}(C)]^2 D + [\mathrm{Im}(B)]^2 F - 2\,\mathrm{Im}(B)\,\mathrm{Im}(C)\,\mathrm{Re}(E)}{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2}, \\ N_{33} &= \frac{[\mathrm{Im}(B)]^2 D + [\mathrm{Im}(C)]^2 F + 2\,\mathrm{Im}(B)\,\mathrm{Im}(C)\,\mathrm{Re}(E)}{[\mathrm{Im}(B)]^2 + [\mathrm{Im}(C)]^2} + \frac{\mathrm{Re}(B)\,\mathrm{Im}(C) - \mathrm{Im}(B)\,\mathrm{Re}(C)}{\mathrm{Im}(E)}.\end{aligned} \tag{28}$$

Then Eq. (15) leads us to

$$\tan 2\theta_{12} = \frac{2c_{12}s_{12}}{c_{12}^2 - s_{12}^2} = \frac{2N_{12}}{N_{22} - N_{11}}, \tag{29}$$

which can be expressed in terms of the elements of H'_ν via Eq. (28). Furthermore, three neutrino masses can simply be obtained from

$$m_1^2 = \frac{1}{2}(N_{11}+N_{22}) - \frac{1}{2}\sqrt{(N_{22}-N_{11})^2 + 4N_{12}^2}, \qquad m_2^2 = \frac{1}{2}(N_{11}+N_{22}) + \frac{1}{2}\sqrt{(N_{22}-N_{11})^2 + 4N_{12}^2}, \qquad m_3^2 = N_{33}. \tag{30}$$

So we complete the derivation of two constraint equations for the elements of H'_ν and their exact relations with seven physical quantities m_i^2 (for $i = 1, 2, 3$), θ_{ij} (for $ij = 12, 13, 23$) and δ.

3. Now we consider an especially interesting case of neutrino mixing and CP violation: $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$. In fact, θ_{23} corresponds to the μ–τ flavor symmetry in the neutrino sector in the chosen flavor basis (e.g., $|V_{\mu i}| = |V_{\tau i}|$ holds (for $i = 1, 2, 3$) in this case [8]); and $\delta = \pm\pi/2$ implies the maximal strength of CP violation in neutrino oscillations for given values of θ_{12}, θ_{13} and θ_{23} (i.e., the leptonic Jarlskog parameter is maximal in this case [9]). From Eqs. (18) and (20), we see that $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ lead us to

$$\mathrm{Im}(B) = \mathrm{Im}(C), \qquad \mathrm{Re}(B) = -\mathrm{Re}(C); \tag{31}$$

or equivalently $B = -C^*$. In this case Eqs. (23) and (27) are simplified to $D = F$ and

$$A = D + \mathrm{Re}(E) + 2\frac{\mathrm{Re}(B)\,\mathrm{Im}(B)}{\mathrm{Im}(E)} - \frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)}. \tag{32}$$

As a consequence,

$$\tan\theta_{13} = \frac{1}{\sqrt{2}}\left|\frac{\mathrm{Im}(E)}{\mathrm{Re}(B)}\right|, \qquad \tan 2\theta_{12} = 2\frac{|\mathrm{Re}(B)|\sqrt{2[\mathrm{Re}(B)]^2 + [\mathrm{Im}(E)]^2}}{|\mathrm{Im}(B)\,\mathrm{Im}(E) - 2\,\mathrm{Re}(B)\,\mathrm{Re}(E)|}; \tag{33}$$

and

$$m_1^2 = \frac{1}{2}\left[2D - \frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)} - \sqrt{\left[\frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)} - 2\,\mathrm{Re}(E)\right]^2 + 4\left\{2\left[\mathrm{Re}(B)\right]^2 + \left[\mathrm{Im}(E)\right]^2\right\}}\right],$$

$$m_2^2 = \frac{1}{2}\left[2D - \frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)} + \sqrt{\left[\frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)} - 2\,\mathrm{Re}(E)\right]^2 + 4\left\{2\left[\mathrm{Re}(B)\right]^2 + \left[\mathrm{Im}(E)\right]^2\right\}}\right],$$

$$m_3^2 = A - \frac{\mathrm{Im}(B)\,\mathrm{Im}(E)}{\mathrm{Re}(B)}. \tag{34}$$

These simplified results are expected to be useful in discussing a specific neutrino mass model with the μ–τ flavor symmetry and maximal CP violation.

Given the μ–τ flavor symmetry and maximal CP violation, the form of H'_ν explicitly reads

$$\mathcal{H}'_\nu = \begin{pmatrix} A & B & -B^* \\ B^* & D & E \\ -B & E^* & D \end{pmatrix}, \tag{35}$$

in which A, B D and E are related to one another through Eq. (32). Hence $\mathcal{H}'_\nu$ totally contains five real and independent parameters. The corresponding form of H_ν defined in Eq. (11) is

$$\mathcal{H}_\nu = P_l \mathcal{H}'_\nu P_l^\dagger = \begin{pmatrix} A & Be^{i(\phi_e-\phi_\mu)} & -B^*e^{i(\phi_e-\phi_\tau)} \\ B^*e^{i(\phi_\mu-\phi_e)} & D & Ee^{i(\phi_\mu-\phi_\tau)} \\ -Be^{i(\phi_\tau-\phi_e)} & E^*e^{i(\phi_\tau-\phi_\mu)} & D \end{pmatrix}. \tag{36}$$

The meaning of this matrix is clear: if the texture of a Majorana neutrino mass matrix $\mathcal{M}_\nu$ derived from a specific neutrino mass model satisfies $\mathcal{M}_\nu\mathcal{M}_\nu^\dagger = \mathcal{H}_\nu$ as given in Eq. (36), then it must predict $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ in the standard parametrization of the MNSP matrix. This prediction is independent of the constraint equation in Eq. (32). Because three unphysical phases ϕ_α (for $\alpha = e, \mu, \tau$) can be arbitrarily rearranged, one may simply compare a model-dependent texture of $\mathcal{H}_\nu$ with Eq. (36) to judge whether they are consistent with each other.

To illustrate, let us consider a typical texture of the effective Majorana neutrino mass matrix:

$$\mathcal{M}_\nu = \begin{pmatrix} a & b & -b^* \\ b & d & e \\ -b^* & e & d^* \end{pmatrix} \tag{37}$$

with a and e being real, which has been discussed in a number of neutrino mass models with discrete flavor symmetries [10]. Therefore,

$$\mathcal{H}_\nu = \mathcal{M}_\nu \mathcal{M}_\nu^\dagger = \begin{pmatrix} \mathcal{A} & \mathcal{B} & -\mathcal{B}^* \\ \mathcal{B}^* & \mathcal{D} & \mathcal{E} \\ -\mathcal{B} & \mathcal{E}^* & \mathcal{D} \end{pmatrix}, \tag{38}$$

where

$$\mathcal{A} = a^2 + 2|b|^2, \qquad \mathcal{B} = (a-e)b^* + bd^*, \qquad \mathcal{D} = |b|^2 + |d|^2 + e^2, \qquad \mathcal{E} = -b^2 + 2de. \tag{39}$$

Comparing Eq. (38) with Eq. (36), we immediately see that they are consistent with each other if

$$2\phi_e = \phi_\mu + \phi_\tau \tag{40}$$

is taken. In this case $\mathcal{H}_\nu$ in Eq. (36) totally contains six real and independent parameters: five of them come from $\mathcal{H}'_\nu$ given in Eq. (35), and the left is just $(\phi_\mu - \phi_\tau)$ because $\phi_e - \phi_\mu = -(\phi_\mu - \phi_\tau)/2$ and $\phi_e - \phi_\tau = (\phi_\mu - \phi_\tau)/2$ hold. In comparison, $\mathcal{H}_\nu$ in Eq. (38) consists of six real and independent parameters too. That is why it is improper to take $\phi_e = \phi_\mu = \phi_\tau$ (or to simply assume all of them to vanish [7]) and then equalize Eqs. (36) and (38). Otherwise, the resultant parameter mismatching problem would violate Eq. (32) and make the results in Eqs. (33) and (34) invalid. We stress that Eq. (40) is the proper phase convention which allows us to reduce $\mathcal{H}_\nu$ to $\mathcal{H}'_\nu$ in Eq. (35) after the transformation $\mathcal{H}'_\nu = P_l^\dagger \mathcal{H}_\nu P_l$. Hence we must be able to arrive at the μ–τ flavor symmetry with $\theta_{23} = \pi/4$ and maximal CP violation with $\delta = \pm\pi/2$. This example clearly shows that $\mathcal{H}_\nu$ and $\mathcal{H}'_\nu$ correspond to two different flavor bases as generally defined in Eqs. (5) and (9), and only in the latter basis the exact analytical results of m_i^2 (for $i = 1, 2, 3$), θ_{ij} (for $ij = 12, 13, 23$) and δ obtained above are safely applicable.

We remark that $\theta_{23} = \pi/4$ is strongly favored by current neutrino oscillation data [11]. Although $\delta = \pm\pi/2$ is purely a phenomenological conjecture, it corresponds to maximal leptonic CP violation and thus is very interesting. In fact, it is possible to obtain $\delta = \pm\pi/2$ from a number of neutrino mass models [12]. If neither the μ–τ flavor symmetry nor maximal CP violation is realistic, one may still use the exact analytical relations between the elements of H'_ν and seven physical quantities to discuss a specific neutrino mass model. To do so, however, one must carefully choose the flavor basis of three neutrino fields so as to eliminate or factor out the relevant unphysical phases hidden in the original Majorana neutrino mass matrix.

Of course, one may follow the same procedure to directly diagonalize $H_\nu = V \hat{M}_\nu^2 V^\dagger = P_l H'_\nu P_l^\dagger$ in a generic flavor basis. In this case the analytical results of m_i (for $i = 1, 2, 3$), θ_{ij} (for $ij = 12, 13, 23$) and δ are more complicated and less useful than the ones obtained above, simply because the phases of P_l must be involved to cancel the unphysical phases hidden in the matrix elements of H_ν. Such an exercise has been done in Ref. [13]. Switching off the unphysical phases and taking account of the constraint equations, we find that it is possible to reach an agreement between the results obtained in Ref. [13] and ours.

Finally, we point out an immediate and interesting application of Eq. (36) to the analysis of terrestrial matter effects on neutrino mixing and CP violation. Assuming a constant matter density profile, we may write out the effective Hamiltonian responsible for the propagation of a neutrino beam in matter in the same way as that in vacuum:

$$\begin{aligned} \mathbf{H}_{\rm v} &= \frac{1}{2\mathbf{E}} V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger, \\ \mathbf{H}_{\rm m} &= \frac{1}{2\mathbf{E}} \tilde{V} \begin{pmatrix} \tilde{m}_1^2 & 0 & 0 \\ 0 & \tilde{m}_2^2 & 0 \\ 0 & 0 & \tilde{m}_3^2 \end{pmatrix} \tilde{V}^\dagger = \frac{1}{2\mathbf{E}} V \begin{pmatrix} m_1^2 & 0 & 0 \\ 0 & m_2^2 & 0 \\ 0 & 0 & m_3^2 \end{pmatrix} V^\dagger + \begin{pmatrix} \mathbf{a} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \end{aligned} \tag{41}$$

where $\mathbf{E}$ is the neutrino beam energy, $\mathbf{a} = \sqrt{2}\, G_{\rm F} n_e$ stands for the terrestrial matter effects [14], $\tilde{m}_i$ (for $i = 1, 2, 3$) denote the effective neutrino masses in matter, and $\tilde{V}$ represents the effective neutrino mixing matrix in matter [15]. Given $\theta_{23} = \pi/4$ and $\delta = \pm\pi/2$ for V in vacuum, the texture of $\mathbf{H}_{\rm v}$ must be the same as $\mathcal{H}_\nu$ in Eq. (36). In this case $\mathbf{H}_{\rm m}$ takes the same texture as $\mathbf{H}_{\rm v}$, but its $(1,1)$ element is different from that of $\mathbf{H}_{\rm v}$. This difference implies that the constraint equation in Eq. (32) does not hold for the matrix elements of $\mathbf{H}_{\rm m}$, and thus we are left with $\tilde{m}_i \neq m_i$, $\tilde{\theta}_{12} \neq \theta_{12}$ and $\tilde{\theta}_{13} \neq \theta_{13}$. In contrast, the basic texture of $\mathbf{H}_{\rm v}$ or $\mathbf{H}_{\rm m}$ can be reduced to $\mathcal{H}'_\nu$ after a proper phase transformation, so $\tilde{\theta}_{23} = \theta_{23} = \pi/4$ and $\tilde{\delta} = \delta = \pm\pi/2$ must hold. This observation is apparently consistent with the Toshev equality $\sin 2\tilde{\theta}_{23} \sin\tilde{\delta} = \sin 2\theta_{23} \sin\delta$ [16]. Our conclusion is that the μ–τ flavor symmetry with $\theta_{23} = \pi/4$ and maximal CP violation with $\delta = \pm\pi/2$ keep unchanged when constant matter effects are taken into account for a long-baseline neutrino oscillation experiment. We shall explore more generic applications of our results obtained in this Letter to the description of neutrino oscillations in matter elsewhere [17].

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